Number Fields and Galois Theory

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MIT PRIMES Circle

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- 14 years old
- Grade 10
Introduction
Overview
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- Number theory from *Elementary Number Theory* by Jones and Jones
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  - Divisibility
  - Prime Numbers
  - Congruences
  - Congruences of Prime-Power Moduli
  - Euler’s Function
  - The Group of Units
  - Quadratic Residues
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- Galois theory, especially in relation to number fields
Number Fields
and Galois Theory

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Number Fields
Factorizing Ideals
Galois Theory

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- Today’s topic: number fields and Galois theory
Number Fields
Definition

A **field** $F$ is a commutative ring containing the multiplicative identity where every non-zero element is a unit (has an inverse).
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### Example

$\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all examples of fields.
Fields

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A field $F$ is a commutative ring containing the multiplicative identity where every non-zero element is a unit (has an inverse).

**Example**

$\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all examples of fields.

**Non-Example**

$\mathbb{Z}$ (the ring of integers) is not a field since only 1 and $-1$ have a multiplicative inverse.
Finite Fields

Definition

A finite field is a field with a finite number of elements.
Finite Fields

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A finite field is a field with a finite number of elements.

Example

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a finite field (p is prime).
Finite Fields

The element 1 in any finite field generates a subfield of size a prime number $p$. 
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**Proposition**

Therefore every finite field is a finite extension of some \( \mathbb{F}_p \).
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**Proposition**

Therefore every finite field is a finite extension of some $\mathbb{F}_p$.

We denote these as $\mathbb{F}_q$ where $q = p^k$. 
Cyclotomic Fields

The \( n \)th roots of unity are the \( n \) (distinct) complex solutions to \( x^n = 1 \). The \( n \)th roots of unity form a regular \( n \)-gon with its vertices on the unit circle. These are the powers of \( \zeta_n := e^{2\pi i/n} \).

**Definition**

The \( n \)th cyclotomic field \( \mathbb{Q}(\zeta_n) \), is the field consisting of \( a_0 + a_1 \zeta_n + a_2 \zeta_n^2 + \cdots + a_{n-1} \zeta_n^{n-1} \) for \( a_0, a_1, \ldots, a_{n-1} \in \mathbb{Q} \).

Remark: it actually has dimension \( \phi(n) \) as a \( \mathbb{Q} \)-vector space, not \( n \).
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Number Fields

Definition
Algebraic number fields $K$, also known as number fields, are finite degree extension fields of $\mathbb{Q}$.
In other words, the following conditions are satisfied:

1. $K$ is a field.
2. $\mathbb{Q} \subseteq K$.
3. $K$ is a finite dimensional vector space over $\mathbb{Q}$. 
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Examples of Number Fields

**Example**

\[ \mathbb{Q}, \mathbb{Q}(i), \mathbb{Q}(\sqrt{d}), \text{ and } \mathbb{Q}(\zeta_{n}) \] are all number fields.
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Non-Example

The finite fields \(\mathbb{F}_q\) are not number fields because they do not contain \(\mathbb{Q}\).
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The finite fields \( \mathbb{F}_q \) are not number fields because they do not contain \( \mathbb{Q} \).

Non-Example

The fields \( \mathbb{R}, \mathbb{C}, \text{ and } \mathbb{Q}(\pi) \) (or any other transcendental number) are not number fields because they are infinite-dimensional vector spaces over \( \mathbb{Q} \) (alternatively, infinite-degree extensions).
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Non-Example

The ring \( \mathbb{Q}[x]/(x^2) \) is not a number field because it is not a field.
Minimal Polynomials

Definition

The minimal polynomial for a constant \( \alpha \) over a given field \( F \) is a monic polynomial \( f(x) \) of minimum degree that is irreducible over \( F \) such that \( f(\alpha) = 0 \).

Essentially, the minimal polynomial is the smallest polynomial which still has \( \alpha \) as a root.

Example

\( x^2 + 1 \) is the minimal polynomial for \( i \) over the field \( \mathbb{R} \).
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Characterizing Number Fields

Theorem (Primitive Element Theorem)

Every finite extension of $\mathbb{Q}$ is $\mathbb{Q}(\alpha)$ where $\alpha$ is a root of its minimal polynomial over $\mathbb{Q}$.

In other words, every number field is realized by adjoining some single element to $\mathbb{Q}$!

Example

$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$ would still be just $\mathbb{Q}$ adjoin some single element.

