ON LU MATRICES AND SPRINGER THEORY

ANDREW GU

Abstract. In this paper, we investigate and find the number of LU matrices in $GL_n(\mathbb{F}_q)$ that are similar to a regular semisimple s in $GL_n(\mathbb{F}_q)$. Linking our results with M.-T. Trinh's study of certain "generalized Steinberg varieties," we expand on his work. Trinh has established certain numerical identities coming from a $P = W$ conjecture of Cataldo-Hausel-Migliorini between affine Springer fibers and these generalized Steinberg varieties. The results of this paper provide numerical evidence of the relation between Springer fibers and LU matrices. Using a linear-algebraic approach, we find a direct relation between LU matrices and Trinh's spaces. Consequently, we derive a closed formula for a point count of LU matrices that is a constant factor from the point count of Trinh's spaces. Furthermore, we identify a common point count among these sets. From this we propose a conjecture that generalizes our results.

Keywords: LU Matrices, Affine Springer Fibers, Springer Theory, Grothendieck-Springer Resolution, Linear Algebra, Braid Group, Flag Variety, Conjugacy Class, Representation Theory

Contents

1. Introduction

1.1. **Background.** For a regular semisimple s in $GL_n(\mathbb{F}_q)$, it has been conjectured by Bezrukavnikov, Boixeda Alvarez, McBreen, and Yun [BBMY] that there exists some relationship between affine Springer fibers and the set of elements from $G = GL_n(\mathbb{F}_q)$ that are similar to s and admit an LU composition, which we define as X_s. The theory of Springer fibers is important for number theory and algebraic geometry, and more specifically for the representation theory of affine Weyl Groups. Affine Springer fibers are related to Hitchin's integrable system from his work on the Yang–Mills theory [H]. In addition, B.-C. Ngo used this relationship to prove the Langlands–Shelstad Fundamental lemma [N].

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Nonabelian Hodge theory relates the Hitchin moduli space and a certain Betti moduli space [S]. The previously mentioned conjecture by Bezrukavnikov regarding LU matrices and affine Springer fibers may be considered as a variant of Nonabelian Hodge theory.

This paper relates LU matrices to certain Betti spaces considered by Minh-Tam Trinh [T]. For GL_n, Trinh establishes certain numerical identities coming from a $P = W$ conjecture of Cataldo-Hausel-Migliorini [CHM] between affine Springer fibers and these Betti spaces, using results of Maulik–Yun [MY]. Therefore, this paper gives numerical evidence of the relation between affine Springer fibers and LU matrices.

To understand X_s, it is important to consider its point count. In this paper, we find the number of elements of X_s, and discuss possible implications. To do this, we need to consider another set. First, let S_n and Br_n be the symmetric group and the braid group, respectively. Let s_i for $1 \leq i \leq n-1$ be the transpositions swapping i and $i+1$. Note that they are the generators for S_n. Similarly, define σ_i as the positive twist swapping i and $i+1$; they are generators for Br_n. Finally, for any two flags $h, f \in G/B$, define the relation $h \overset{s_i}{\text{---}} f$ to mean $f^{-1}h \in Bs_iB$, where we interpret s_i as a permutation matrix in G. Now, we can define the aforementioned set:

Definition 1.1. Suppose a braid β can be written as $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$. Define

$$Y(\beta) := \{h_0B, h_1B, \ldots, h_kB \in G/B, g \in G \mid h_0 \overset{s_{i_1}}{\text{---}} h_1 \overset{s_{i_2}}{\text{---}} \ldots \overset{s_{i_k}}{\text{---}} h_k, gh_0B = h_kB\}.$$

These spaces have been studied by Lusztig [L2] and Trinh [T].

Let π be the braid $(\sigma_1\sigma_2\ldots\sigma_{n-1})^n$. This is known as the *full twist*. Most of this paper is dedicated to the case when $\beta = \pi$.

1.2. **Our Results.** Here, we give a preview of our main results. Besides the set X_s, another set of interest is $Y(\pi)_s$, which is just the subset of $Y(\pi)$ for when g is s. A large portion of this paper is directed to this set and its derivatives. In order to relate it to X_s, we must break down its definition, which turns out to be related to LU decomposition. We then find the following:

Corollary 1.2. *We have*

$$|Y(\pi)_s| = |X_s|.$$

This is also referred to as Corollary 3.23 in Section 3. It will be an important result for later purposes. We then introduce the Grothendieck-Springer versions of X_s and $Y(\pi)$:

Definition 1.3. Consider the maps $B^-B \xrightarrow{id} G$ and $\tilde{G} \xrightarrow{p} G$ where $p(g, hB) = g$. Define

$$\tilde{X} := B^-B \underset{G}{\times} \tilde{G} = \{(g, hB) \in \tilde{G} \mid g \in BB^-\},$$

i.e. g has an LU decomposition.

Definition 1.4. Consider the map $\mathfrak{q}$ from $\tilde{X}$ to T where (g, hB) is mapped to the diagonal entries of $h^{-1}gh \in B$. For $t \in T$, let $\tilde{X}(t)$ be the preimage of t under $\mathfrak{q}$.

Definition 1.5. Then, we define

$$\tilde{Y}(\beta) := \{hB, h_0B, h_1B, \ldots, h_kB \in G/B,$$
$$g \in G \mid (g, hB) \in \tilde{G}, h_0 \overset{s_{i_1}}{\text{---}} h_1 \overset{s_{i_2}}{\text{---}} \cdots \overset{s_{i_k}}{\text{---}} h_k, gh_0B = h_kB\}.$$

We then define $\tilde{Y}(\beta; t)$ for a diagonal matrix t as the subset of $\tilde{Y}(\beta)$ for which the main diagonal of $h^{-1}gh$ is t.

We now can obtain the Grothendieck-Springer version of our first theorem. Namely:

Theorem 1.6. *For all diagonal matrices t, we have*
$$|\tilde{Y}(\pi;t)| = \frac{|G|}{(q-1)^n}|\tilde{X}(t)|.$$

This is our first main result, also referred to as Theorem 3.1. We then reduce the computation of $|X_s|$ to computing certain bitraces using Deligne-Lusztig theory, resulting in the following:

Theorem 1.7. *For regular semisimples s, we have*
$$|X_s| = \sum_{\lambda \in Irr(S_n)} q^{c(1)+c(\lambda)}(\chi^\lambda_{S_n}(1))^2.$$

This is also referred to as Theorem 6.1. It then implies the following:

Theorem 1.8. *For regular semisimples s, we have*
$$|\tilde{Y}(\pi;s)| = \frac{|G|}{(q-1)^n}\sum_{\lambda \in Irr(S_n)} q^{c(1)+c(\lambda)}(\chi^\lambda_{S_n}(1))^2.$$

This is also referred to as Theorem 6.3. The expression on the right side appears in Minh-Tam Trinh's work in [T, Theorem 7]. In particular, he finds
$$|\tilde{Y}(\pi;1)| = \frac{|G|}{(q-1)^n}\sum_{\lambda \in \mathrm{Irr}(S_n)} q^{c(1)+c(\lambda)}(\chi^\lambda_{S_n}(1))^2.$$
Note that the version of his theorem presented here uses our notation. This then implies the following:

Corollary 1.9. *For all regular semisimples s, we have*
$$|\tilde{Y}(\pi;1)| = |\tilde{Y}(\pi;s)|.$$

We then make an important conjecture:

Conjecture 1.10. *We have $|\tilde{Y}(\pi;t)|$ constant in t.*

This is also referred to as Conjecture 6.5, and it is powerful because it unites Theorem 1.8 and [T, Theorem 7] in Trinh's paper, and further generalizes to all cases. It is what our research will focus on in the future.

1.3. **Contents.** Section 2 contains all of the definitions necessary to understand Section 3. Subsection 2.1 involves the definition of several relevant subsets of $G = GL_n(\mathbb{F}_q)$, as well as defining one of the main subjects of this paper: X_s. We make a few remarks about the relationships between certain sets that will be useful in later sections. Subsection 2.2 serves to define everything related to braids, in particular $Y(\beta)$ and $\tilde{Y}(\beta)$. We also define π, which is also known as the full twist, as in most cases, we will have $\beta = \pi$ in this paper. Finally, in Subsection 2.3, we introduce terminology used in the study of partitions. It is only through these terms that we can write a formula for X_s, as we later see.

