The Sperner Property and Quotients of Boolean Algebras

Jason Tang
Mentor: Luis Kumanduri

May 22, 2021

Partially-ordered Sets

Definition

A finite set P is called a **partially-ordered set (poset)** if it is equipped with relation $\leq$ with the following properties for $a, b, c \in P$:

- $a \leq a$ for all a.
- $a \leq b$ and $b \leq a$ implies $a = b$.
- $a \leq b$ and $b \leq c$ implies $a \leq c$.

Partially-ordered Sets

Definition

A finite set P is called a **partially-ordered set (poset)** if it is equipped with relation $\leq$ with the following properties for $a, b, c \in P$:

- $a \leq a$ for all a.
- $a \leq b$ and $b \leq a$ implies $a = b$.
- $a \leq b$ and $b \leq c$ implies $a \leq c$.

Example: The Boolean algebra B_n, which consists of all subsets of the set $\{1, 2, \ldots, n\}$.

- Relation $\leq$ is defined such that sets $a, b \in B_n$ satisfy $a \leq b$ if a is a subset of b.
- $\{1\} \leq \{1, 2\}$, whereas $\{1\}$ and $\{3\}$ are incomparable.

Chains

Definition

For a poset P, a **chain** is a subset $C \subseteq P$ for which all elements of C are pairwise comparable.

Chains

Definition

For a poset P, a **chain** is a subset $C \subseteq P$ for which all elements of C are pairwise comparable.

- Define the overall **rank** of P as the length of the maximum chain within P, and define the **rank of element** $p \in P$, denoted by $\rho(p)$, as the length of the maximum chain with greatest element p.

Chains

Definition

For a poset P, a **chain** is a subset $C \subseteq P$ for which all elements of C are pairwise comparable.

- Define the overall **rank** of P as the length of the maximum chain within P, and define the **rank of element** $p \in P$, denoted by $\rho(p)$, as the length of the maximum chain with greatest element p.
- Define the ***n*th level** of P as the subset $P_n \subseteq P$ consisting of all elements $p \in P$ for which $\rho(p) = n$.

Chains

Definition

For a poset P, a **chain** is a subset $C \subseteq P$ for which all elements of C are pairwise comparable.

- Define the overall **rank** of P as the length of the maximum chain within P, and define the **rank of element** $p \in P$, denoted by $\rho(p)$, as the length of the maximum chain with greatest element p.
- Define the ***n*th level** of P as the subset $P_n \subseteq P$ consisting of all elements $p \in P$ for which $\rho(p) = n$.

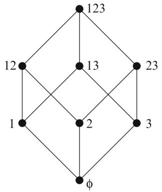

Figure: Hasse Diagram of B_3; from Stanley's *Algebraic Combinatorics*

Antichains

Definition

For a poset P, an **antichain** is a subset $A \subseteq P$ for which no elements of A are pairwise comparable.

Antichains

Definition

For a poset P, an **antichain** is a subset $A \subseteq P$ for which no elements of A are pairwise comparable.

- Note that each level of P is an individual antichain.

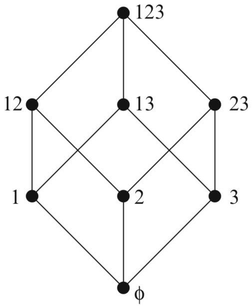

Figure: Hasse Diagram of B_3; from Stanley's *Algebraic Combinatorics*

Sperner Problem

Problem

What is the size of the largest antichain in B_n?

Sperner Problem

Problem

What is the size of the largest antichain in B_n?

- A reasonable guess would be that the largest antichain is the largest level of B_n, i.e. $(B_n)_{\lfloor n/2 \rfloor}$.

Sperner Problem

Problem

What is the size of the largest antichain in B_n?

- A reasonable guess would be that the largest antichain is the largest level of B_n, i.e. $(B_n)_{\lfloor n/2 \rfloor}$.

Definition

A poset P has the **Sperner property** if the maximum size of an antichain is equal to the size as the largest level of P.

We seek to prove the Sperner property for B_n.

Order-raising operators

Consider a poset P with levels denoted by P_j.

Definition

A linear transformation $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is an **order-raising operator** if for any $x \in P_i$, the image $U(x)$ can be expressed as a linear combination of elements in P_{i+1} comparable to x.

Order-raising operators

Consider a poset P with levels denoted by P_j.

Definition

A linear transformation $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is an **order-raising operator** if for any $x \in P_i$, the image $U(x)$ can be expressed as a linear combination of elements in P_{i+1} comparable to x.

- We can assign a $p_{i+1} \times p_i$ matrix $[U]$ to each order-raising operator U, where the columns are lined by bases $x_1, \ldots, x_{p_i}$ of P_i and the rows are lined by bases $y_1, \ldots, y_{p_{i+1}}$ of P_{i+1}.

