Bidding Games

Matvey Borodin, Kaylee Ji, Yifan Kang
Mentor: Chun Hong Lo

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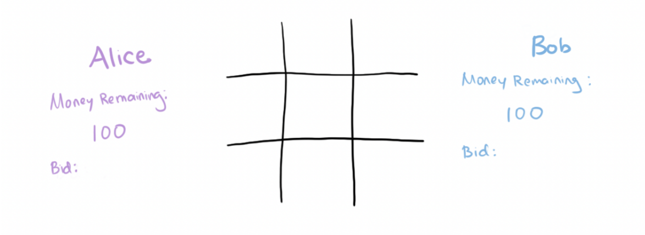

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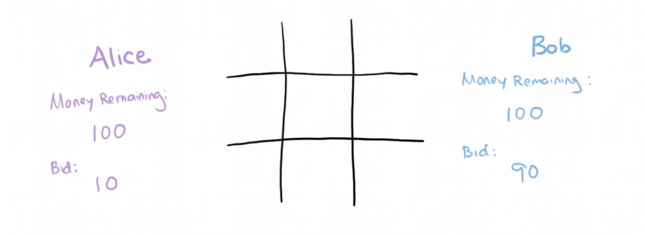

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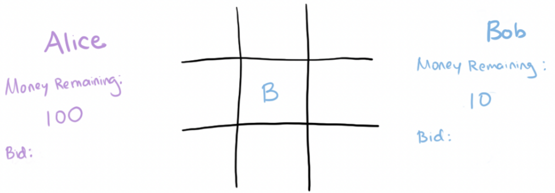

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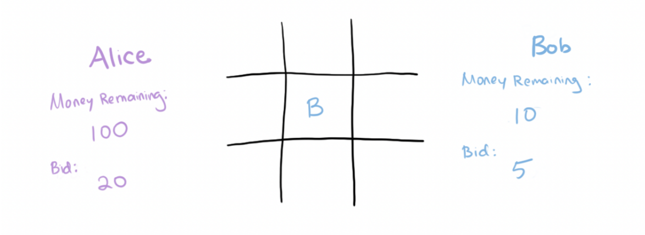

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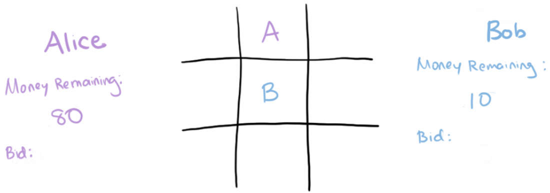

What are Bidding Games?

Definition (Bidding Games).

- two player zero sum games on a graph where each player has an objective node
- each turn, highest bidding player moves
- players bid simultaneously
- players know each other's bidding history and budgets

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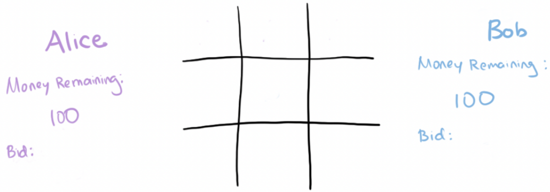

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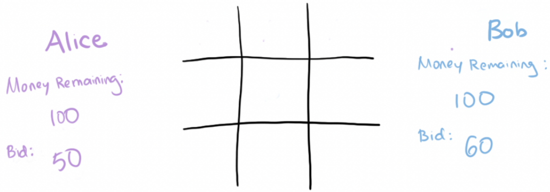

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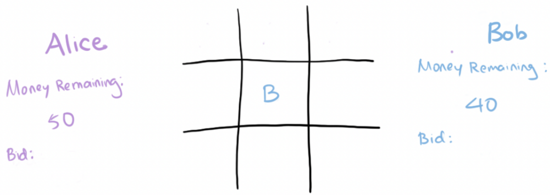