Section 3 presents useful results that directly relate the sizes of multiple sets. The first theorem relates the sizes of $\tilde{X}_s$ and $\tilde{Y}(\pi;s)$, with a key lemma relating a sequence of flags with LU matrices. We then relate $\tilde{Y}(\pi;s)$ to $Y(\pi)_s$, the latter being more understood and providing a bridge to the next section.

Section 4 introduces multiple concepts that relate $|Y(\pi)_g|$ to a sum of irreducible characters evaluated at g. Among these is the Iwahori-Hecke Algebra, whose characters' evaluations are well known. This leaves the irreducible characters for G. We propose that $\chi_G^\lambda(s) = \chi_{S_n}^\lambda(1)$, which we prove in the next section.

Section 5 sets out to prove the previously mentioned proposition. We introduce a formula for $\chi_G^\lambda(s)$, and find its exact value by considering maximal tori fixed by the Frobenius endomorphism.

Section 6 contains the final results. The first is a theorem that finds the point count of X_s, which involve the partition formulas we previously mentioned. The second theorem makes the observation that $Y(\pi; s) = Y(\pi; 1)$ by matching results of this paper and [T, Theorem 7]. Along with a computer program written for $n = 3$, this lends itself to an interesting conjecture: $Y(\pi; t)$ is the same for all $t \in T$.

2. Preliminary Definitions

2.1. **Defining X_s.** Let $G = GL_n(\mathbb{F}_q)$ be the general linear group. Let B be the set of upper triangular matrices and B^- be the set of lower triangular matrices. Let N be the set of strictly upper triangular matrices and N^- the set of strictly lower triangular matrices.

A matrix admits an *LU decomposition* if it can be expressed as the product of a strictly lower triangular matrix and an upper triangular matrix, and this representation is unique if it exists. This is well known from Bruhat decomposition. The set of matrices that admit an LU decomposition is B^-B.

Definition 2.1. Let s be a diagonal matrix in $G := GL_n(\mathbb{F}_q)$ with distinct eigenvalues in $\mathbb{F}_q$. Let $\mathbb{O}_s$ be all matrices in the same conjugacy class as s. Then, we define

$$X_s = \mathbb{O}_s \cap B^-B.$$

Example 2.2. For $SL_2(\mathbb{F}_q)$, we first find the general form of an LU matrix:

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c^{-1} \end{bmatrix} = \begin{bmatrix} c & d \\ bc & bd + c^{-1} \end{bmatrix},$$

where $b, c, d \in \mathbb{F}_q$. We now need to make sure this matrix is similar to

$$s := \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix},$$

where $e \neq e^{-1} \in \mathbb{F}_q$ are the eigenvalues of s. In other words, we want the characteristic polynomials to be the same, so $x^2 - (c + c^{-1} + bd)x + 1 = x^2 - (e + e^{-1})x + 1$.

Let $\tilde{G} := \{(g, hB) \in G \times G/B \mid h^{-1}gh \in B\}$ be the Grothendieck-Springer Resolution. Let T the set of diagonal matrices in $GL_n(\mathbb{F}_q)$, which is the same as $(\mathbb{F}_q^\times)^n$.

Definition 2.3. Consider the maps $B^-B \xrightarrow{id} G$ and $\tilde{G} \xrightarrow{p} G$ where $p(g, hB) = g$. Define

$$\tilde{X} := B^-B \underset{G}{\times} \tilde{G} = \{(g, hB) \in \tilde{G} \mid g \in BB^-\},$$

i.e. g has an LU decomposition.

Definition 2.4. Consider the map $\mathfrak{q}$ from $\tilde{X}$ to T where (g, hB) is mapped to the diagonal entries of $h^{-1}gh \in B$. For $t \in T$, let $\tilde{X}(t)$ be the preimage of t under $\mathfrak{q}$.

Remark 2.5. Recall that a regular semisimple is a diagonalizable matrix with distinct eigenvalues. Note that when s is a regular semisimple, $|\tilde{X}(s)| = |X_s|$. This is because there is a bijection between $\mathbb{O}_s$ and preimage of s under the map $\tilde{G} \to T$.

2.2. **Braids.** Let S_n be the symmetric group, and let its generators be s_i for $1 \leq i \leq n-1$, where s_i is the transposition swapping i and $i+1$. Let Br_n be the braid group, and let its generators be σ_i for $1 \leq i \leq n-1$, where σ_i is the positive twist swapping i and $i+1$.

For any two flags $h, f \in G/B$, define the relation $h \overset{s_i}{—} f$ to mean $f^{-1}h \in Bs_iB$, interpreting s_i as a permutation matrix in G.

Definition 2.6. The *full twist* is defined to be the braid $\pi = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n$.

Definition 2.7. Suppose a braid β can be written as $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$. Define

$$Y(\beta) := \{h_0B, h_1B, \dots, h_kB \in G/B, g \in G \mid h_0 \overset{s_{i_1}}{—} h_1 \overset{s_{i_2}}{—} \dots \overset{s_{i_k}}{—} h_k, gh_0B = h_kB\}.$$

Consider also the map τ to G where $(h_0B, \dots, h_kB, g)$ is mapped to g. Let $Y(\beta)_g$ be the preimage of g under τ.

Definition 2.8. Then, we define

$$\begin{aligned}\tilde{Y}(\beta) := \{&hB, h_0B, h_1B, \dots, h_kB \in G/B, \\ &g \in G \mid (g, hB) \in \tilde{G}, h_0 \overset{s_{i_1}}{—} h_1 \overset{s_{i_2}}{—} \cdots \overset{s_{i_k}}{—} h_k, gh_0B = h_kB\}.\end{aligned}$$

Definition 2.9. Consider the map ρ from $\tilde{Y}(\beta)$ to T where $(hB, h_0B, \dots, h_kB, g)$ is mapped to the diagonal entries of $h^{-1}gh \in B$. For $t \in T$, let $\tilde{Y}(\beta; t)$ be the preimage of t under ρ.

Definition 2.10. Consider also the map to G where $(hB, h_0B, \dots, h_kB, g)$ is mapped to g. Let $\tilde{Y}(\beta)_g$ be the preimage of g under this map.

For most of this paper, we focus on the case $\beta = \pi$.

Remark 2.11. Note that

$$\tilde{Y}(\beta) = Y(\beta) \underset{G}{\times} \tilde{G}$$

with the maps τ and p from Definition 2.7 and Definition 2.3, respectively. When g is similar to a regular semisimple, we then have $|\tilde{Y}(\beta)_g| = n!|Y(\beta)_g|$ because there exist exactly $n!$ flags h for which $h^{-1}gh \in B$, corresponding to the $n!$ ways to permute the eigenvectors of g.

2.3. **Partitions.** A partition of a positive integer n is a way to write it as a sum of unordered positive integers. We can write a partition λ as $(\lambda_1, \lambda_2, \dots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.

The *content* of a partition λ is $c(\lambda) = \sum_i \sum_{j=1}^{\lambda_i}(j-i)$.

Recall that the irreducible representations of S_n are in bijection with partitions of n. For a partition λ, let $\chi^\lambda_{S_n}$ denote the corresponding irreducible character of S_n.

Represent λ as a Young Diagram. For (i,j) the square in row i, column j, we let $h(i,j)$ denote the number of squares (i', j') in the Young Diagram λ such that $i' \geq i, j' = j$ or $i' = i, j' \geq j$. Then, $\chi^\lambda_{S_n}(1)$ is the dimension of the corresponding irreducible representation, which is equal to

$$\chi^\lambda_{S_n}(1) = \frac{n!}{\prod_{i,j} h(i,j)},$$

where the product is over all (i,j) in the Young diagram. Note that this is the Hook Length Formula.

3. Size Relations

In this section, we break down a complex part of the definition of $Y(\beta)$ when $\beta = \pi$; namely, the following:

$$h_0 \overset{s_{i_1}}{—} h_1 \overset{s_{i_2}}{—} \cdots \overset{s_{i_k}}{—} h_k.$$

We begin by investigating what $h \overset{s_i}{—} h'$ means exactly, by translating it into basic matrix multiplication. From there, we obtain a product of $n(n-1)$ matrices, which end up having a product that is reminiscent of properties of a matrix that admits an LU decomposition. Indeed, it turns out that the product admits an *UL decomposition*. Seeing this, we look further and find a bijection between the $n(n-1)$ matrices and NN^-.