Order-matchings

Definition

A one-to-one function μ is called an **order-matching** if across every $x \in P_i$, the relationship between x and $\mu(x)$ is the same. That is, if μ is from $P_i \rightarrow P_{i+1}$, then all $x < \mu(x)$, and if μ is from $P_{i+1} \rightarrow P_i$, then all $x > \mu(x)$.

Order-matchings

Definition

A one-to-one function μ is called an **order-matching** if across every $x \in P_i$, the relationship between x and $\mu(x)$ is the same. That is, if μ is from $P_i \to P_{i+1}$, then all $x < \mu(x)$, and if μ is from $P_{i+1} \to P_i$, then all $x > \mu(x)$.

Lemma

If an order-raising operator $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is one-to-one, then there is an order-matching from P_i to P_{i+1}. Similarly, if U is onto, then there is an order-matching from P_{i+1} to P_i.

- The lemma can be proved by examining the determinant of a submatrix of $[U]$, whose terms will allow us to directly construct the order-matching we seek.

Application to B_n

- Define order-raising operators $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ where $U_i(x)$ is the sum of all elements in $(B_n)_{i+1}$ comparable to x.

Application to B_n

- Define order-raising operators $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ where $U_i(x)$ is the sum of all elements in $(B_n)_{i+1}$ comparable to x.
- Define dual operators $D_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}$, where $D_i(y)$ is the sum of all elements in $(B_n)_{i-1}$ comparable to y.

Application to B_n

- Define order-raising operators $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ where $U_i(x)$ is the sum of all elements in $(B_n)_{i+1}$ comparable to x.
- Define dual operators $D_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}$, where $D_i(y)$ is the sum of all elements in $(B_n)_{i-1}$ comparable to y.
- Note that $[D_i] = [U_{i-1}]^T$, which implies that $[U_{i-1}][D_i]$ is positive semidefinite. Thus, it has nonnegative eigenvalues.

Application to B_n

- Define order-raising operators $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ where $U_i(x)$ is the sum of all elements in $(B_n)_{i+1}$ comparable to x.
- Define dual operators $D_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}$, where $D_i(y)$ is the sum of all elements in $(B_n)_{i-1}$ comparable to y.
- Note that $[D_i] = [U_{i-1}]^T$, which implies that $[U_{i-1}][D_i]$ is positive semidefinite. Thus, it has nonnegative eigenvalues.

Lemma

For any $i \in [0, n]$, the property $D_{i+1}U_i - U_{i-1}D_i = (n-2i)I_i$ holds for identity transformation I_i on $\mathbb{R}(B_n)_i$.

Application to B_n

- Define order-raising operators $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ where $U_i(x)$ is the sum of all elements in $(B_n)_{i+1}$ comparable to x.
- Define dual operators $D_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}$, where $D_i(y)$ is the sum of all elements in $(B_n)_{i-1}$ comparable to y.
- Note that $[D_i] = [U_{i-1}]^T$, which implies that $[U_{i-1}][D_i]$ is positive semidefinite. Thus, it has nonnegative eigenvalues.

Lemma

For any $i \in [0, n]$, the property $D_{i+1}U_i - U_{i-1}D_i = (n-2i)I_i$ holds for identity transformation I_i on $\mathbb{R}(B_n)_i$.

- Thus, $[D_{i+1}][U_i] = [U_{i-1}][D_i] + (n-2i)[I_i]$. Then, for all $i < n/2$, the matrix $[D_{i+1}][U_i]$ has all positive eigenvalues and $D_{i+1}U_i$ is invertible. Hence, U_i is one-to-one for $i < n/2$.

Application to B_n

- Define order-raising operators $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ where $U_i(x)$ is the sum of all elements in $(B_n)_{i+1}$ comparable to x.
- Define dual operators $D_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}$, where $D_i(y)$ is the sum of all elements in $(B_n)_{i-1}$ comparable to y.
- Note that $[D_i] = [U_{i-1}]^T$, which implies that $[U_{i-1}][D_i]$ is positive semidefinite. Thus, it has nonnegative eigenvalues.

Lemma

For any $i \in [0, n]$, the property $D_{i+1}U_i - U_{i-1}D_i = (n-2i)I_i$ holds for identity transformation I_i on $\mathbb{R}(B_n)_i$.

- Thus, $[D_{i+1}][U_i] = [U_{i-1}][D_i] + (n-2i)[I_i]$. Then, for all $i < n/2$, the matrix $[D_{i+1}][U_i]$ has all positive eigenvalues and $D_{i+1}U_i$ is invertible. Hence, U_i is one-to-one for $i < n/2$.
- Employing a similar lemma allows us to conclude that U_iD_{i+1} is invertible for $i \geq n/2$ and that U_i is onto for those i.