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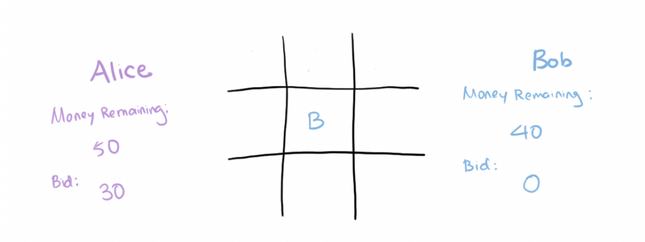

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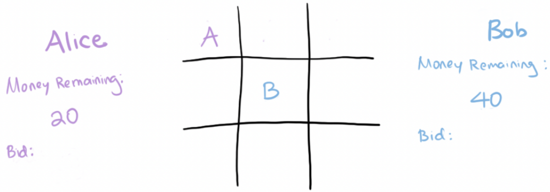

Win n Times in a Row

Definition (Win n Times in a Row Game).

- all-pay bidding game with $\leq n$ turns
- player 1 wins if they out-bids player 2 n times in a row
- player 2 wins if they out-bids player 1 any turn
- assumes money is infinitely divisible
- tie breaking: if both players bid the same value, we consider player 1's bid higher

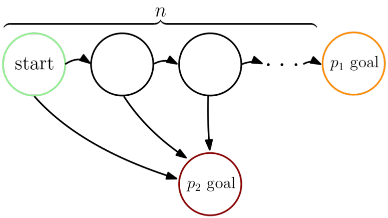

Figure: Visualizing WnR(n) on a graph

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Important notes:

- same game if Alice has budget 2 and Bob has budget 1 and each player halves their bids
- *budget ratio* - ratio of player 1's budget to player 2's budget
- we will set players 2's budget as 1 in later games

Value

To analyze the game, we assume both players use randomized strategies (eg. a strategy for Player 1 on the their first turn is to bid 1 or 0.5, each with probability $\frac{1}{2}$).

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When the Lower Value is equal to the Upper Value, we call this quantity Value.

Simple cases in WnR(2)

- $B_1 = 2$: Bid 1 on both turns guarantees winning, so the value of the game is 1.

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- $B_1 = 2$: Bid 1 on both turns guarantees winning, so the value of the game is 1.
- $B_1 = 1$: If player 1 wins the first round, player 2 will win the second bidding. Player 1 has no chance of winning two times in a row so the value of the game is 0.

The value of the game

Theorem

In the "win twice in a row" game, given initial budget ratio B_1, the value of the game is 1 for $B_1 \geq 2$, 0 for $B_1 \leq 1$ and $\frac{1}{n}$ for $B_1 \in [1 + \frac{1}{n}, 1 + \frac{1}{n-1})$ with $n \in \mathbb{Z}_{\geq 2}$.

Proof.

- Let $B_1 = 1 + \frac{1}{n} + \epsilon$ with $n \in \mathbb{Z}_{\geq 2}$ and $\epsilon \in [0, \frac{1}{n-1} - \frac{1}{n})$.

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- Let $B_1 = 1 + \frac{1}{n} + \epsilon$ with $n \in \mathbb{Z}_{\geq 2}$ and $\epsilon \in [0, \frac{1}{n-1} - \frac{1}{n})$.
- Next, we want to show a strategy for player 1 that has at least $\frac{1}{n}$ chance of winning.

Player 1's strategy in WnR(2)

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- If player 1 plays $\frac{k+1}{n}$:
  $B_1' = B_1 - b_1 = \frac{n-k}{n} + \epsilon > \frac{n-k}{n} \geq 1 - b_2 = B_2'$
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- Since player 1 would pick $\frac{k+1}{n}$ with probability $\frac{1}{n}$, the lower value is $\frac{1}{n}$.

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- If $b_1 < b_2$, player 1 loses immediately.