This then allows us to relate $\tilde{Y}(\beta; t)$ with $X(t)$ for any $t \in T$. In particular, we will later define $Y(\beta)\tilde{\times}B$, and find multiple bijections using auxiliary sets to relate point counts. At the end of the section, we find that

$$|X_s| = |Y(\pi)_s|$$

in Corollary 3.23, which is relevant to later sections.

The following theorem is one of the main results in this section. It is more "general" in a sense; while this paper focuses on regular semisimples for now, this theorem holds true for all $t \in T$.

Theorem 3.1. *For all $t \in T$, we have*

$$|\tilde{X}(t)| \cdot \frac{|G|}{(q-1)^n} = |Y(\pi; t)|.$$

Definition 3.2. Define

$$Y(\beta)\tilde{\times}B := \{h_0B, h_1B, \ldots, h_kB \in G/B, \eta \in h_0B,$$
$$g \in G \mid h_0 \overset{s_{i_1}}{—} h_1 \overset{s_{i_2}}{—} \cdots \overset{s_{i_k}}{—} h_k, gh_0B = h_kB\}.$$

We introduce $Y(\beta)\tilde{\times}B$ to relate $\tilde{Y}(\pi; t)$ to $\tilde{X}(t)$. The addition of η allows us to more easily see a relationship between the relation $h_0 \overset{s_{i_1}}{—} h_1 \overset{s_{i_2}}{—} \cdots \overset{s_{i_k}}{—} h_k$ and LU matrices when $\beta = \pi$ by "allowing" right matrix multiplication.

We next define $\mathcal{Y}'$, and make an important proposition regarding $Y(\pi)\tilde{\times}B$ and $\mathcal{Y}'$.

Definition 3.3. Define

$$\mathcal{Y}' := \{u \in N, \bar{u} \in N^-, h_0B \in G/B, g \in G, \eta \in h_0B \mid \eta u\bar{u}B = g\eta B\}.$$

Proposition 3.4. *There exists a bijection between $Y(\pi)\tilde{\times}B$ and $\mathcal{Y}'$. We call one specific map Ω, which is outlined later.*

With its definition, $\mathcal{Y}'$ looks closely related to LU matrices. We will show that this is indeed the case with the following proposition.

Proposition 3.5. *There exists a bijection between $\mathcal{Y}'$ and $(N^- \times B) \times G \times N$. In particular, the map we want is*

$$(u, \bar{u}, h_0B, g, \eta) \to (\bar{u}, b := \bar{u}^{-1}u^{-1}\eta^{-1}g\eta u, \eta, u).$$

Proof. The map we gave above is injective and well defined since

$$\eta u\bar{u}B = g\eta B \Rightarrow \bar{u}^{-1}u^{-1}\eta^{-1}g\eta \in B \Rightarrow \bar{u}^{-1}u^{-1}\eta^{-1}g\eta u \in B.$$

It suffices to provide an injective map in the other direction. Note that given any $(\bar{u}, b, \eta, u) \in (N^- \times B) \times G \times N$, we have

$$(u, \bar{u}, h_0B, \eta u\bar{u}bu^{-1}\eta^{-1}, \eta) \in \mathcal{Y}'$$

where h_0B is such that $\eta \in h_0B$. This is because

$$\eta u\bar{u}B = (\eta u\bar{u}bu^{-1}\eta^{-1})\eta B.$$

This is once again injective. Therefore, we have found the bijection as desired. $\square$

Corollary 3.6. *Since we know $N^- \times B = B^-B$, we can also say $\mathcal{Y}'$ is in bijection with $(B^-B) \times G \times N$. In that case, the map would be*

$$(u, \bar{u}, h_0B, g, \eta) \to (u^{-1}\eta^{-1}g\eta u, \eta, u)$$

and $u^{-1}\eta^{-1}g\eta u \in B^-B$.

Now, we describe the relation

$$h_0 \overset{s_{i_1}}{\text{---}} h_1 \overset{s_{i_2}}{\text{---}} \cdots \overset{s_{i_k}}{\text{---}} h_k.$$

Let $e_{i,j}$ be the matrix with a 1 at position (i, j) and zero everywhere else.

Lemma 3.7. *If $h \overset{s_i}{\text{---}} h'$, then for each element $\eta \in hB$, there exists a unique $a \in \mathbb{F}_q$ for which $\eta(s_i + ae_{i,i}) \in h'B$.*

Proof. We first prove that such an a is unique. Note that $(s_i + ae_{i,i})^{-1} = s_i - ae_{i+1,i+1}$. This is clear since

$$\begin{aligned}(s_i + ae_{i,i})(s_i - ae_{i+1,i+1}) &= s_i^2 + ae_{i,i}s_i - s_iae_{i+1,i+1} - a^2e_{i,i}e_{i+1,i+1} \\ &= I_n + ae_{i,i+1} - ae_{i,i+1} - 0 = I_n\end{aligned}$$

where I_n is the n-by-n identity matrix. Suppose that for $a \neq b$, we have $\eta(s_i + ae_{i,i}) \in h'B$ and $\eta(s_i + be_{i,i}) \in h'B$. Then, note that

$$\begin{aligned}(\eta(s_i + ae_{i,i}))^{-1}\eta(s_i + be_{i,i}) &= (s_i + ae_{i,i})^{-1}\eta^{-1}\eta(s_i + be_{i,i}) = (s_i + ae_{i,i})^{-1}(s_i + be_{i,i}) \\ &= (s_i - ae_{i+1,i+1})(s_i + be_{i,i}) \\ &= s_i^2 + s_ibe_{i,i} - ae_{i+1,i+1}s_i - abe_{i+1,i+1}e_{i,i} \\ &= I_n + be_{i+1,i} - ae_{i+1,i} = I_n + (b - a)e_{i+1,i}.\end{aligned}$$

In particular, $I_n + (b - a)e_{i+1,i} \notin B$, contradiction. Thus, we have proven that a is unique if it exists.

Now, we show that a exists. Note that

$$h \overset{s_i}{\text{---}} h' \Leftrightarrow \eta^{-1}h \overset{s_i}{\text{---}} \eta^{-1}h' \Leftrightarrow I_n \overset{s_i}{\text{---}} \eta^{-1}h'.$$

We then would like to prove that if $I_n \overset{s_i}{\text{---}} f$ for some f, then $fB = (s_i + ae_{i,i})B$ for some $a \in \mathbb{F}_q$. First, note that $fB \in Bs_iB$, so it suffices to prove that each element of Bs_i is in $(s_i + ae_{i,i})B$ for some $a \in \mathbb{F}_q$, or even simpler, that for each $b \in B$, we have

$$(s_i + ae_{i,i})^{-1}bs_i \in B$$

for some a. Let $b_{x,y}$ be the entry at position (x, y) of b. Note that

$$(s_i + ae_{i,i})^{-1}bs_i = (s_i - ae_{i+1,i+1})bs_i = s_i(I_n - ae_{i,i+1})bs_i.$$

Choose $a = \frac{b_{i,i+1}}{b_{i+1,i+1}}$. This guarantees that $(I_n - ae_{i,i+1})b$ is 0 at the position $(i, i+1)$. Then, $s_i(I_n - ae_{i,i+1})bs_i$ swaps the ith and $i+1$th row of $(I_n - ae_{i,i+1})b$, and then the ith and $i+1$th column. Since we guaranteed that the entry at the the position $(i, i+1)$ was 0 before the swaps, it follows that $s_i(I_n - ae_{i,i+1})bs_i \in B$. Below is an example for a 4-by-4 matrix when $i = 2$:

$$(I_n - ae_{i,i+1})b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-b_{2,3}}{b_{3,3}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & b_{2,2} & b_{2,3} & b_{2,4} \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & 0 & b_{4,4} \end{pmatrix}$$
$$= \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & b_{2,2} & 0 & b_{2,4} - \frac{b_{3,4}b_{2,3}}{b_{3,3}} \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & 0 & b_{4,4} \end{pmatrix}.$$

And then we have

$$s_i(I_n - ae_{i,i+1})s_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & b_{2,2} & 0 & b_{2,4} - \frac{b_{3,4}b_{2,3}}{b_{3,3}} \\ 0 & 0 & b_{3,3} & b_{3,4} \\ 0 & 0 & 0 & b_{4,4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} b_{1,1} & b_{1,3} & b_{1,2} & b_{1,4} \\ 0 & b_{3,3} & 0 & b_{3,4} \\ 0 & 0 & b_{2,2} & b_{2,4} - \frac{b_{3,4}b_{2,3}}{b_{3,3}} \\ 0 & 0 & 0 & b_{4,4} \end{pmatrix},$$

which is an upper triangular matrix as desired. Therefore, we have proven the existence and uniqueness of a and we are done.