Sperner's Theorem

Lemma

If an order-raising operator $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ *is one-to-one, then there is an order-matching from* P_i *to* P_{i+1}*. Similarly, if* U *is onto, then there is an order-matching from* P_{i+1} *to* P_i*.*

- Combining the above results with our previous lemma, we conclude there is a sequence of order-matchings

$$(B_n)_0 \to (B_n)_1 \to \cdots \to (B_n)_{\lfloor n/2 \rfloor} \leftarrow B_{\lfloor n/2 \rfloor + 1} \cdots \leftarrow (B_n)_n$$

Sperner's Theorem

Lemma

If an order-raising operator $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is one-to-one, then there is an order-matching from P_i to P_{i+1}. Similarly, if U is onto, then there is an order-matching from P_{i+1} to P_i.

- Combining the above results with our previous lemma, we conclude there is a sequence of order-matchings

$$(B_n)_0 \to (B_n)_1 \to \cdots \to (B_n)_{\lfloor n/2 \rfloor} \leftarrow B_{\lfloor n/2 \rfloor + 1} \cdots \leftarrow (B_n)_n$$

- Note that due to the sequence of order-matchings, all chains of B_n contain an element of $(B_n)_{\lfloor n/2 \rfloor}$.

Sperner's Theorem

Lemma

If an order-raising operator $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is one-to-one, then there is an order-matching from P_i to P_{i+1}. Similarly, if U is onto, then there is an order-matching from P_{i+1} to P_i.

- Combining the above results with our previous lemma, we conclude there is a sequence of order-matchings

$$(B_n)_0 \to (B_n)_1 \to \cdots \to (B_n)_{\lfloor n/2 \rfloor} \leftarrow B_{\lfloor n/2 \rfloor + 1} \cdots \leftarrow (B_n)_n$$

- Note that due to the sequence of order-matchings, all chains of B_n contain an element of $(B_n)_{\lfloor n/2 \rfloor}$.
- An antichain can intersect each chain at most once, so it cannot exceed the size of $(B_n)_{\lfloor n/2 \rfloor}$.

Sperner's Theorem

Lemma

If an order-raising operator $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is one-to-one, then there is an order-matching from P_i to P_{i+1}. Similarly, if U is onto, then there is an order-matching from P_{i+1} to P_i.

- Combining the above results with our previous lemma, we conclude there is a sequence of order-matchings

$$(B_n)_0 \to (B_n)_1 \to \cdots \to (B_n)_{\lfloor n/2 \rfloor} \leftarrow B_{\lfloor n/2 \rfloor + 1} \cdots \leftarrow (B_n)_n$$

- Note that due to the sequence of order-matchings, all chains of B_n contain an element of $(B_n)_{\lfloor n/2 \rfloor}$.
- An antichain can intersect each chain at most once, so it cannot exceed the size of $(B_n)_{\lfloor n/2 \rfloor}$.

Theorem (Sperner's Theorem)

B_n is Sperner.

Quotient Posets of B_n

While it's nice to know B_n is Sperner, we often want to understand how sets like B_n behave under group action.

Definition

The **quotient poset**, denoted by B_n/G, of B_n under a subgroup G of S_n consists of the orbits of G and is equipped with relation $\leq$ such that sets $a, b \in B_n/G$ satisfy $a \leq b$ if an element of a is less than or equal to an element of b under B_n's relation.

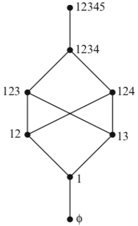

Figure: Hasse Diagram of B_5/G where G is generated by $(5, 1, 2, 3, 4)$; from Stanley's *Algebraic Combinatorics*

Sperner Property on Quotient Posets

Theorem

B_n/G is Sperner for any subgroup G of the symmetric group S_n.

Sperner Property on Quotient Posets

Theorem

B_n/G is Sperner for any subgroup G of the symmetric group S_n.

- We essentially recreate the proof used for B_n with some additions and modifications.

Sperner Property on Quotient Posets

Theorem

B_n/G is Sperner for any subgroup G of the symmetric group S_n.

- We essentially recreate the proof used for B_n with some additions and modifications.

Definition

Define $\mathbb{R}(B_n)_i^G$ as the subset of $\mathbb{R}(B_n)_i$ whose elements are fixed under the action of G.

Sperner Property on Quotient Posets

Theorem

B_n/G is Sperner for any subgroup G of the symmetric group S_n.

- We essentially recreate the proof used for B_n with some additions and modifications.

Definition

Define $\mathbb{R}(B_n)_i^G$ as the subset of $\mathbb{R}(B_n)_i$ whose elements are fixed under the action of G.

Key claims:

- Lemma: For any order-raising operator U_i and $v \in \mathbb{R}(B_n)_i^G$, the image $U_i(v)$ lies in $\mathbb{R}(B_n)_{i+1}^G$.