In fact, $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}) = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{11}$. 
Characterizing Number Fields

**Theorem (Primitive Element Theorem)**

*Every finite extension of \( \mathbb{Q} \) is \( \mathbb{Q}(\alpha) \) where \( \alpha \) is a root of its minimal polynomial \( f(x) \) over \( \mathbb{Q} \).*

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Ring of Integers

Definition

The ring of integers of a number field $K$, denoted $\mathcal{O}_K$, is the subset of $K$ whose minimal polynomial over $\mathbb{Q}$ is monic and integer. $\mathbb{Q}$ is the fractions of using $\mathbb{Z}$, and $\mathbb{Z}$ is the "integer" part of $\mathbb{Q}$. In the same way, for a number field $K$, $\mathcal{O}_K$ is the "integer" part of $K$, and $K$ is the fractions of using $\mathcal{O}_K$.

Proposition

$K \subset L$, where $L$ is an extension of the field $K$, implies $\mathcal{O}_K \subset \mathcal{O}_L$. 
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The ring of integers of $\mathbb{Q}$ is $\mathbb{Z}$. 
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The ring of integers of \( \mathbb{Q}(\sqrt{d}) \) for \( d \equiv 1 \pmod{4} \) (and \( d \) squarefree) is actually \( \mathbb{Z} \left[ \frac{1+\sqrt{d}}{2} \right] \).
Factorizing Ideals
Prime Ideals

Definition

A prime ideal of a commutative ring \( R \) is a proper ideal \( p \) such that for two elements \( a_1, a_2 \in R \) and \( a_1 a_2 \in p \) implies \( a_1 \in p \), \( a_2 \in p \), or \( a_1, a_2 \in p \).

Example

The prime ideals of \( \mathbb{Z} \) are \((0)\) and \((p)\) for all prime integers \( p \).

Example

The only prime ideal of a field \( F \) is the zero ideal \((0)\).

Non-Example

The ideal \((3, x^2 + 11)\) of \( \mathbb{Z}[x] \) is not prime since \( x^2 + 11 - 3 \cdot 4 = x^2 - 1 = (x - 1)(x + 1) \), but neither \( x - 1 \) nor \( x + 1 \) is in the ideal.
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Factorizing Ideals in $\mathcal{O}_K$

All rings of integers $\mathcal{O}_K$ are Dedekind domains. All prime ideals are maximal ideals. Crucially, all ideals have unique factorization into prime ideals.

$\mathbb{Q} \subset K \Rightarrow \mathcal{O}_\mathbb{Q} = \mathbb{Z} \subset \mathcal{O}_K$.

Prime ideal $p \mathbb{Z} \subset \mathbb{Z}$; lifting to $\mathcal{O}_K$, have $p \mathcal{O}_K$ (multiples of $p$ in $\mathcal{O}_K$).

This is an ideal, but unlike $p \mathbb{Z}$, it is usually not prime.

We will study its prime factorization.
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**Theorem**

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- We will study its prime factorization.
General Factorization Properties

Because $p\mathcal{O}_K$ is an ideal, it has prime factorization

$$p\mathcal{O}_K = \prod_{i=1}^{r} Q_i^{e_i},$$

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Just as how $\mathbb{Z}$ is a subring of $\mathcal{O}_K$, $\mathbb{Z}/p\mathbb{Z}$ is a subfield of $\mathcal{O}_K/Q_i$. 
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**Definition**

We will denote $f_i$ to be the degree of the extension. In other words, $f_i := [\mathcal{O}_K/Q_i : \mathbb{Z}/p\mathbb{Z}]$. 
Relationship of dimension with factorization

Theorem

We have

\[ [K : \mathbb{Q}] = \sum_{i=1}^{r} e_i f_i. \]
Relationship of dimension with factorization

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\[ [K : \mathbb{Q}] = \sum_{i=1}^{r} e_i f_i. \]

Even better, when \( K/\mathbb{Q} \) is Galois (which we will define later):

**Theorem**

Let \( K/\mathbb{Q} \) be Galois. Then all of the \( e_i \) and \( f_i \) are the same, so

\[ [K : \mathbb{Q}] = r. \]
Computing The Factorization

By the Primitive element theorem, \( K = \mathbb{Q}(\alpha) \). Let \( f(x) \) be the minimal polynomial of \( \alpha \). It turns out that factorization of \( p \mathcal{O}_K \) is as easy as factorizing \( f(x) \) modulo \( p \) (for all but finitely many \( p \)).

Example ▶ In \( \mathbb{Q}(\sqrt{2})/\mathbb{Q} \), \( \alpha = \sqrt{2} \), and \( f(x) = x^2 - 2 \).

▶ To factor \( 7 \mathcal{O}_{\mathbb{Q}(\sqrt{2})} \), we just factor \( x^2 - 2 \) (mod 7).

▶ \( x^2 - 2 \equiv (x - 3)(x - 4) \) (mod 7).

