REFINEMENTS OF PRODUCT FORMULAS FOR VOLUMES OF FLOW POLYTOPES

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Abstract. Flow polytopes are an important class of polytopes in combinatorics whose lattice points and volumes have interesting properties and relations. The Chan–Robbins–Yuen (CRY) polytope is a flow polytope with normalized volume equal to the product of consecutive Catalan numbers. Zeilberger proved this by evaluating the Morris constant term identity, but no combinatorial proof is known. There is a refinement of this formula that splits the largest Catalan number into Narayana numbers, which Mészáros gave an interpretation as the volume of a collection of flow polytopes. We introduce a new refinement of the Morris identity with combinatorial interpretations both in terms of lattice points and volumes of flow polytopes. Our results generalize Mészáros's construction and a recent flow polytope interpretation of the Morris identity by Corteel–Kim–Mészaros. We prove the product formula of our refinement following the strategy of the Baldoni–Vergne proof of the Morris identity with a combinatorial approach.

1. Introduction

1.1. **Foreword.** Flow polytopes play a fundamental role in combinatorial optimization through their relation to maximum matching and minimum cost problems (e.g. see [16, Ch. 13]). Flow polytopes have been used in various fields like toric geometry [9] and representation theory [2]. More recently, they have been related to geometric and algebraic combinatorics thanks to connections with Schubert polynomials [6], diagonal harmonics [14], and generalized permutahedra [12].

Given a graph G with vertex set $\{0, 1, \ldots, n, n+1\}$ and edges (i, j) oriented $i \to j$ if $i < j$, we associate with G a net flow vector $\mathbf{a} = (a_0, a_1, \ldots, a_n, -\sum_{i=0}^{n} a_i)$ such that vertex i has net flow a_i for $i = 0, 1, \ldots, n$. The set of all flows with net flow vector $\mathbf{a}$, called the *flow polytope*, is denoted by $\mathcal{F}_G(\mathbf{a})$. Define $K_G(\mathbf{a})$ as the number of lattice points (integer flows) of $\mathcal{F}_G(\mathbf{a})$, called the *Kostant vector partition function.* The name comes from the fact that for the complete graph k_{n+2}, $K_G(\mathbf{a})$ is a vector partition function studied by Kostant in the context of Lie algebras (e.g. [10]). The following theorem, which appears in unpublished work of Postnikov and Stanley and in the work of Baldoni-Vergne [2], relates the volume of a flow polytope to a Kostant partition function.

Theorem 1.1 (Postnikov-Stanley, Baldoni-Vergne [2])**.** *For a loopless digraph G with vertices $\{0, 1, \ldots, n+1\}$ having unique source 0 and unique sink $n+1$,*

$$\operatorname{vol} \mathcal{F}_G(1, 0, \ldots, 0, -1) = K_G(0, d_1, \ldots, d_n, -\sum_{i=1}^{n} d_i), \tag{1.1}$$

where $d_i = \operatorname{indeg}_G(i) - 1$.

In Section 3, we provide a new recursive proof of this theorem by extending a well-known subdivision map of flow polytopes to integer flows.

An important example of a flow polytope is the Chan-Robbins-Yuen (CRY) polytope [4], defined as $CRY_{n+1} := \mathcal{F}_{k_{n+2}}(1, 0, \ldots, 0, -1)$. Zeilberger calculated the volume of CRY_{n+1} algebraically using the Morris constant term identity, equivalent to the famous Selberg integral formula (see [7]). For convenience, we use the term *volume* in this paper to refer to normalized volume. For instance, a d-dimensional simplex has volume 1.

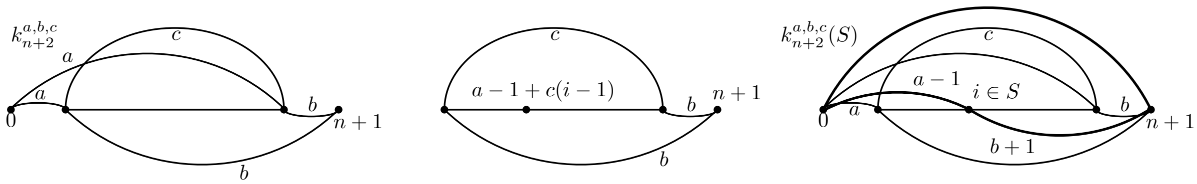

FIGURE 1. The graph $k_{n+2}^{a,b,c}$ and graph of the Kostant partition function corresponding to the volume of $\mathcal{F}_{k_{n+2}^{a,b,c}}$.

Theorem 1.2 (Zeilberger's Morris Identity [20]). *For positive integers n, a, and b, and nonnegative integer c, define the constant term*

$$M_n(a,b,c) := \mathrm{CT}_x \prod_{i=1}^{n} (1-x_i)^{-b} x_i^{-a+1} \prod_{1 \le i < j \le n} (x_j - x_i)^{-c},$$

where $\mathrm{CT}_x := \mathrm{CT}_{x_n} \cdots \mathrm{CT}_{x_1}$. *Then*

$$M_n(a,b,c) = \prod_{j=0}^{n-1} \frac{\Gamma(a-1+b+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2}+1)}{\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})\Gamma(\frac{c}{2}(j+1)+1)}. \tag{1.2}$$

By specializing this identity, Zeilberger proved that the volume of CRY_{n+1} is the product of the first $n-1$ Catalan numbers.

Theorem 1.3 (Zeilberger [20]). *The volume of the polytope CRY_{n+1} is given by $M_n(1,1,1) = \prod_{i=1}^{n-1} C_i$, where $C_i = \frac{1}{2i+1}\binom{2i}{i}$ is the ith Catalan number.*

However, no combinatorial proof of Theorem 1.3 is known.

Corteel-Kim-Mészáros [5, Theorem 1.2] also showed that for any positive a, b, and c, $M_n(a,b,c)$ gives the volume of the flow polytope on the following graph. For positive integer n, let $k_{n+2}^{a,b,c}$ denote the graph on vertex set $\{0, 1, \ldots, n+1\}$ with edge $(0,i)$, $i \in [1,n]$ appearing with multiplicity a, edge $(i, n+1)$, $i \in [1,n]$, appearing with multiplicity b, and $(i,j), 1 \le i < j \le n$, appearing with multiplicity c (see Figure 1). Then they showed the following.