Sperner Property on Quotient Posets

Theorem

B_n/G is Sperner for any subgroup G of the symmetric group S_n.

- We essentially recreate the proof used for B_n with some additions and modifications.

Definition

Define $\mathbb{R}(B_n)_i^G$ as the subset of $\mathbb{R}(B_n)_i$ whose elements are fixed under the action of G.

Key claims:

- Lemma: For any order-raising operator U_i and $v \in \mathbb{R}(B_n)_i^G$, the image $U_i(v)$ lies in $\mathbb{R}(B_n)_{i+1}^G$.
- A basis for $\mathbb{R}(B_n)_i^G$ is comprised of

$$v_o = \text{sum of all elements in orbit } o$$

for all orbits $o \in B_n/G$.

Pairing Order-Raising Operators

- Start with the classic $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$, defined such that

$$U_i(v_o) = \sum v_{o'}$$

across all $o \in (B_n/G)_i$ and $o' \in (B_n/G)_{i+1}$.

Pairing Order-Raising Operators

- Start with the classic $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$, defined such that

$$U_i(v_o) = \sum v_{o'}$$

  across all $o \in (B_n/G)_i$ and $o' \in (B_n/G)_{i+1}$.
- Define corresponding operators $\hat{U}_i : \mathbb{R}(B_n/G)_i \to \mathbb{R}(B_n/G)_{i+1}$ defined by

$$\hat{U}_i(o) = \sum o'$$

  across all $o' \in \mathbb{R}(B_n/G)_{i+1}$.

Pairing Order-Raising Operators

- Start with the classic $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$, defined such that

  $$U_i(v_o) = \sum v_{o'}$$

  across all $o \in (B_n/G)_i$ and $o' \in (B_n/G)_{i+1}$.
- Define corresponding operators $\hat{U}_i : \mathbb{R}(B_n/G)_i \to \mathbb{R}(B_n/G)_{i+1}$ defined by

  $$\hat{U}_i(o) = \sum o'$$

  across all $o' \in \mathbb{R}(B_n/G)_{i+1}$.
- There is an isomorphism from $\mathbb{R}B_n^G \to \mathbb{R}B_n/G$ by mapping v_o to o.

Pairing Order-Raising Operators

- Start with the classic $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$, defined such that

$$U_i(v_o) = \sum v_{o'}$$

  across all $o \in (B_n/G)_i$ and $o' \in (B_n/G)_{i+1}$.
- Define corresponding operators $\hat{U}_i : \mathbb{R}(B_n/G)_i \to \mathbb{R}(B_n/G)_{i+1}$ defined by

$$\hat{U}_i(o) = \sum o'$$

  across all $o' \in \mathbb{R}(B_n/G)_{i+1}$.
- There is an isomorphism from $\mathbb{R}B_n^G \to \mathbb{R}B_n/G$ by mapping v_o to o.
  - The fact that U_i is order-raising implies that $\widehat{U}_i$ is order-raising.

Pairing Order-Raising Operators

- Start with the classic $U_i : \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$, defined such that

$$U_i(v_o) = \sum v_{o'}$$

  across all $o \in (B_n/G)_i$ and $o' \in (B_n/G)_{i+1}$.
- Define corresponding operators $\hat{U}_i : \mathbb{R}(B_n/G)_i \to \mathbb{R}(B_n/G)_{i+1}$ defined by

$$\hat{U}_i(o) = \sum o'$$

  across all $o' \in \mathbb{R}(B_n/G)_{i+1}$.
- There is an isomorphism from $\mathbb{R}B_n^G \to \mathbb{R}B_n/G$ by mapping v_o to o.
  - The fact that U_i is order-raising implies that $\widehat{U}_i$ is order-raising.
  - U_i being one-to-one for $i < n/2$ implies that $\widehat{U}_i$ is one-to-one for $i < n/2$.

Concluding the Sperner Property

- We define dual operators D_i and $\widehat{D_i}$ analogously, and we can use the same reasoning to conclude that $\widehat{D_i}$ is order-lowering and one-to-one for $i \geq n/2$.

Concluding the Sperner Property

- We define dual operators D_i and $\widehat{D_i}$ analogously, and we can use the same reasoning to conclude that $\widehat{D_i}$ is order-lowering and one-to-one for $i \geq n/2$.

Lemma

If an order-raising operator $U : \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is one-to-one, then there is an order-matching from P_i to P_{i+1}. Similarly, if U is onto, then there is an order-matching from P_{i+1} to P_i.

Combining the above results with this lemma that we used before, we can finish off the proof the exact same way.

Theorem

B_n/G is Sperner for any subgroup G of the symmetric group S_n.

Acknowledgements

I would like to thank:

- My mentor, Luis Kumanduri
- MIT PRIMES

References

Richard Stanley.
Algebraic Combinatorics.
Springer, 2013.