The binomial theorem and related identities

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Common mistake

The box method:

\[
\begin{array}{c|c|c}
\text{a} & \text{+b} \\
\hline
\text{a} & a^2 & ab \\
\text{+b} & ab & b^2 \\
\end{array}
\]

The foil method:

\[
(a + b)(a + b) = a^2 + 2ab + b^2
\]

\[
(a + b)^2 = a^2 + b^2
\]

\[
(a + b)^0 = 1
\]

\[
(a + b)^1 = a + b
\]

\[
(a + b)^4 = ??
\]

??
Background information - binomial theorem

Help us to expand \((x + y)^n\) expression
Explains how to express the coefficient in \((x + y)^n\)
Can prove the result in combinatorics
Explore probability
The binomial theorem

The binomial theorem formula

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

Reminder:

\[\binom{n}{k} = \frac{n!}{k! (n-k)!}\]
Example

Expand \((x + y)^5\)

\[
\binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5
\]

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]

\[
\binom{5}{0} = \frac{5!}{0!5!} = 1
\]

Answer: \(x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1y^5\)
Prove: \( 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \)

Set \( X=-1 \) and \( y=1 \) in the binomial theorem

\[
(-1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}
\]

\[
0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k
\]
The pascal’s triangle

Help you to calculate the binomial theorem and find combinations way faster and easier.

We start with 1 at the top and start adding number slowly below the triangular.
Example

Let's look at an example

\[ (3x - 6)^4 \]

Now let's solve this problem by using Pascal's triangle:

\[
\begin{array}{cccccc}
1 & 4 & 6 & 4 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

\[
\begin{align*}
1(3x)^4(-6)^0 & + 4(3x)^3(-6)^1 + 6(3x)^2(-6)^2 + 4(3x)^1(-2)^3 + 1(3x)^0(-6)^4 \\
& + 1(3x)^0(-6)^4
\end{align*}
\]

Answer: \[ 3x^4 - 648x^3 + 1944x^2 - 96x + 1296 \]
How can people come up with Pascal’s triangle

All nonnegative integers $n$ and $k$

\[
\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}
\]
Why it is true?

\[
\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}
\]

Numbers of ways to choose \(k\) elements from \(n\)  
Numbers of ways to choose \(k+1\) elements from \(n\)  
Counts number of ways to choose \(k+1\) elements from \(n+1\) elements

\(\binom{n}{k}\) choose \(n+1\) as one of \(k+1\) elements

\(\binom{n}{k+1}\) do not choose \(n+1\)
Generalized binomial theorem

The binomial theorem is only true when \( n = 0,1,2,... \).
So what is \( n \) is a negative number or fractions, how can we solve.

The binomial theorem: \( (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \)

The generalized binomial theorem:

\[
(1 + b)^r = \sum_{k=0}^{\infty} \binom{r}{k} b^k, \quad r \in \mathbb{R}
\]
Example

What does \( r \) choose \( k \) mean when \( r \) is not a positive integer \( \binom{r}{k} \)?

\[
\binom{r}{k} = \frac{r(r-1)(r-2)(r-3)\ldots(r-(k-1))}{k!}
\]

\[
\binom{-3}{6} = \frac{(-3)(-4)(-5)(-6)(-7)(-8)}{6!5!4!3!2!1} = 28
\]
Example

$$b = -x$$

$$\sum_{k=0}^{\infty} \binom{-1}{k} (-x)^k = (1 - x)^{-1}$$
Trinomial theorem

\[(a + b + c)^n = \sum_{i+j+k} \binom{n}{i,j,k} a^i b^j c^k\]

Where \(i, j, k\) will be non-negative number

\[\binom{n}{i, j, k} = \frac{n!}{i! j! k!}\]
Example

How many terms in \((a + b + c)^7\) ?

\[i+j+k=7\]

\[I=2\]

\[J=0\]

\[K=5\]

\[\binom{9}{2}=36\]
Background information - multinomial theorem

How can we expand \((x_1 + \ldots + x_k)^n\)?

William L Hosch created the multinomial theorem

Multinomial theorem originally taken from binomial theorem

It consists of the sum of many terms
Multinomial theorem

\[(x_1 + \cdots + x_k)^n = \sum \frac{n!}{e_1!e_2!\cdots e_k!} x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}\]

Where: \(e_1, e_2 \ldots e_k \geq 0\),

\(e_i\) is exponent of \(x_i\) in a monomial

\(e_1 + \cdots + e_k = n\)
Vandermonde’s identity

\[ \binom{n + m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k - i} \quad r, m, n \in \mathbb{N}_0 \]
**Example**

Prove Vanderdoes's identity using combinatorics:

- $|X| = n$
- $|Y| = m$
- $|X \cap Y| = |X| + |Y| - 0$
- $|X \cup Y| = |X| + |Y| = m + n$

**One side:** $\binom{m+n}{r}$

**Other side:** let $0 \leq k \leq r$

Choose $K$ elements from $X$ $\binom{n}{k}$

Choose $r-k$ elements from $Y$ $\binom{m}{r-k}$

We get $\sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}$
Binomial theorem, general version

Formula:

\[(1 + x)^m = \sum_{n \geq 0} \binom{m}{n} x^n\]

Where m must be any real number

Sum taken all non-negative integer n
Example

Find the power series expansion of $\sqrt{1 - 4x}$

\[
(1 - 4x)^{\frac{1}{2}} = \sum_{n \geq 0} \binom{1}{n} (-4x)^n
\]

\[
\binom{1}{n} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-2n + 3}{2}}{n!} = (-1)^{n-1} \frac{(2n - 3)!!}{2^n \cdot n!}
\]
Continue the example

\[
\sqrt{1 - 4x} = 1 - 2x - \sum_{n \geq 2} \frac{2^n \cdot (2n - 3)!!}{n!} \cdot x^n
\]

\[
\frac{2^n \cdot (2n - 3)!!}{n!} = 2 \cdot \frac{(2n - 2)!}{n! \cdot (n - 1)!}
\]

We got

\[
\sqrt{1 - 4x} = 1 - 2x - \frac{2}{n} \sum_{n \geq 2} \left(\frac{2n - 2}{n - 1}\right) x^n
\]
Source

A walk-through combinatorics
An introduction to enumeration and graph theory
Fourth edition
Miklos Bona