Theorem 1.4 (Corteel-Kim-Mészáros [5]). *Let n, a and b be positive integers and c be a nonnegative integer. Then*

$$\operatorname{vol} \mathcal{F}_{k_{n+2}^{a,b,c}}(1,0,\ldots,0,-1) = K_{k_{n+2}^{a,b,c}}(0, a_1, \ldots, a_n, -\sum_{i=1}^{n} a_i) = M_n(a,b,c), \tag{1.3}$$

where $a_i = a-1+c(i-1)$.

Another interesting property of the CRY polytope was a refinement of the product formula. Namely, the following conjecture of Chan-Robbins-Yuen [4, Conj. 2], settled by Zeilberger [20], refines the product $C_n C_{n-1} \cdots C_1$ by splitting C_n into a sum of Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

Theorem 1.5 (Zeilberger [20]). *The sum of Kostant partition functions $K_{k_{n+2}}(0, a_1, \ldots, a_n, -\sum_{j=1}^{n} a_j)$ such that for $i \in [n]$, $a_i \le i-1$, with $a_i = i-1$ holding for exactly k values of i, is given by the product*

$$N(n,k) \prod_{i=1}^{n-1} C_i.$$

Mészáros [11, Thm. 11] also provides a collection of interior disjoint polytopes whose volumes sum to $N(n,k)C_{n-1} \cdots C_1$, thus giving a combinatorial interpretation to Conjecture 2 of Chan-Robbins-Yuen.

1.2. **A new refinement of $M_n(a,b,c)$.** In [20], Zeilberger sketched the proof of Theorem 1.5 using Aomoto's refinement of the Selberg integral [1], but no explicit refinement of $M_n(a,b,c)$ was given (see also [19]). In this paper we give such a refinement and give a Kostant partition function interpretation generalizing Theorem 1.5. We also give our refinement a geometric interpretation as a collection of interior disjoint polytopes subdividing $\mathcal{F}_{k_{n+2}^{a,b,c}}(1,0,\ldots,0,-1)$, extending Mészáros's interpretation [11] and Theorem 1.4. Our refinement is inspired by a related refinement of $M_n(a,b,c)$ introduced by Baldoni-Vergne [3] to prove the Morris identity (Theorem 1.2), for which we extend a Kostant partition function interpretation (see Section 5) but which did not imply Theorem 1.5.

To state our results, define the constant term

$$\Psi_n(k,a,b,c) := \mathrm{CT}_x[t^k]\prod_{i=1}^{n}(1-x_i)^{-b}x_i^{-a+1}(1+t\frac{x_i}{1-x_i})\prod_{1\le i<j\le n}(x_j-x_i)^{-c}. \tag{1.4}$$

In the case that $k=0$, $\Psi_n(0,a,b,c)=M_n(a,b,c)$. We now give Kostant partition function and polytope volume interpretations for $\Psi_n(k,a,b,c)$, as well as an explicit product formula.

To state these interpretations, we introduce some notation. Denote by $\binom{[n]}{k}$ the set of k-element subsets of $[n]$. For $S\in\binom{[n]}{k}$, let $k_{n+2}^{a,b,c}(S)$ be the set of graphs obtained from taking $k_{n+2}^{a,b,c}$, adding n edges $(0,n+1)$, and for each $i\in S$ deleting one of the a incoming edges $(0,i)$ and adding an outgoing edge $(i,n+1)$ (See Figure 1).

Theorem 1.6. *For positive integers n,a, and b, nonnegative integer c, and nonnegative integer $k\le n$, the constant term $\Psi_n(k,a,b,c)$ equals the following:*

(i) the sum of Kostant partition functions of the form $K_{k_{n+2}^{a,b,c}}(0,a_1,\ldots,a_n,-\sum_{j=1}^n a_j)$ such that for $i\in[n]$, $a_i\le a-1+c(i-1)$, with $a_i=a-1+c(i-1)$ holding for exactly $n-k$ values of i.

(ii) the volume of the interior disjoint polytopes $\{\mathcal{F}_{k_{n+2}^{a,b,c}(S)}\mid S\in\binom{[n]}{k}\}$. Thus,

$$\Psi_n(k,a,b,c)=\sum_{S\in\binom{[n]}{k}}\mathrm{vol}\,\mathcal{F}_{k_{n+2}^{a,b,c}(S)}.$$

We see that when $a=b=c=1$, the Kostant partition function interpretation of $\Psi_n(k,a,b,c)$ reduces to Conjecture 2 of Chan-Robbins-Yuen, giving that

$$\Psi_n(k,1,1,1)=N(n,k+1)\cdot C_{n-1}\cdots C_1.$$

As a corollary, our constant term $\Psi_n(k,a,b,c)$ refines the Morris constant term $M_n(a,b+1,c)$.

Corollary 1.7. *Let n,a, and b be positive integers, and let c be a nonnegative integer. Then*

$$M_n(a,b+1,c)=\sum_{k=0}^{n}\Psi_n(k,a,b,c). \tag{1.5}$$

We also compute the following explicit product formula for $\Psi_n(k,a,b,c)$ that completes our refinement of the Morris identity.

Theorem 1.8. *For positive integers n,a, and b, nonnegative integer c, and nonnegative integer $k\le n$, the constant term $\Psi_n(k,a,b,c)$ is given by*

$$\Psi_n(k,a,b,c)=\binom{n}{k}M_n(a,b,c)\prod_{j=1}^{k}\frac{a-1+(n-j)\frac{c}{2}}{b+(j-1)\frac{c}{2}}.$$

We prove Theorem 1.8 by proving four recurrence relations satisfied by $\Psi_n(k,a,b,c)$, by proving these relations uniquely define $\Psi_n(k,a,b,c)$, and by proving the product formula also satisfies these relations. This closely follows the approach of Baldoni-Vergne [3, p. 10] in their proof of the Morris identity. However, our proofs are combinatorial rather than algebraic, with the notable exception of the proof of the relation (4.9), which states that for $1\le k\le n$,

$$k(b+(k-1)c/2)\cdot\Psi_n(k,a,b,c)=(n-k+1)(a-1+(n-k)c/2)\cdot\Psi_n(k-1,a,b,c). \tag{$\star$}$$

We leave as an open problem to prove this relation combinatorially, which would then imply a combinatorial proof of the volume formula for the CRY polytope (Theorem 1.3).