▶ Plug in \( x = \alpha \) to get product of ideals: \( 7 \mathcal{O}_{\mathbb{Q}(\sqrt{2})} = (7, \alpha - 3)(7, \alpha - 4) \).

▶ Degree of terms are all 1, so all \( f_i = 1 \).
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Example

▶ In $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, $\alpha = \sqrt{2}$, and $f(x) = x^2 - 2$.
▶ To factor $7\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$, we just factor $x^2 - 2 \pmod{7}$.
▶ $x^2 - 2 \equiv (x - 3)(x - 4) \pmod{7}$. 
Computing The Factorization

By the Primitive element theorem, \( K = \mathbb{Q}(\alpha) \). Let \( f(x) \) be the minimal polynomial of \( \alpha \). It turns out that factorization of \( p\mathcal{O}_K \) is as easy as factorizing \( f(x) \) modulo \( p \) (for all but finitely many \( p \)).

Example

- In \( \mathbb{Q}(\sqrt{2})/\mathbb{Q} \), \( \alpha = \sqrt{2} \), and \( f(x) = x^2 - 2 \).
- To factor \( 7\mathcal{O}_{\mathbb{Q}(\sqrt{2})} \), we just factor \( x^2 - 2 \) (mod 7).
- \( x^2 - 2 \equiv (x - 3)(x - 4) \) (mod 7).
- Plug in \( x = \alpha \) to get product of ideals:
  \( 7\mathcal{O}_{\mathbb{Q}(\sqrt{2})} = (7, \alpha - 3)(7, \alpha - 4) \).
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- Plug in \( x = \alpha \) to get product of ideals: \( 7\mathcal{O}_{\mathbb{Q}(\sqrt{2})} = (7, \alpha - 3)(7, \alpha - 4) \).
- Degree of terms are all 1, so all \( f_i = 1 \).
Galois Theory
Motivation

Is \( i \) or \( -i \) the square root of \(-1\)? We arbitrarily choose \( i \), but there is no real reason to pick one over another. In this case, let's look at the automorphisms of \( \mathbb{C} \) preserving \( \mathbb{R} \). These consist of \( \{1, \sigma\} \) where 1 is the identity on \( \mathbb{C} \) and \( \sigma \) is complex conjugation. Because complex conjugation is in here, we cannot tell \( i \) and \( -i \) apart. Galois theory aims to quantify these issues.
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Example $\mathbb{Q}(i)/\mathbb{Q}$, $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are all Galois extensions.
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Galois group

Definition
Let $F \subset E$ be a Galois extension. The Galois group of $E/F$, denoted as $G = \text{Gal}(E/F)$, is the set of all automorphisms of $E$ that map every element of $F$ to itself.

Example
The automorphisms of $\mathbb{C}$ fixing $\mathbb{R}$ means that $i$ must be sent to $\pm i$. If $i \mapsto i$, then it is the identity on $\mathbb{C}$. If $i \mapsto -i$, it is complex conjugation on $\mathbb{C}$. 

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Examples of Galois groups

Example
▶ Consider Gal\((\mathbb{Q}(\sqrt{2})/\mathbb{Q})\).
▶ Minimal polynomial: \(x^2 - 2\), roots \(\pm \sqrt{2}\).
▶ Galois group: \(\{1, f\} \cong \mathbb{Z}/2\mathbb{Z}\), with 1 is the identity automorphism and \(f\) mapping \(\sqrt{2}\) to \(-\sqrt{2}\).

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▶ Consider Gal\((\mathbb{Q}(i, \sqrt{2})/\mathbb{Q})\).
▶ Galois group: \(\{1, \alpha, \beta, \alpha\beta\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\).
▶ 1 is identity; \(\alpha\) fixes \(\sqrt{2}\) and sends \(i\) to \(-i\); \(\beta\) fixes \(i\) and sends \(\sqrt{2}\) to \(-\sqrt{2}\).

We now look at a visual way to represent this.
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We now look at a visual way to represent this.
Galois Correspondence

\[ \text{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}) = \{1, \alpha, \beta, \alpha\beta\} \]

\[ \alpha(\sqrt{2}) = \sqrt{2}, \quad \alpha(i) = -i, \]
\[ \beta(\sqrt{2}) = -\sqrt{2}, \quad \beta(i) = i, \]
\[ \alpha\beta(\sqrt{2}) = -\sqrt{2}, \quad \alpha\beta(i) = -i. \]
Fundamental Theorem of Galois Theory

Definition

Every finite Galois Extension and its subfields share a 1 to 1 correspondence with the Galois Group and its subgroups.
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Every finite Galois Extension and its subfields share a 1 to 1 correspondence with the Galois Group and its subgroups. These subfields and subgroups are in an inclusion reversing bijection.
We would like to thank the following:

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