□

From here on, we focus only on the case when $\beta = \pi$, when the braid is the full twist. Consider a single element $(h_0, h_1 \ldots, h_{n(n-1)}, \eta, g)$ from $Y(\pi)\tilde{\times}B$. Suppose $\eta A_1 A_2 \cdots A_k \in h_k B$ where $A_k = s_{i_k} + a_k e_{i_k,i_k}$ for $1 \le k \le n(n-1)$. Note specifically that $s_{i_k} = s_k$ where we take indices mod $n-1$ appropriately. We refer to s_{i_k} as s_k now since we are dealing specifically with $\beta = \pi$.

Next, let $k = x(n-1)+y$ for integers x and $1 \le y \le n-1$. Define $B_k = I_n + a_k e_{x+1,x+y+1}$, where I_n is the identity matrix, and $x+y+1$ is taken mod n (row/column 0 is assumed to be row/column n).

We have defined B_k to make the product in the following lemma easier to analyze. In particular, while A_k has a "permutation" component to it, B_k has an identity matrix. In addition, B_1, B_2,..., $B_{n(n-1)}$ has a nice pattern, shown for $n = 3$:

$$\begin{pmatrix} 1 & a_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ a_4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_5 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_6 & 1 \end{pmatrix}.$$

Consider how the position of the a_k moves as we go from $k = 1$ to $n(n-1)$. When we begin on a row, we start at the position to the right of the 1 (wrapping around for the last row). Then, we move right until all the positions without a 1 have been visited. We then move on to the next row. This description of B_k is helpful for later lemmas.

Lemma 3.8. *We have*

$$A_1 A_2 \cdots A_{n(n-1)} = B_1 B_2 \cdots B_{n(n-1)}.$$

Proof. We claim that we can express $B_k = \sigma_k A_k \omega_k$ where σ_k and ω_k are permutation matrices. In particular, we claim $\sigma_k = s_1 s_2 \dots s_{k-1}$ and $\omega_k = s_k s_{k-1} \dots s_1$, where k is taken mod $n-1$ appropriately. Since $\omega_{k-1}\sigma_k = I$ for all $2 \le k \le n(n-1)$, $\sigma_1 = I$, and $\omega_{n(n-1)} = I$, it follows trivially from our claim that

$$\begin{aligned} B_1 B_2 \cdots B_{n(n-1)} &= (\sigma_1 A_1 \omega_1)(\sigma_2 A_2 \omega_2) \cdots (\sigma_{n(n-1)} A_{n(n-1)} \omega_{n(n-1)}) \\ &= \sigma_1 A_1 A_2 \cdots A_{n(n-1)} \omega_{n(n-1)} = A_1 A_2 \cdots A_{n(n-1)}. \end{aligned}$$

Thus, it suffices to prove our claim. It may be useful to also think of B_k as $\sigma_k A_k s_k \sigma_k^{-1}$.

Note that $A_k s_k = A_k s_y = I + a_k e_{(y,y+1)}$, and that we are then conjugating with σ_k to obtain B_k, which is just applying the permutation σ_k to both the rows and the columns. Note that $s_1 s_2 \dots s_{x(n-1)}$ maps each m to $m + x \pmod n$. Next,

$$s_{x(n-1)+1} \dots s_{x(n-1)+y-1} = s_1 s_2 \dots s_{y-1}$$

maps each $1 \le m \le y-1$ to $m+1$, y to 1 and fixes everything else. Therefore,

$$\begin{aligned} \sigma_k(y) &= s_1 s_2 \dots s_{x(n-1)}(s_1 s_2 \dots s_{y-1}(y)) = s_1 s_2 \dots s_{x(n-1)}(1) = x+1 \\ \sigma_k(y+1) &= s_1 s_2 \dots s_{x(n-1)}(s_1 s_2 \dots s_{y-1}(y+1)) = s_1 s_2 \dots s_{x(n-1)}(y+1) \\ &= x+y+1 (\mathrm{mod}\ n). \end{aligned}$$

It follows that position $(y, y+1)$ is mapped to position $(x+1, x+y+1)$, and so

$$\sigma_k A_k s_k \sigma_k^{-1} = \sigma_k(I + a_k e_{y,y+1})\sigma_k^{-1} = I + a_k e_{x+1,x+y+1} = B_k,$$

as desired. Below is an example for $n = 4$, $k = 8$:

$$A_k s_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_8 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, $\sigma_k A_k s_k \sigma_k^{-1}$ is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & a_8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_8 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = B_k$$

as desired. $\square$

Lemma 3.9. *We have*

$$B_1 B_2 \cdots B_{n(n-1)} \in NN^-.$$

Proof. In this proof, we rely on the fact that the matrices whose bottom right square submatrices all have determinant 1 are precisely elements of NN^-. This is a well known fact.

We do this via induction on the size of the matrix, which is n-by-n. The case $n = 1$ is clear.

Suppose we have proven this claim for $n = m$. We now prove it for $n = m + 1$. First, let $C_k = B_k - I$. Namely, $C_k = a_k e_{x+1,x+y+1}$ is a matrix with at most one nonzero entry. Then, we can express the product as

$$B_1 B_2 \cdots B_{m(m+1)} = (I + C_1)(I + C_2) \cdots (I + C_{m(m+1)}).$$

Note that in the expansion of $(I + C_1)(I + C_2) \cdots (I + C_{m(m+1)})$, we have the sum of $C_{i_1} C_{i_2} \cdots C_{i_d}$ where $1 \le i_1 < i_2 < \cdots < i_d \le m(m + 1)$. Furthermore, note that $C_{i_1} C_{i_2} \cdots C_{i_d}$ is a matrix with at most one nonzero entry.

We claim that all nonzero products of this form involving C_1, C_2,..., C_m or C_{2m}, C_{3m-1},..., $C_{m(m+1)-m+1}$ are not in the bottom right m-by-m submatrix. To clarify, we mean all nonzero products $C_{i_1} C_{i_2} \cdots C_{i_d}$ for which there exists $1 \le j \le d$ such that

$$i_j \in \{1, 2, \ldots, m, 2m, 3m - 1, \ldots, m(m + 1) - m + 1\}.$$

This claim finishes the inductive step since the bottom right m-by-m matrix is not affected by those C_k. Note that C_1, C_2,..., C_m are in the top row and C_{2m}, C_{3m-1},..., $C_{m(m+1)-m+1}$ are in the first column. Thus, the bottom right m-by-m matrix is then the product of matrices that are the same as the matrices we obtain for $n = m$, which then implies that the determinants are 1 for each of the bottom right square submatrices. The matrix itself trivially has determinant 1 since B_k has determinant 1 for all k. Below, the case for $m = 2$ is shown:

$$\begin{pmatrix} 1 & \boldsymbol{a}_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \boldsymbol{a}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{a}_4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boldsymbol{a}_5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_6 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{a_1 a_4} + \boldsymbol{a_5}\,(\boldsymbol{a_1} a_3 + \boldsymbol{a_2}) + 1 & \boldsymbol{a_1} + a_6\,(\boldsymbol{a_1} a_3 + \boldsymbol{a_2}) & \boldsymbol{a_1} a_3 + \boldsymbol{a_2} \\ a_3 \boldsymbol{a_5} + \boldsymbol{a_4} & \underline{a_3 a_6 + 1} & \underline{a_3} \\ \boldsymbol{a_5} & \underline{a_6} & \underline{1} \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \underline{1} & \underline{a_3} \\ 0 & \underline{0} & \underline{1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \underline{1} & \underline{0} \\ 0 & \underline{a_6} & \underline{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \underline{a_3 a_6 + 1} & \underline{a_3} \\ 0 & \underline{a_6} & \underline{1} \end{pmatrix}.$$

Remark 3.10. In the above diagrams, the bolded symbols are C_1, C_2,. . ., C_m and C_{2m}, C_{3m-1},. . . , $C_{m(m+1)-m+1}$ for $m = 2$. Note that the product $B_1 B_2 \cdots B_{m(m+1)}$ only contains bolded symbols in the first column and row. In particular, the bottom right m-by-m matrix only depends on the B_i that are not bolded, which returns to the case $n = m$, as claimed; note how the bottom right m-by-m matrices, which are underlined, are the same in the second and third lines.