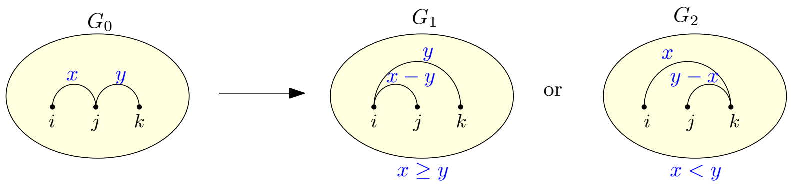

FIGURE 2. The subdivision lemma reduces a flow polytope to two interior disjoint polytopes whose union is integrally equivalent to the original flow polytope.

1.3. **Outline.** The rest of this paper is structured as follows. In Section 2, we establish basic theory surrounding flow polytopes, Kostant partition functions, and the Morris constant term identity. Section 3 gives a recursive proof of Theorem 1.1. In Section 4, we prove our results for $\Psi_n(k, a, b, c)$, including Theorem 1.6, Corollary 1.7, and Theorem 1.8. In Section 5, we apply our methods for $\Psi_n(k, a, b, c)$ to the Baldoni-Vergne constant term and prove Theorem 5.2, and in Section 6 we provide final remarks and potential avenues for future research.

2. Background and Notation

2.1. **Flow polytopes and their subdivisions.** Given a loopless acyclic connected digraph G with vertex set $\{0, 1, \ldots, n, n+1\}$ and m edges, we orient edge (i, j) from i to j if $i < j$. We can then represent each edge (i, j) by the positive type A_n root $\alpha(i, j) = e_i - e_j$. We also define M_G to be the $(n+2) \times m$ matrix whose columns are given by the multiset $\{\{\alpha(e)\}\}_{e \in E(G)}$.

Then given a net flow vector $\mathbf{a} = (a_0, a_1, \ldots, a_n, -\sum_{i=0}^{n} a_i)$, where a_i represents the net flow at vertex i, we define an $\mathbf{a}$-*flow* $\mathbf{f}_G$ as a vector $\mathbf{f}_G = (f(e))_{e \in E(G)}$ satisfying $M_g \mathbf{f}_G = \mathbf{a}$. We now define the *flow polytope* $\mathcal{F}_G(\mathbf{a})$ as the set of all $\mathbf{a}$-flows on G. More precisely, $\mathcal{F}_G(\mathbf{a}) := \{\mathbf{f}_G \in \mathbb{R}^m_{\geq 0} \mid M_G \mathbf{f}_g = \mathbf{a}\}$. In the absence of an explicit vector $\mathbf{a}$, it is implied that $\mathbf{a} = (1, 0, \ldots, 0, -1)$. In other words, $\mathcal{F}_G := \mathcal{F}_G(1, 0, \ldots, 0, -1)$. If G has a unique source 0 and sink $n+1$, then the *dimension* of $\mathcal{F}_G$ is $m - n - 1$.

Next we define a notion of equivalence for flow polytopes. Let $\mathrm{aff}(\cdot)$ denote affine span. For two flow polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, we say that P and Q are *integrally equivalent*, denoted $P \equiv Q$, if there exists an affine transformation $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ that is a bijection both when restricted between P and Q and when restricted between $\mathrm{aff}(P) \cap \mathbb{Z}^n$ and $\mathrm{aff}(Q) \cap \mathbb{Z}^m$. Polytopes that are integrally equivalent share many similar properties, including the same volume and Ehrhart polynomials.

We now give a recursive subdivision of flow polytopes used by Postnikov-Stanley in their unpublished work and which appears in the work of Mészáros-Morales [13].

Let $G = (\{0, 1, \ldots, n, n+1\}, E)$. We now repeatedly apply the following algorithmic step, called the *reduction rule*: starting with a graph G_0 on vertex set $\{0, 1, \ldots, n, n+1\}$ and $(i, j), (j, k) \in E(G_0)$ for some $i < j < k$, we reduce G_0 to two graphs G_1 and G_2 with vertex set $\{0, 1, \ldots, n, n+1\}$ and edge sets

$$E(G_1) := E(G_0) \setminus \{(j, k)\} \cup \{(i, k)\}, \tag{2.1}$$

$$E(G_2) := E(G_0) \setminus \{(i, j)\} \cup \{(i, k)\}. \tag{2.2}$$

Proposition 2.1 (Subdivision Lemma, Postnikov, Stanley [17] (e.g. [11, Prop. 1]))**.** *Given a graph G_0 on the vertex set $\{0, 1, \ldots, n, n+1\}$ and $(i, j), (j, k) \in E(G_0)$ for arbitrary $i < j < k$, define G_1 and G_2 by the above reduction rule. Then we have*

$$\mathcal{F}_{G_0} \equiv \mathcal{F}_{G_1} \cup \mathcal{F}_{G_2}, \qquad \mathcal{F}^{\circ}_{G_1} \cap \mathcal{F}^{\circ}_{G_2} = \varnothing,$$

where $\mathcal{F}^{\circ}_G$ denotes the interior of the polytope $\mathcal{F}_G$.

The subdivision lemma is illustrated in Figure 2. The proof can be found in [13]. Define a graph G to be *reducible* if we can apply the reduction rule to two of its edges (that is, there exists $(i, j), (j, k) \in E(G)$). Otherwise, the graph G is *irreducible*. We now define the reduction tree $\mathcal{T}(G)$ of a graph G. The root of $\mathcal{T}(G)$ is G, and each node G_0 has two children G_1 and G_2 described by the reduction rule. Each leaf of $\mathcal{T}(G)$ is hence irreducible. $\mathcal{T}(G)$ is not unique and depends on the order of reductions applied, but the number of leaves is always the same.