Player 2's strategy in WnR(2)

- Else if $b_1 > b_2 + \frac{1}{n} + \epsilon$. The budget ratio would be

$$\frac{B_1 - b_1}{1 - b_2} < \frac{(1 + \frac{1}{n} + \epsilon) - (b_2 + \frac{1}{n} + \epsilon)}{1 - b_2} < 1$$

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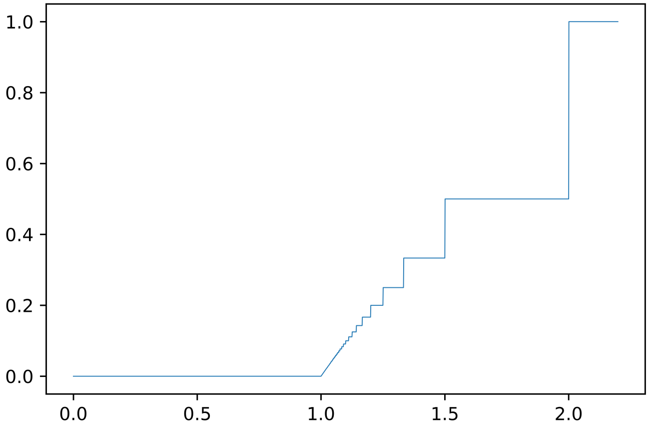
- However, $\frac{1}{n} + \epsilon < \frac{1}{n} + \epsilon'$, which means that for every b_1 there's at most 1 value of b_2 that player 1 could win.
- This shows us that the upper value of the game is $\frac{1}{n}$. Thus, the value is $\frac{1}{n}$.

Graph



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- Computer algorithm to approximate lower value
- Simplify by assuming strategies consider finitely many bid values
- Uses linear programming to solve for optimal strategy

Example with $\epsilon = 1$

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Table: Payoff matrix A

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- Note we consider min, not weighted average for player 2 strategy

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Another example

- $n = 3$, $B_1 = 2$, $\epsilon = 0.25$
  - $\max_{\mathbf{p}} \min(A \cdot \mathbf{p})$
  - $\mathbf{p} = \begin{pmatrix} 0.368 \\ 0.158 \\ 0.158 \\ 0.0 \\ 0.316 \end{pmatrix}$

	0	0.25	0.5	0.75	1
0	0.5	0.5	0.33	0.2	0
0.25	0	1	0.5	0.5	0.25
0.5	0	0	1	1	0.5
0.75	0	0	0	1	1
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function VALUE(n, ϵ, B)

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    return min_i Σ_j payoff(j, i) · p(j)
end function
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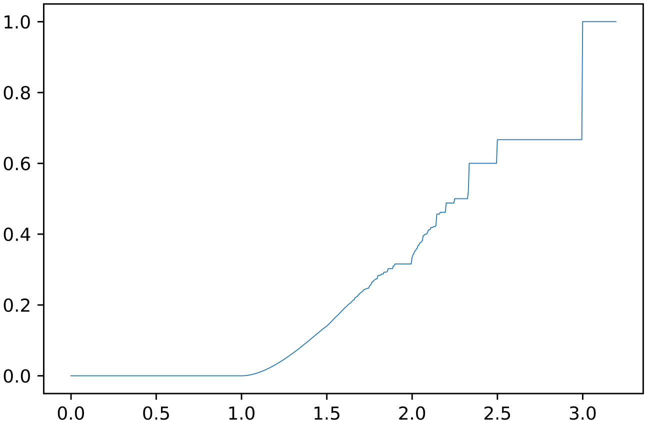
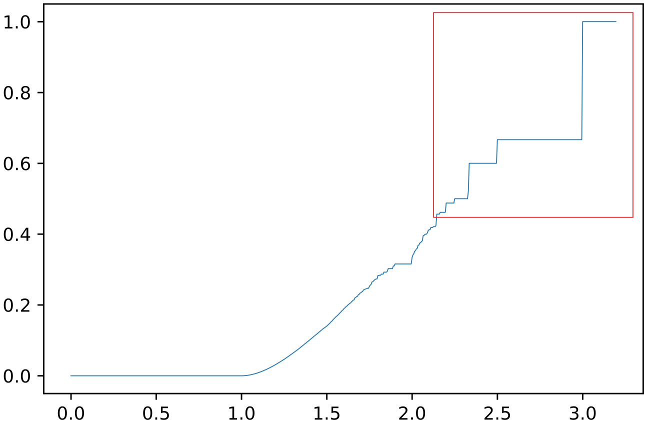
Graph



Graph



Graph