Now, we prove the claim. Note that C_1, C_2,. . . , C_m are all matrices with a single entry on the top row. Also note that any product involving at least one of them will start with one of them at the very left. This means, according to matrix multiplication rules, the product can only have an entry in the top row, so they will not have an effect on the bottom m-by-m. That is, in block matrices,

$$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Below is an explicit example:

$$C_1 C_3 C_5 = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a_3 a_5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, the partial products from the left always have the nonzero entry in the top row.

We apply similar logic for C_{2m}, C_{3m-1},. . . , $C_{m(m+1)-m+1}$, which are matrices with the single entry on the first column. We claim that among any nonzero product involving at least one of these, it must be the last matrix. Consider the following:

$$\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, any matrix multiplied to the right of C_{2m}, C_{3m-1},. . . , or $C_{m(m+1)-m+1}$ must have a nonzero entry in the top row in order to have a nonzero product by matrix multiplication rules. However, those matrices are the first m matrices, so it is not possible for them to be multiplied to the right of C_{2m}, C_{3m-1},. . . , $C_{m(m+1)-m+1}$. Thus, if C_{2m}, C_{3m-1},. . . , $C_{m(m+1)-m+1}$ are involved in a nonzero product, they must be the last matrices. This then implies that any nonzero product they have is contained in the first column, which does not affect the bottom m-by-m and our claim is proven.

Therefore, all the determinants of the bottom square submatrices are 1, and it follows that $B_1 B_2 \cdots B_{n(n-1)}$ can always be written as the product of a strictly upper triangular matrix and a strictly lower triangular matrix. □

We will refer to a matrix that can be written as the product of a strictly upper triangular matrix and strictly lower triangular matrix interchangeably as a UL matrix or an element of NN^{-}.

Lemma 3.11. *There exists a bijection between the set of tuples* $(a_1, a_2, \ldots, a_{n(n-1)})$ *and* NN^-*. In other words, there is also a bijection between the set of tuples* $(A_1, A_2, \ldots, A_{n(n-1)})$ *and* NN^-*.*

Proof. Indeed, both have cardinality $q^{n(n-1)}$, so it remains to show that there is an injective map from one to the other.

Consider $(A_1A_2\cdots A_{n-1})(A_n\cdots A_{2n-2})\cdots(A_{(n-1)(n-1)+1}\cdots A_{n(n-1)})$. Note this grouping of the matrices creates a product of n matrices, each with a similar form; namely, the top row for $A_{x(n-1)+1}A_{x(n-1)+2}\cdots A_{(x+1)(n-1)}$ is $\begin{bmatrix} a_{x(n-1)+1} & a_{x(n-1)+2} & \cdots & a_{(x+1)(n-1)} & 1\end{bmatrix}$, and the rest is just 1's diagonally starting from the position $(2, 1)$, as shown in the following example:

Example 3.12. Shown below is $A_1A_2A_3$ for $n = 4$:

$$\begin{pmatrix} a_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let these n matrices be M_1, M_2,..., M_n. Then, I claim $M_1M_2\cdots M_j$ for $0 \le j \le n$ has a 1 in the first entry of the $j+1$th row (or not if $j = n$), and a diagonal of 1's from there. Every other entry in the $j+1$th row or below is a zero. In particular, it has the following form:

$$\begin{pmatrix} * & * \\ I_{n-j} & 0 \end{pmatrix}$$

where there are j rows of $*$'s. Here is an example for $j = 2$ when $n = 4$:

$$\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_4 & a_5 & a_6 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a_1a_4+a_2} & \boldsymbol{a_1a_5+a_3} & \boldsymbol{a_1a_6+1} & \boldsymbol{a_1} \\ \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{a_6} & \mathbf{1} \\ \underline{1} & \underline{0} & 0 & 0 \\ \underline{0} & \underline{1} & 0 & 0 \end{pmatrix}.$$

Remark 3.13. In the above diagram, the bolded section represents the top j rows, and the underlined section represents I_{n-j}.

Now, onto the induction. The cases $j = 0$ (which is just the identity matrix I_n) and $j = 1$ are clear. Now, suppose our claim is true for $j = m$ for $m \le n-1$. We now prove it is true for $j = m+1$. Note that M_{m+1} has two parts: the top row, and the rest. Note that $M_1M_2\cdots M_m$ multiplied by the top row just fills in the top $m+1$ rows, so it is irrelevant to our induction hypothesis. This is because only the first column of $M_1M_2\cdots M_m$ can form nonzero products, and only the top $m+1$ entries can be nonzero there.

Now we focus on the second part: the diagonal of 1's starting at $(2, 1)$, or

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in Example 3.12.

This basically just shifts the columns of $M_1M_2\cdots M_m$ to the left, as shown below:

$$\begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a_2} & \boldsymbol{a_3} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \underline{1} & \underline{0} & 0 & 0 \\ \underline{0} & \underline{1} & 0 & 0 \end{pmatrix}.$$

This then creates the diagonal pattern starting from the $m+2$th row as desired, and our claim is proven. By induction, our claim is true for all $0 \le j \le n$.

As a corollary, the jth row must be

$$\begin{bmatrix} a_{(n-1)j+1} & a_{(n-1)j+1} & \cdots & a_{(n-1)(j+1)} & 1 \end{bmatrix},$$

which is the top row of M_j. This is a simple consequence of matrix multiplication; namely, consider the product

$$(M_1M_2\cdots M_{j-1})\cdot M_j$$

for $1 \le j \le n$. The jth row of $(M_1M_2\cdots M_{j-1})$ is $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$. The product of this 1-by-n vector and M_j is simply the top row of M_j. It follows that the jth row of the product $(M_1M_2\cdots M_{j-1})\cdot M_j$ is the top row of M_j as desired.

In this way, we can go from each UL matrix back to a $n(n-1)$-tuple, since we can start by observing the bottom row of $M_1M_2\cdots M_n$, from which we can determine M_n. Once we have M_n, we divide by it to obtain $M_1M_2\cdots M_{n-1}$, observe the $n-1$th row, and repeat. This process gives us all the numbers in the tuple from our UL matrix, which implies this process is injective. It follows that there exists a bijection between the tuple $(a_1, a_2, \dots, a_{n(n-1)})$ and the UL matrices, and we are done. Here is an example of this process being run for $n = 3$:

$$\begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 28 & 16 & 5 \\ 22 & 13 & 4 \\ 5 & 3 & 1 \end{pmatrix}$$
$$\Rightarrow a_5 = 5, a_6 = 3$$
$$\begin{pmatrix} 28 & 16 & 5 \\ 22 & 13 & 4 \\ 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\Rightarrow a_3 = 4, a_4 = 2$$
$$\begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\Rightarrow a_1 = 1, a_2 = 1.$$

□

In other words, each sequence of flags $h_0, h_1, \dots h_{n(n-1)}$ and η represents $h_{n(n-1)}B = \eta u\bar{u}B$ for a unique pair $u \in N$ and $\bar{u} \in N^-$. We can now prove Proposition 3.4.

Proof of Proposition 3.4. By Lemma 3.7, there first exists a bijection between the set of tuples $(\eta, h_0B, h_1B, \dots h_{n(n-1)}B)$ for which $\eta \in h_0B$ and

$$h_0 \overset{s_1}{-} h_1 \overset{s_2}{-} \cdots \overset{s_{n(n-1)}}{-} h_{n(n-1)}$$

and the set of tuples $(\eta, A_1, A_2, \ldots, A_{n(n-1)})$, where we use similar notation as before. By Lemma 3.11, there exists a bijection between set of tuples $(\eta, A_1, A_2, \ldots, A_{n(n-1)})$ with the set of tuples $(\eta, u, \bar{u})$ where $u \in N$ and $\bar{u} \in N^-$. In particular, this second bijection requires $u\bar{u} = A_1 A_2 \cdots A_{n(n-1)}$.