2.2. **Kostant partition functions.** We now examine an important subset of $\mathcal{F}_G(\mathbf{a})$. For a graph G on vertex set $\{0,1,\dots,n,n+1\}$ and (i,j) oriented $i \to j$ if $i<j$, denote by $\mathcal{F}_G^{\mathbb{Z}}(\mathbf{a})$ the set of lattice points of the flow polytope $\mathcal{F}_G(\mathbf{a})$, and define $K_G(\mathbf{a}) := \#F_G^{\mathbb{Z}}(\mathbf{a})$ to be the number of such lattice points, called the *Kostant partition function.* The name comes from interpreting the function as giving the number of ways of writing $\mathbf{a}$ as a $\mathbb{N}$-linear combination of the type A positive roots $e_i - e_j$, where e_i is the ith standard vector and $i<j$.

The generating function of Kostant partition functions on G is given by

$$\sum_{\mathbf{a}} K_G(\mathbf{a})\mathbf{x}^{\mathbf{a}} = \prod_{(i,j)\in E(G)} (1-x_i x_j^{-1})^{-1},$$

where the term $x_i x_j^{-1}$ represents a single flow along the edge (i,j), and the number of flows with net flow of j at vertex i is represented by the coefficient of x_i^j. In particular, for the graph $k_{n+2}^{a,b,c}$, the generating function can be simplified to

$$\sum_{\mathbf{a}} K_{k_{n+2}^{a,b,c}}(\mathbf{a})\mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{n}(1-x_0x_i^{-1})^{-a}(1-x_ix_{n+1}^{-1})^{-b}\prod_{1\le i<j\le n}(1-x_ix_j^{-1})^{-c}. \tag{2.3}$$

Theorem 1.1 relates the volume of a flow polytope to a Kostant partition function with a certain net flow vector. Using the generating function for Kostant partition functions, this has very useful implications, such as Theorem 1.4. To prove Theorem 1.4, first apply Theorem 1.1 for $k_{n+2}^{a,b,c}$. Since the net flow at the source is zero, we can ignore the term $\prod_{i=1}^{n}(1-x_0x_i^{-1})^{-a}$. Because the total flow is conserved, the flow at vertex $n+1$ is already determined, so we can simplify the product by setting $x_{n+1}=1$. The result follows by extracting the appropriate coefficient in (2.3), and expressing it as a constant term extraction (see [5, Theorem 1.2]). This approach thus gives a way to express Kostant partition functions as a constant term.

2.3. **Catalan numbers, Narayana numbers, and Proctor's formula.** The Catalan numbers satisfy the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$, and are one of the most ubiquitous sequences in combinatorics. For instance, the Catalan number C_n counts the number of lattice paths from $(0,0)$ to (n,n) that do not pass above the line $y = x$. Stanley [18] even gives an exercise in 66 parts asking readers to show the number of various combinatorial structures was the Catalan sequence, and his online addendum adds over 200 additional combinatorial interpretations. The Catalan numbers are refined by the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ such that

$$C_n = \sum_{k=1}^{n} N(n,k).$$

In analogy to the Catalan numbers, the Narayana number $N(n,k)$ counts the number of lattice paths from $(0,0)$ to (n,n) that do not pass above the line $y=x$ and has $2k-1$ turns. Notably, both Narayana and Catalan numbers appear in Theorem 1.5, where the Narayana refine the volume of the CRY polytope. Proctor's formula describes another form in which Catalan numbers can appear. In [15], Proctor shows that

$$\prod_{1\le i<j\le n} \frac{2(a-1)+i+j-1}{i+j-1} = \det[C_{n-2+i+j}]_{i,j=1}^{a-1}.$$

We will see Catalan numbers appear in several forms in Section 2.4 for special cases of the Morris identity, including through Proctor's formula.

2.4. **The Morris constant term identity $M_n(a,b,c)$.** We first formalize the notion a constant term extraction. For a Laurent series $f(x_i)$, we denote the coefficient of x_i^j by $[x_i^j]f(x_i)$, and we denote the constant term in x_i by $\mathrm{CT}_{x_i} f(x_i)$. Similarly, for a Laurent series $f(x_1,x_2,\dots,x_n)$, we denote the constant term by $\mathrm{CT}_x f(x_1,x_2,\dots,x_n) := \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1} f(x_1,x_2,\dots,x_n)$.

Similary, we define the *residue* of $f(x_i)$ with respect to x_i as the coefficient of x_i^{-1}. We denote this by $\mathrm{Res}_{x_i} f(x_i) := [x_i^{-1}]f(x_i)$, and we also use the notation $\mathrm{Res}_x f(x_1,x_2,\dots,x_n) := \mathrm{Res}_{x_n}\cdots\mathrm{Res}_{x_1} f(x_1,x_2,\dots,x_n)$. A useful property of residues is that for an analytic function $f(x_1,x_2,\dots x_n)$, the residue of a partial derivative is always zero. That is,

$$\mathrm{Res}_{x_i}\frac{\partial}{\partial x_i} f(x_1,x_2,\dots x_n) = 0.$$

We now give some special properties and cases of the Morris constant term identity (1.2). Note that for $c > 0$, we can substitute $\Gamma(x+1) = x\Gamma(x)$ to obtain the following alternate form of Morris identity

$$M_n(a,b,c) = \frac{1}{n!}\prod_{j=0}^{n-1} \frac{\Gamma(a-1+b+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2})}{\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})\Gamma(\frac{c}{2}(j+1))}.$$

This form of the Morris identity is used in most of our computational proofs. Note also the following fundamental symmetry of $M_n(a,b,c)$ with respect to a and b.

Proposition 2.2. *$M_n(a,b,c) = M_n(b,a,c)$.*

Proof. This can directly be seen from symmetry of a and b in the product formula of $M_n(a,b,c)$.

On a polytope level, consider a bijection $\varphi : \mathcal{F}_{k^{a,b,c}_{n+2}} \to \mathcal{F}_{k^{b,a,c}_{n+2}}$ defined as follows. For each flow $f \in \mathcal{F}_{k^{a,b,c}_{n+2}}$, $\varphi(f)$ is obtained by reversing the direction of flow of each edge in $k^{a,b,c}_{n+2}$. It is easy to see that this is an involution, so the volume of the two polytopes must be equal. □

Recall that $M_n(1,1,1)$ is a product of consecutive Catalan numbers. Interestingly, the case $M_n(a,1,1)$ strongly resembles $M_n(1,1,1)$, and is, by Proctor's formula, a product of Catalan numbers times a determinant of Catalan numbers.