Now, consider $Y(\pi)\tilde{\times}B$ and $\mathcal{Y}'$. For each

$$(h_0B, h_1B, \ldots h_{n(n-1)}B, \eta, g) \in Y(\pi)\tilde{\times}B,$$

we can map one-to-one to a tuple $(g, \eta, A_1, A_2, \ldots, A_{n(n-1)})$ for which

$$\eta A_1 A_2 \cdots A_{n(n-1)}B = g\eta B.$$

We can further map this using a bijection to a tuple $(u, \bar{u}, h_0B, g, \eta)$ for which $\eta u\bar{u}B = g\eta B$ and h_0B is such that $\eta \in h_0B$. Note this last tuple is an element of $\mathcal{Y}'$. It follows that we have found a bijection between $Y(\pi)\tilde{\times}B$ and $\mathcal{Y}'$. We then let Ω be the composition of these two maps we just used.

□

We now define the "Springer" versions of $Y(\beta)\tilde{\times}B$ and $\mathcal{Y}'$. This is necessary to find a relationship to $\tilde{X}(t)$, which involves the Grothendieck–Springer resolution as well.

Definition 3.14. Define

$$\begin{aligned}\tilde{Y}(\beta)\tilde{\times}B := \{&hB, h_0B, h_1B, \ldots, h_kB \in G/B, \eta \in h_0B,\\ &g \in G \mid (g, hB) \in \tilde{G}, h_0 \overset{s_{i_1}}{—} h_1 \overset{s_{i_1}}{—} \cdots \overset{s_{i_k}}{—} h_k, gh_0B = h_kB\}.\end{aligned}$$

Definition 3.15. Consider the map p from $\tilde{Y}(\beta)\tilde{\times}B$ to T where $(hB, h_0B, \ldots, h_kB, \eta, g)$ is mapped to the diagonal entries of $h^{-1}gh \in B$. For $t \in T$, let $\tilde{Y}(\beta; t)\tilde{\times}B$ be the preimage of t under p.

Definition 3.16. Define

$$\begin{aligned}\tilde{\mathcal{Y}}' := \{&u \in N, \bar{u} \in N^-, hB \in G/B, h_0B \in G/B,\\ &g \in G, \eta \in h_0B \mid (g, hB) \in \tilde{G}, \eta u\bar{u}B = g\eta B\}.\end{aligned}$$

Definition 3.17. Consider the map p$'$ from $\tilde{\mathcal{Y}}'$ to T where $(u, \bar{u}, h_0B, h_0B, g, \eta)$ is mapped to the diagonal entries of $h^{-1}gh \in B$. For $t \in T$, let $\tilde{\mathcal{Y}}'(t)$ be the preimage of t under p$'$.

Note that $\tilde{Y}(\pi)\tilde{\times}B$ and $\tilde{\mathcal{Y}}'$ are still in bijection with each other. We thus define a new map between them.

Definition 3.18. Define $\tilde{\Omega}$ to be the bijective map that naturally arises from Ω taking $\tilde{Y}(\pi)\tilde{\times}B$ to $\tilde{\mathcal{Y}}'$. It is clear that $\tilde{Y}(\pi; t)\tilde{\times}B$ is mapped to $\tilde{\mathcal{Y}}'(t)$ under $\tilde{\Omega}$.

As noted before, $\tilde{\mathcal{Y}}'(t)$ is related to LU matrices, and thus, should be related to $\tilde{X}(t)$. We find this exact relation in the following lemma. We then use a series of bijections to obtain our main result: Theorem 3.1.

Lemma 3.19. *There exists a bijection θ_t from $\tilde{\mathcal{Y}}'(t)$ to $\tilde{X}(t) \times G \times N$ given by*

$$(u, \bar{u}, hB, h_0B, g, \eta) \to (u^{-1}\eta^{-1}g\eta u, u^{-1}\eta^{-1}hB, \eta, u).$$

Proof. We can check that θ_t is well defined and injective, so let us show that

$$(u^{-1}\eta^{-1}g\eta u, u^{-1}\eta^{-1}hB, \eta, u) \in \tilde{X}(t) \times G \times N.$$

By Corollary 3.6, we know $u^{-1}\eta^{-1}g\eta u \in B^-B$. Furthermore, we have

$$(u^{-1}\eta^{-1}g\eta u)u^{-1}\eta^{-1}hB = u^{-1}\eta^{-1}ghB = u^{-1}\eta^{-1}hB,$$

so $(u^{-1}\eta^{-1}g\eta u, u^{-1}\eta^{-1}hB) \in \tilde{G}$. Finally,

$$(u^{-1}\eta^{-1}h)^{-1}(u^{-1}\eta^{-1}g\eta u)(u^{-1}\eta^{-1}h) = h^{-1}gh \in B,$$

so the values on the main diagonal are indeed t. Thus, θ_t is indeed injective map from $\tilde{\mathcal{Y}}'(t)$ to $\tilde{X}(t)$.

Now, it suffices to find an injective map θ'_t in the other direction. Consider

$$(g', hB, \eta, u) \to (u, \bar{u}, \eta uhB, h_0B, \eta ug'u^{-1}\eta, \eta)$$

where $g' \in B^-B$, h_0B is such that $\eta \in h_0B$, and $\bar{u}$ is the unique element of N^- in the LU decomposition of g' (Definition 2.1). Thus, this map is injective. Now, we prove that

$$(u, \bar{u}, \eta uhB, h_0B, \eta ug'u^{-1}\eta^{-1}, \eta) \in \tilde{\mathcal{Y}}'(t).$$

Note that

$$(\eta ug'u^{-1}\eta^{-1})\eta B = \eta ug'B = \eta u\bar{u}(\bar{u}^{-1}g')B = \eta u\bar{u}B,$$

since $\bar{u}^{-1}g' \in B$. Next, we have

$$(\eta uh)^{-1}(\eta ug'u^{-1}\eta^{-1})(\eta uh) = h^{-1}g'h \in B,$$

so the values on the main diagonal are indeed t. Thus, θ_t is indeed injective map from $\tilde{X}(t)$ to $\tilde{\mathcal{Y}}'(t)$.

Therefore, θ_t is actually a bijection and we are done. ☐

Theorem 3.20. *There exists a bijection between $\tilde{X}(t) \times G \times N$ and $\tilde{Y}(\pi; t)\tilde{\times}B$.*

Proof. Combine the bijection $\tilde{\Omega}$ as defined in Definition 3.18 and the bijection θ_t from Lemma 3.19. ☐

Proof of Theorem 3.1. By definition,

$$|\tilde{Y}(\pi; t)| = \frac{|\tilde{Y}(\pi; t)\tilde{\times}B|}{|B|}.$$

By Definition 3.18,

$$|\tilde{Y}(\pi; t)\tilde{\times}B| = \frac{|\tilde{\Omega}(\tilde{Y}(\pi; t)\tilde{\times}B)|}{|B|} = \frac{|\tilde{\mathcal{Y}}'(t)|}{|B|}.$$

Finally, by Lemma 3.19,

$$\frac{|\tilde{\mathcal{Y}}'(t)|}{|B|} = \frac{|\theta_t(\tilde{\mathcal{Y}}'(t))|}{|B|} = \frac{|\tilde{X}(t)||G||N|}{|B|} = |\tilde{X}(t)|\frac{|G|}{(q-1)^n}.$$

Thus,

$$\begin{aligned} |\tilde{Y}(\pi; t)| &= \frac{|\tilde{Y}(\pi; t)\tilde{\times}B|}{|B|} = \frac{|\tilde{\Omega}(\tilde{Y}(\pi; t)\tilde{\times}B)|}{|B|} = \frac{|\tilde{\mathcal{Y}}'(t)|}{|B|} \\ &= \frac{|\theta_t(\tilde{\mathcal{Y}}'(t))|}{|B|} = \frac{|\tilde{X}(t)||G||N|}{|B|} = |\tilde{X}(t)|\frac{|G|}{(q-1)^n} \end{aligned}$$

and we are done. ☐

We have now found a relationship between $\tilde{X}(t)$ and $\tilde{Y}(\pi;t)$ by proving Theorem 3.1. However, $\tilde{Y}(\pi;t)$ is not as well understood as $Y(\pi)_t$. Therefore, our next goal is to find the relationship between the sizes of $\tilde{Y}(\pi;t)$ and $Y(\pi)_t$, specifically when t is a regular semisimple s. We do this via an auxiliary set $\tilde{Y}(\pi;s)\tilde{\times}T$ defined below.