Corollary 2.3. [2, 11] *The constant term $M_n(a,1,1)$ can be expressed as a product of consecutive Catalan numbers times a determinant of Catalan numbers.*

$$\begin{aligned} M_n(a,1,1) &= C_1C_2\cdots C_{n-1}\prod_{1\le i<j\le n}\frac{2(a-1)+i+n-1}{i+n-1} \\ &= \det[C_{n-2+i+j}]_{i,j=1}^{a-1}C_{n-1}C_{n-2}\cdots C_1. \end{aligned}$$

To investigate possible generalizations, here we list simplified identities for some other special cases of the Morris identities. Proofs of these formulas and other special cases, namely $M_n(a,b,1)$ and $M_n(a,b,2k)$, are rather computational and are hence provided in the Appendix. Intriguingly, the explicit formula for $M_n(a,b,1)$ strongly resembles the formula for $M_n(a,1,1)$.

Corollary 2.4. *For positive integers $n, a,$ and b, the constant term $M_n(a,b,1)$ is given by*

$$M_{2n}(a,b,1) = C_1C_2\cdots C_{2n-1}\prod_{1\le i<j\le 2n}\frac{2(a+b-2)+i+j-1}{i+j-1}\prod_{1\le i\le n}\frac{\binom{2a+2b+4i-6}{2a+2i-3}}{\binom{2a+2b+4i-6}{2i-1}}$$

$$M_{2n-1}(a,b,1) = \binom{a+b-2}{a-1}C_1C_2\cdots C_{2n-2}\prod_{1\le i<j\le 2n-1}\frac{2(a+b-2)+i+j-1}{i+j-1}\prod_{1\le i\le n-1}\frac{\binom{2a+2b+4i-4}{2a+2i-2}}{\binom{2a+2b+4i-4}{2i}}.$$

We also give a formula for $M_n(a,b,c)$ for even c, which curiously differs significantly from other computed special cases.

Corollary 2.5. *For positive integers n, a, b and k, the constant term $M_n(a,b,2k)$ is given by the product*

$$M_n(a,b,2k) = \prod_{i=1}^{n}\frac{(a+b-2+(2i-3)k)!k!}{((i-2)k)!(ik)!}\binom{a+b-2+(2i-2)k}{a-1+(i-1)k}.$$

To prove the Morris identity, Baldoni-Vergne defined the generating function

$$\phi'_n(k,a,b,c) := k!(n-k)!e_k\prod_{i=1}^{n}(1-x_i)^{-b}x_i^{-a+1}\prod_{1\le i<j\le n}(x_j-x_i)^{-c},$$

where $e_k = [t^k]\prod_{i=1}^{n}(1+tx_i)$ is the kth elementary symmetric polynomial. They proved several recurrence relations that computed the constant term $\Phi'_n(k,a,b,c) := \mathrm{CT}_x\,\phi'_n(k,a,b,c)$, which implies the Morris identity when $k=0$.

Theorem 2.6 (Baldoni-Vergne [3])**.** *For positive integer n and nonnegative integer k, a, b, c, with $a+b\ge 2$, the constant term $\Phi'_n(k,a,b,c)$ is given by the formula*

$$\Phi'_n(k,a,b,c) \;=\; n!\cdot M_n(a,b,c)\prod_{j=1}^{k}\frac{a-1+(n-j)\frac{c}{2}}{a+b-2+(2n-j-1)\frac{c}{2}}.$$

Interestingly, the Baldoni-Vergne constant term does not generalize the refinement of $M_n(1,1,1)$ given by Theorem 1.5 which helped motivate our new refinement of $M_n(a,b,c)$ in Section 4.

3. A recursive proof of Theorem 1.1

In this section we give a new recursive proof of Theorem 1.1 by introducing a subdivision map for the right-hand side of (1.1). To give our proof, we first show that all subdivisions reduce to a similar form.

Lemma 3.1. *Every connected directed graph G on vertex set $\{0,1,\ldots,n+1\}$ with unique source 0 and unique sink $n+1$ can be reduced to subdivisions G' with the same vertex set, unique source and sink and for $i\in[n]$, $\mathrm{outdeg}_{G'}(i)=1$.*

Proof. We apply the following algorithm:

(1) Consider if graph G has a non-empty set S of vertices i such that $\mathrm{indeg}_G(i)>1$ and $\mathrm{outdeg}_G(i)>1$. Then we apply the reduction rule at any vertex in S.

(2) Consider if graph G has a non-empty set T of vertices i such that $\mathrm{indeg}_G(i)=1$ and $\mathrm{outdeg}_G(i)>1$. Then we apply the reduction rule at any vertex in T.

We note the net flow for a vertex in T is zero, so the flow along the incoming edge must be at least the flow along any of the outgoing edges. Obtaining G_1 and G_2 as in (2.1) and (2.2), applying the map on flows as shown in Figure 2, gives that $\mathcal{F}_{G_1}=\emptyset$, and can be disregarded. Hence, we see that the uniqueness of the sink and source are also preserved in G_2.

(3) We continually apply steps (1) and (2) until $S=T=\varnothing$, at which point we conclude $\mathrm{outdeg}_G(i)=1$ for $i\in[n]$.

Since the graph is finite, we see the algorithm must terminate. □

We now prove the following lemma, which establishes the base case for our induction.

Lemma 3.2 (Base Case)**.** *For a graph G on vertex set $\{0,1,\ldots,n+1\}$ with m edges, unique source 0, unique sink $n+1$, and where $\mathrm{outdeg}_G(i)=1$ for $i\in[n]$, we have that*

$$\mathrm{vol}\,\mathcal{F}_G(1,0,\ldots,0,-1)=K_G(0,d_1,\ldots,d_n,-\sum_{i=1}^n d_i)=1,$$

where $d_i=\mathrm{indeg}_G(i)-1$.

Proof. First we show $\mathrm{vol}\,\mathcal{F}_G(1,0,\ldots,0,-1)=1$. Since $\mathrm{outdeg}_G(i)=1$ for $i\in[n]$ then the source has outdegree $m-n$, and the flows along these $m-n$ edges determines a unique flow on G. To see this, note that the flows of the outgoing edges of vertices in the set $\{0,1,\ldots,i\}$ for $i\in[n]$ determine recursively the outgoing flow at vertex $i+1$. We see that $\mathcal{F}_G$ is equivalent to a $(m-n-1)$-dimensional simplex and has normalized volume 1.