Definition 3.21. For a regular semisimple s, define

$$\tilde{Y}(\pi;s)\tilde{\times}T := \{(hB, h_0B, h_1B, \ldots, h_{n(n-1)}B, g) \in \tilde{Y}(\pi;s), \eta \in G \mid \eta^{-1}g\eta = s\}.$$

The following lemma sets the desired relationship in stone.

Lemma 3.22. *For a regular semisimple s, we have a bijection between $\tilde{Y}(\pi;s)\tilde{\times}T$ and $Y(\pi)_s \times G$.*

Proof. For each $(h_0B, h_1B, \ldots, h_{n(n-1)}B, s, h') \in Y(\pi)_s \times G$, consider the map to

$$(h'B, h'h_0B, h'h_1B, \ldots, h'h_{n(n-1)}B, h'sh'^{-1}, h') \in \tilde{Y}(\pi;s)\tilde{\times}T.$$

This is well defined and is injective. Consider also the map

$$(hB, h_0B, h_1B, \ldots, h_{n(n-1)}B, g, \eta) \in \tilde{Y}(\pi;s)\tilde{\times}T$$
$$\Downarrow$$
$$(\eta^{-1}h_0B, \eta^{-1}h_1B, \ldots, \eta^{-1}h_{n(n-1)}B, s, \eta).$$

It is not hard to see that this map is injective and that

$$(\eta^{-1}h_0B, \eta^{-1}h_1B, \ldots, \eta^{-1}h_{n(n-1)}B, s, \eta) \in Y(\pi)_s \times G.$$

Therefore, the two maps we discussed were both bijections and we are done. □

We now have all the bijections necessary to obtain this next important result.

Corollary 3.23. *For a regular semisimple s, we have*

$$|X_s| = |Y(\pi)_s|.$$

Proof. Lemma 3.22 implies that

$$|\tilde{Y}(\pi;s)\tilde{\times}T| = |G||Y(\pi)_s|.$$

Note that $|\tilde{Y}(\pi;s)\tilde{\times}T| = |T||\tilde{Y}(\pi;s)|$, because for any g similar to s, there are exactly $|T|$ values of η for which $\eta^{-1}g\eta = s$. Thus, we have

$$|\tilde{Y}(\pi;s)| = \frac{|\tilde{Y}(\pi;s)\tilde{\times}T|}{|T|} = \frac{|G|}{(q-1)^n}|Y(\pi)_s|.$$

By Theorem 3.1, it follows that we have

$$|\tilde{X}(s)| = |Y(\pi)_s|.$$

Furthermore, by Remark 2.5, we have

$$|X_s| = |\tilde{X}(s)| = |Y(\pi)_s|,$$

as desired.

□

Over the next sections, we focus on the point count of $Y(\pi)_s$, which then gives us $|X_s|$.

4. Additional Definitions

In this section, we introduce terminology commonly used with $Y(\pi)_s$, as well as citing a few results from related papers. While the previous section was more related to linear algebra, this section bridges us over to representation theory and group theory. By delving into a different field, we can find other ways to express $|Y(\pi)_s|$ that may be easier to calculate. In particular, we find that

$$|Y(\pi)_s| = \sum_{\lambda\in \mathrm{Irr}(S_n)} \chi_G^\lambda(s)\chi_{\mathrm{Hecke}}^\lambda(T_\pi).$$

It turns out that $\chi_{\mathrm{Hecke}}^\lambda(T_\pi)$ is already well known, so it then suffices to find $\chi_G^\lambda(g)$. We propose that

$$\chi_G^\lambda(s) = \chi_{S_n}^\lambda(1)$$

at the end of the section. Now, let us begin with definitions.

Let R_1 be the set of functions from G/B to $\mathbb{C}$. Note that R_1 is a vector space over $\mathbb{C}$. Let l_g for $g \in G$ be the group action of g on R_1. In other words, for any $f \in R_1$ and $xB \in G/B$, we have

$$l_g(f)(xB) = f(g^{-1}xB).$$

Let

$$R_1(g) := \mathrm{tr}(l_g) = |\{xB \in G/B \mid gxB = xB\}|.$$

Let $\mathcal{H}$ be the Iwahori–Hecke algebra for S_n with $\mathbb{C}$ coefficients.

Theorem 4.1 ([GP, Theorem 8.1.7])**.** *Irreducible representations of $\mathcal{H}$ are in bijection with irreducible representations of S_n (which are in bijection with partitions of n).*

For $T \in \mathcal{H}$, let $r_T : R_1 \to R_1$ be such that for any $f \in R_1$, we have

$$r_T(f) = f * T.$$

In other words, for any $xB \in G/B$, we have

$$r_T(f) = \frac{1}{|B|}\sum_{\substack{g\in G, hB\in G/B\\ ghB=xB}} f(g)T(h).$$

For $g \in G$ and $T \in \mathcal{H}$, let

$$\mathrm{tr}(g, T \mid R_1) := \mathrm{tr}(l_g \circ r_T) = \mathrm{tr}(r_T \circ l_g).$$

Let χ_G^λ be the irreducible unipotent character of G corresponding to λ. Let $\chi_{\mathrm{Hecke}}^\lambda$ be the irreducible character of the Iwahori–Hecke algebra corresponding to λ.

By Iwahori's Theorem ([GP, Corollary 8.4.7]) and the double centralizer theorem, we have:

Theorem 4.2. *For all $g \in G$, $T \in \mathcal{H}$, we have the equality*

$$\mathrm{tr}(g, T \mid R_1) = \sum_{\lambda\in \mathrm{Irr}(S_n)} \chi_G^\lambda(g)\chi_{Hecke}^\lambda(T).$$

Definition 4.3. Let $\mathbb{1}_{Bs_iB} \in \mathcal{H}$ for $1 \le i \le n-1$ be such that for all $g \in G$, $\mathbb{1}_{Bs_iB}(g) = 1$ if $g \in Bs_iB$ and $\mathbb{1}_{Bs_iB}(g) = 0$ otherwise. For $\beta \in \mathrm{Br}_n^+$, if $\beta = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$, define

$$T_\beta := \mathbb{1}_{Bs_{i_1}B} * \mathbb{1}_{Bs_{i_2}B} * \cdots * \mathbb{1}_{Bs_{i_k}B}.$$

Remark 4.4. It is well known that T_β does not depend on our choice of $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$.

Note that by [HR, Equation 2.2], we have

$$\operatorname{tr}(g, T_\pi \mid R_1) = |Y(\pi)_g|. \tag{4.5}$$

By [GP, Theorem 9.4.3], we have

$$\chi^\lambda_{\mathrm{Hecke}}(T_\pi) = q^{c(1)+c(\lambda)}(\chi^\lambda_{S_n}(1)). \tag{4.6}$$

Proposition 4.7. *For any regular semisimple s, we have*

$$\chi^\lambda_G(s) = \chi^\lambda_{S_n}(1).$$

We prove this in the next section.

5. Evaluating a Character

In this section, we use the theory of maximal tori to find a formula for $\chi^\lambda_G(s)$ and then evaluate it.

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and let $\mathbf{G} = GL_n(\overline{\mathbb{F}}_q)$. Let $\mathbf{T} \subset \mathbf{G}$ be the set of diagonal matrices. Let Fr be the Frobenius endomorphism. For any $\mathbf{H} \subset \mathbf{G}$, let

$$\mathbf{H}^{\mathrm{Fr}} = \{g \in \mathbf{H} \mid \mathrm{Fr}(g) = g\} = \{g \in \mathbf{H} \mid g \in G\}.$$

Note that $\mathbf{T}$ is a maximal torus, and for any element $g \in \mathbf{G}$, $g^{-1}\mathbf{T}g$ is also a maximal torus. For any regular semisimple s in $\mathbf{T}$, note that the only maximal torus that s lies in is $\mathbf{T}$.