Next we show that $K_G(0,d_1,\ldots,d_n,-\sum_{i=1}^n d_i)=1$. We recursively show that there is only one integer flow f with the desired net flow. Since the source has net flow zero, then $f(0,i)=0$ for $i\in[n]$. Then the flows of the outgoing edges of vertices in the set $\{0,1,\ldots,i\}$ recursively determine the outgoing flow from vertex $i+1$ since $\mathrm{outdeg}_G(i+1)=1$. Thus, only a single integer flow f is possible. □

We now define some notation. For a reducible graph G_0 on vertex set $\{0,1,\ldots,n,n+1\}$, let G_1 and G_2 be obtained by equations (2.1) and (2.2) for fixed $(i,j),(j,k)\in E(G_0)$. Let $d_i'=\mathrm{indeg}_{G_1}(i)-1$, and let $d_i''=\mathrm{indeg}_{G_2}(i)-1$. Also, let $\mathbf{d}=(0,d_2,\cdots,d_n,-\sum_{i=2}^n d_i)$ and likewise define $\mathbf{d_1}=(0,d_2',\cdots,d_n',-\sum_{i=2}^n d_i')$ and $\mathbf{d_2}=(0,d_2'',\cdots,d_n'',-\sum_{i=2}^n d_i'')$.

We prove that if G_1 and G_2 satisfy Theorem 1.1, so does G_0. By the subdivision lemma, we have that

$$\mathrm{vol}\,\mathcal{F}_{G_0}=\mathrm{vol}\,\mathcal{F}_{G_1}+\mathrm{vol}\,\mathcal{F}_{G_2}.$$

Hence, it suffices we show the following lemma.

Lemma 3.3 (Inductive Step)**.** *Let G_0,G_1,G_2 and $\mathbf{d},\mathbf{d_1},\mathbf{d_2}$ be as defined above. Then,*

$$K_{G_0}(\mathbf{d})=K_{G_1}(\mathbf{d_1})+K_{G_2}(\mathbf{d_2}).$$

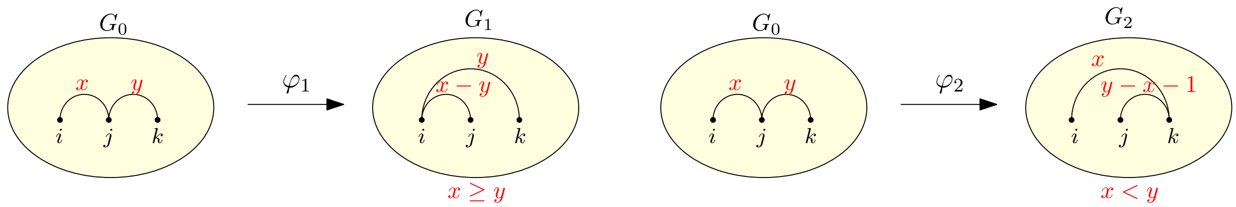

FIGURE 3. The maps φ_1 and φ_2 for integer flows of a subdivided graph.

Proof. Note that since $\mathbf{d_1} \neq \mathbf{d_2}$, we have that $\mathcal{F}^{\mathbb{Z}}_{G_1}(\mathbf{d_1})$ and $\mathcal{F}^{\mathbb{Z}}_{G_2}(\mathbf{d_2})$ are disjoint. We give a bijection

$$\varphi : \mathcal{F}^{\mathbb{Z}}_{G_0}(\mathbf{d}) \to \mathcal{F}^{\mathbb{Z}}_{G_1}(\mathbf{d_1}) \,\dot{\cup}\, \mathcal{F}^{\mathbb{Z}}_{G_2}(\mathbf{d_2}).$$

For an integer flow $f \in F^{\mathbb{Z}}_{G_0}(\mathbf{d})$, let $x = f(i,j)$, and let $y = f(j,k)$. Then we denote by $F^{\mathbb{Z}}_{G_0}(\mathbf{d}; y \leq x)$ the subset of $F^{\mathbb{Z}}_{G_0}(\mathbf{d})$ where $y \leq x$, and likewise let $F^{\mathbb{Z}}_{G_0}(\mathbf{d}; y > x)$ denote the the subset of $F^{\mathbb{Z}}_{G_0}(\mathbf{d})$ where $y > x$.

We let φ_1 be restriction of φ to $F^{\mathbb{Z}}_{G_0}(\mathbf{d}; y \leq x)$, and φ_2 the restriction of φ to $F^{\mathbb{Z}}_{G_0}(\mathbf{d}; y > x)$. We now construct φ_1 and φ_2 as bijections with disjoint codomains where the union is the codomain of φ. We define φ_1 and φ_2 as illustrated in Figure 3.

More formally, we define

$$\varphi_1 : F^{\mathbb{Z}}_{G_0}(\mathbf{d}; y \leq x) \to \mathcal{F}^{\mathbb{Z}}_{G_1}(\mathbf{d_1}),$$

where $f \mapsto f'$ given by

$$f'(e) = \begin{cases} x - y, & e = (i,j) \\ y, & e = (i,k) \\ f(e), & e \in E(G_1) \setminus \{(i,j),(i,k)\}. \end{cases}$$

Since the indegrees in G_0 and G_1 are the same, we see that the net flow vector is $\mathbf{d}_1 = \mathbf{d}$, so the map is well-defined. We now construct the inverse map φ_1^{-1} with $f \mapsto f'$ given by

$$f'(e) = \begin{cases} f(i,j) + f(i,k), & e = (i,j) \\ f(i,k), & e = (j,k) \\ f(e), & e \in E(G_0) \setminus \{(i,j),(j,k)\}. \end{cases}$$

The net flow vector is again unchanged, so the map is well-defined and therefore φ_1 is a bijection.

We now construct a second bijection

$$\varphi_2 : F^{\mathbb{Z}}_{G_0}(\mathbf{d}; y > x) \to \mathcal{F}^{\mathbb{Z}}_{G_2}(\mathbf{d_2}),$$

with $f \mapsto f'$ given by

$$f'(e) = \begin{cases} y - x - 1, & e = (j,k) \\ x, & e = (i,k) \\ f(e), & e \in E(G_2) \setminus \{(j,k),(i,k)\}. \end{cases}$$

The only change in indegrees is that $d''_j = d_j - 1$ and $d''_k = d_k + 1$. However, the outgoing flow at vertex j also decreases by 1, whereas the incoming flow at vertex k also decreases by 1, so the net flow vector is indeed $\mathbf{d}_2$. We similarly construct the inverse map φ_2^{-1} with $f \mapsto f'$ given by

$$f'(e) = \begin{cases} f(i,k), & e = (i,j) \\ f(j,k) + f(i,k) + 1, & e = (j,k) \\ f(e), & e \in E(G_0) \setminus \{(i,j),(j,k)\}. \end{cases}$$

The only indegrees that change are $d_j = d''_j + 1$ and $d_k = d''_k - 1$, but since the outgoing flow at vertex j increases by 1 and the incoming flow at vertex k decreases by 1, we see the graph is locally unchanged. Hence, the map is well-defined and φ_2 is a bijection as a well.

Since $\mathcal{F}^{\mathbb{Z}}_{G_1}(\mathbf{d_1})$ and $\mathcal{F}^{\mathbb{Z}}_{G_2}(\mathbf{d_2})$ are disjoint, we have that φ is a bijection, and the result follows. ◻

Recursive proof of Theorem 1.1. The proof follows from the base case given in Lemma 3.1 and Lemma 3.2, and the inductive step established in Lemma 3.3. ◻

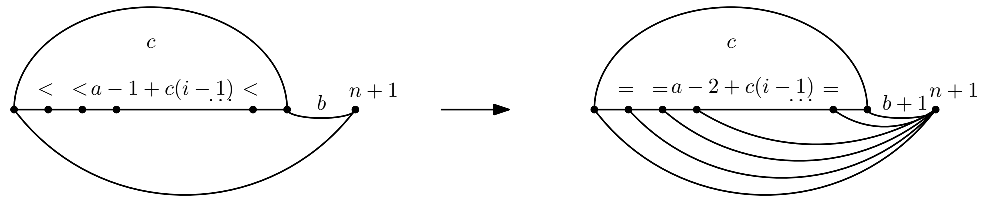

FIGURE 4. For each strict inequality where $a_i < \mathrm{indeg}(i) - 1$, the inequality can be weakened by decreasing a by 1 to obtain $a_i \leq \mathrm{indeg}(i) - 1$. We then add an additional edge $(i, n+1)$ to carry the necessary flow such that $a_i = \mathrm{indeg}(i) - 1$.

4. A NEW REFINEMENT OF $M_n(a,b,c)$

We define the following constant term that generalizes the case of Conjecture 2 to $M_n(a,b,c)$.

Definition 4.1. Define the following constant term

$$\Psi_n(k,a,b,c) := \mathrm{CT}_x[t^k] \prod_{i=1}^{n} (1-x_i)^{-b} x_i^{-a+1} (1 + t\frac{x_i}{1-x_i}) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c}.$$

4.1. **Volume and Kostant partition function interpretations for $\Psi_n(k,a,b,c)$.** We prove both parts of Theorem 1.6.

Proof of Theorem 1.6 (i). We first prove the Kostant partition interpretation of $\Psi_n(k,a,b,c)$. We specialize the generating function in equation (2.3) as in the proof of Theorem 1.4 in Section 2.2. Note that compared with $M_n(a,b,c)$, $\Psi_n(k,a,b,c)$ has an extra term $[t^k] \prod_{i=1}^{n}(1+t\frac{x_i}{1-x_i})$. This term selects k values of $\{1,\ldots,n\}$ and for each selected i multiplies the generating series by $\frac{x_i}{1-x_i}$.

By linearity of constant term extraction,

$$\mathrm{CT}_{x_i} \frac{x_i}{1-x_i} f(x_1,\ldots,x_n) = \sum_{j=1}^{\infty} \mathrm{CT}_{x_i}\, x_i^j f(x_1,\ldots,x_n) = \sum_{j=1}^{\infty} [x_i^{-j}] f(x_1,\ldots,x_n). \tag{4.1}$$

When the generating function for $M_n(a,b,c)$ is substituted for $f(x_1,\ldots,x_n)$ in the RHS of (4.1), this is equivalent to strictly decreasing the net flow at vertex i in the Kostant partition function interpretation of $M_n(a,b,c)$. Since we take the coefficient of t^k, there are exactly k vertices with net flow $a_i < a-1+c(i-1)$, and $n-k$ vertices with net flow $a_i = a-1+c(i-1)$. The result follows.

□

We now prove the volume interpretation. To do so, we first define a modification of $k_{n+2}^{a,b,c}$.

Definition 4.2. For a set $S \subseteq [n]$, let $k_{n+2}^{a,b,c}(S)$ be the graph obtained from $k_{n+2}^{a,b,c}$ by adding n edges $(0, n+1)$ and for each $i \in S$ we delete one of the a incoming edges $(0,i)$ and add an outgoing edge $(i, n+1)$.

For a set $S \subseteq [n]$, define also the set $T(S)$ as the set of vectors $\mathbf{a} = (0, a_1, a_2, \ldots, a_n, -\sum_{j=1}^{n} a_j)$ with $a_i < a-1+c(i-1)$ for $i \in S$ and $a_i = a-1+c(i-1)$ for $i \notin S$.

Proof of Theorem 1.6 (ii). First we show that

$$\sum_{\mathbf{a} \in T(S)} K_{k_{n+2}^{a,b,c}}(\mathbf{a}) = \mathrm{vol}\, \mathcal{F}_{k_{n+2}^{a,b,c}(S)}. \tag{4.2}$$

Consider the Kostant partition functions on the left-hand side. For each vertex $i \in S$, we remove an incoming edge $(0,i)$ (decreasing a by 1) to create a weak inequality instead of a strict equality. We then add an outgoing edge $(i, n+1)$ to carry the necessary flow to force the equality $a_i = \mathrm{indeg}(i) - 1$. This process is shown in Figure 4.