Note by the Lang-Steinberg Theorem, we can find $g \in \mathbf{G}$ for which $g^{-1}\mathrm{Fr}(g) = w$ for each $w \in S_n$ (where we interpret w as a permutation matrix). Set $\mathbf{T}_w = g\mathbf{T}g^{-1}$ and $T_w = (g\mathbf{T}g^{-1})^{\mathrm{Fr}}$. Let the identity permutation be $w = 1$.

Let $R_{T_w,1}$ denote the virtual Deligne–Lusztig character corresponding to the torus $\mathbf{T}_w$ and the trivial character. We find an expression for χ^λ_G involving $R_{T_w,1}$.

Definition 5.1. Define the *almost character* R_λ as

$$R_\lambda := \frac{1}{|S_n|}\sum_{w \in S_n} \chi^\lambda_{S_n}(w) R_{T_w,1}.$$

Remark 5.2. By [L1, Section 4.4], it is known that for $G = GL_n(\mathbb{F}_q)$, we have $\chi^\lambda_G = R_\lambda$.

Now, we find what $R_{T_w,1}$ is so that we can calculate χ^λ_G.

Lemma 5.3. *$R_{T_w,1}(s) = 0$ if $w \neq 1$. Otherwise, $R_{T_w,1}(s) = n!$.*

Proof. By [C, Proposition 7.5.3],

$$R_{T_w,1}(s) = \frac{1}{|T|}\sum_{g \in G, g^{-1}sg \in T_w} 1.$$

Suppose for $w \neq 1$, there exists $g \in G$ for which $g^{-1}sg \in T_w$. Then, $g\mathbf{T}_w g^{-1}$ is a maximal torus that contains s. This then implies that $g\mathbf{T}_w g^{-1} = \mathbf{T}_1$. By construction, there exists $g' \in \mathbf{G}$ such that $g'^{-1}\mathrm{Fr}(g') = w$ and $\mathbf{T}_w = g'\mathbf{T}_1 g'^{-1}$. Thus, $g^{-1}\mathbf{T}_1 g = g'\mathbf{T}_1 g'^{-1}$, or $\mathbf{T}_1 = gg'\mathbf{T}_1(gg')^{-1}$, which means gg' is in the centralizer of $\mathbf{T}_1$. Thus, $gg' \in S_n \times \mathbf{T}_1$, say $gg' = w't$ for $w' \in S_n$ and $t \in \mathbf{T}_1$. Note then that

$$(gg')^{-1}\mathrm{Fr}(gg') = g'^{-1}g^{-1}\mathrm{Fr}(g)\mathrm{Fr}(g') = w,$$

yet

$$(gg')^{-1}\mathrm{Fr}(gg') = t^{-1}w'^{-1}\mathrm{Fr}(w')\mathrm{Fr}(t) = t^{-1}\mathrm{Fr}(t) \in \mathbf{T}.$$

This then implies that $w \in \mathbf{T}$, which is impossible. Thus, $R_w(s) = 0$ when $w \neq 1$.

On the other hand, when $w = 1$, g can be any element from the centralizer of T_1, which is $S_n \times T_1$. This has $n!|T_1| = n!|T|$ elements, so

$$R_{T_1,1}(s) = \frac{1}{|T|} \sum_{g \in G, g^{-1}sg \in T_1} 1 = n!$$

as desired. ☐

Finally, we can find $\chi_G^\lambda(s)$.

Proof of Proposition 4.7. By Lemma 5.3, Definition 5.1, and Remark 5.2,

$$\chi_G^\lambda(s) = R_\lambda(s) = \frac{1}{|S_n|} \sum_{w \in S_n} \chi_{S_n}^\lambda(w) R_{T_w,1}(s) = \frac{1}{n!} \chi_{S_n}^\lambda(1) n! = \chi_{S_n}^\lambda(1).$$

☐

6. Main Results

In this section, we present several theorems that follow from our work above, as well as a conjecture.

Theorem 6.1. *For any regular semisimple s, we have*

$$|X_s| = \sum_{\lambda \in Irr(S_n)} q^{c(1)+c(\lambda)} (\chi_{S_n}^\lambda(1))^2.$$

Proof. We have $|X_s| = |Y(\pi)_s|$ by Corollary 3.23, and Equation 4.5 tells us that

$$\operatorname{tr}(s, T_\pi \mid R_1) = |Y(\pi)_s| = |X_s|.$$

Next, by Theorem 4.2,

$$|X_s| = \operatorname{tr}(s, T_\pi \mid R_1) = \sum_{\lambda \in \operatorname{Irr}(S_n)} \chi_G^\lambda(s) \chi_{\text{Hecke}}^\lambda(T_\pi).$$

Equation 4.6 and Proposition 4.7 then give

$$\begin{aligned} |X_s| &= \sum_{\lambda \in \operatorname{Irr}(S_n)} \chi_G^\lambda(s) \chi_{\text{Hecke}}^\lambda(T_\pi) = \sum_{\lambda \in \operatorname{Irr}(S_n)} \chi_{S_n}^\lambda(1) \cdot q^{c(1)+c(\lambda)} \chi_{S_n}^\lambda(1) \\ &= \sum_{\lambda \in \operatorname{Irr}(S_n)} q^{c(1)+c(\lambda)} (\chi_{S_n}^\lambda(1))^2 \end{aligned}$$

and we are done. ☐

Here, we show that the formula does indeed hold for $n = 2$ by returning to 2.2.

Example 6.2. For $SL_2(\mathbb{F}_q)$, we first find the general form of an LU matrix:

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c^{-1} \end{bmatrix} = \begin{bmatrix} c & d \\ bc & bd + c^{-1} \end{bmatrix}$$

where $b, c, d \in \mathbb{F}_q$. We now need to make sure this matrix is similar to

$$s := \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}$$

where $e \neq e^{-1} \in \mathbb{F}_q$ are the eigenvalues of s. In other words, we want the characteristic polynomials to be the same, so $x^2 - (c + c^{-1} + bd)x + 1 = x^2 - (e + e^{-1})x + 1$. When $c = e, e^{-1}$, there are $2q - 1$ solutions for the pair (b, d), since $bd = 0$. Otherwise, bd is fixed and nonzero, giving $q - 1$ pairs for bd. It follows from Definition 2.1 that $|X_s(\mathbb{F}_q)| = 2(2q-1) + (q-3)(q-1) = q^2 + 1$, which matches with the formula we found for $|X_s|$.

Theorem 6.3. *For any regular semisimple s, we have*

$$|\tilde{Y}(\pi; s)| = \frac{|G|}{(q-1)^n} \sum_{\lambda \in Irr(S_n)} q^{c(1)+c(\lambda)} (\chi^{\lambda}_{S_n}(1))^2.$$

Proof. By Theorem 3.1, Remark 2.5, and Theorem 6.1, we have

$$|\tilde{Y}(\pi; s)| = \frac{|G|}{(q-1)^n} |\tilde{X}(s)| = \frac{|G|}{(q-1)^n} |X_s| = \frac{|G|}{(q-1)^n} \sum_{\lambda \in \mathrm{Irr}(S_n)} q^{c(1)+c(\lambda)} (\chi^{\lambda}_{S_n}(1))^2.$$

□

Theorem 6.4. *For any regular semisimple s, we have*

$$|\tilde{Y}(\pi; s)| = |\tilde{Y}(\pi; 1)|.$$

Proof. Minh-Tam Trinh has proven in [T, Theorem 7] that

$$|\tilde{Y}(\pi; 1)| = \frac{|G|}{(q-1)^n} \sum_{\lambda \in \mathrm{Irr}(S_n)} q^{c(1)+c(\lambda)} (\chi^{\lambda}_{S_n}(1))^2.$$

This is the same as $|\tilde{Y}(\pi; s)|$. □

Considering the theorem above, we might conjecture the following:

Conjecture 6.5. *The point count $|\tilde{Y}(\pi; t)|$ is the same regardless of t.*

Indeed, this has been confirmed by computations using Python for $n = 3$. Given all this circumstantial evidence, this conjecture is something highly worth investigating.

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