We note that if we were to add the edge $(0, n+1)$ n times, the volume would not change. This is due to Theorem 1.1, which gives the volume as a Kostant partition function where the source has zero net flow. Since an edge $(0, n+1)$ also would not affect the indegree of any internal vertex, it has no effect on the Kostant partition function or volume. By adding the edge $(0, n+1)$ n times, the graph becomes $k_{n+2}^{a,b,c}(S)$.

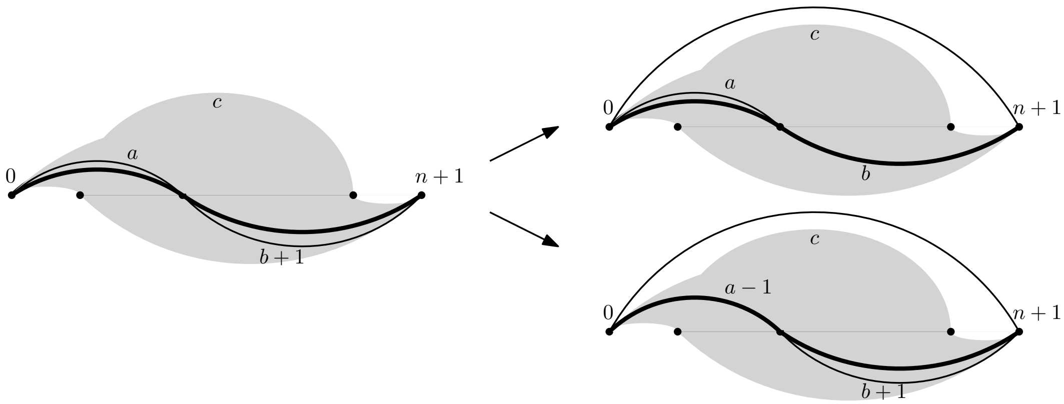

FIGURE 5. The graph $k_{n+2}^{a,b,c}$ is shown in gray on the left with some highlighted edges in black. The graphs on the right give the two subdivisions obtained from applying the subdivision lemma on a single internal vertex.

Hence, since each internal vertex has net flow $a_i = \text{indeg}(i) - 1$, we apply Theorem 1.1 again to obtain that this Kostant partition function is equal to $\text{vol}\,\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$, thus proving equation (4.2).

We can now sum both sides of equation (4.2) over $S \in \binom{[n]}{k}$. Since the Kostant partition function interpretation of $\Psi_n(k,a,b,c)$ counts all flows with exactly k strict inequalities $a_i < a-1+c(i-1)$, we see that the left-hand side is now $\Psi_n(k,a,b,c)$, and the result follows. $\square$

On a polytope level, this volume interpretation translates to the following result.

Lemma 4.3. *For $S \subseteq [n]$, the polytopes $\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$ are interior disjoint and satisfy*

$$\mathcal{F}_{k_{n+2}^{a,b+1,c}} \equiv \bigcup_{S\subseteq[n]} \mathcal{F}_{k_{n+2}^{a,b,c}(S)}.$$

Proof. We apply the subdivision lemma (2.1) and (2.2) at each internal vertex of $F_{k_{n+2}^{a,b+1,c}}$ exactly once. For each internal vertex i, the edge $(0,n+1)$ is added, and either an incoming edge $(0,i)$ or outgoing edge $(i,n+1)$ is deleted. This is shown in Figure 5.

That is, for each vertex i, one of two cases must hold:

(i) Edge $(0,i)$ appears $a-1$ times and edge $(i,n+1)$ appears $b+1$ times.
(ii) Edge $(0,i)$ appears a times and edge $(i,n+1)$ b times.

Each reduced graph is the polytope $\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$, where S is the set of vertices satisfying case (i). Since the graphs $\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$ are obtained different subdivisions, these polytopes are interior disjoint by Proposition 2.1, and the result follows. $\square$

As an application of these interpretations we now prove Corollary 1.7, which refines the product $M_n(a,b,c)$.

Proof of Corollary 1.7 via Kostant partition function. The sum on the right-hand side of (1.5) over k of $\Psi_n(k,a,b,c)$ is the sum of all Kostant partition functions $K_{k_{n+2}^{a,b,c}}(0,a_1,\ldots,a_n,-\sum_{j=1}^n a_j)$ such that for $i \in [n]$, $a_i \le a-1+c(i-1)$. This is equivalent to adding another edge between each vertex i and the sink with flows such that each net flow satisfies $a_i = a-1+c(i-1)$. Hence we see this sum is $M_n(a,b+1,c)$. $\square$

Proof of Corollary 1.7 via volumes. Using Lemma 4.3 and computing the volume on both sides gives

$$\text{vol}\,\mathcal{F}_{k_{n+2}^{a,b+1,c}} = \sum_{S\subseteq[n]} \text{vol}\,\mathcal{F}_{k_{n+2}^{a,b,c}(S)}.$$

The result follows by applying Theorem 1.4 to the left-hand side and Theorem 1.6 to the right-hand side. $\square$

4.2. **Recurrence Relations of $\Psi_n(k,a,b,c)$.** In this section we prove recurrence relations that $\Psi_n(k,a,b,c)$ satisfies and that are instrumental to our proof of Theorem 1.8. First we show two cases where $\Psi_n(k,a,b,c)$ is equivalent to the Morris identity.

Proposition 4.4. *Let n,a,b,c be positive integers. Then*

$$\Psi_n(0,a,b,c) = M_n(a,b,c) \tag{4.3}$$

$$\Psi_n(n,a,b,c) = M_n(a-1,b+1,c). \tag{4.4}$$

Proof. The first equation holds since n equalities implies the exact same constraints as those of $M_n(a,b,c)$. The second equation holds since 0 equalities implies the upper bound of the inequalities can be decreased by 1 (by decreasing a by 1) to make a weak inequality, and another edge from each vertex to the sink can be added with the necessary flow to force equality. This transformation gives a bijection with $M_n(a-1,b+1,c)$. □

We now give a bijective contraction on certain integer flows.

Proposition 4.5. *For a net flow vector $\mathbf{a} = (0,0,a_2,\dots,a_n,-\sum_{j=2}^n a_j)$, define $\tilde{\mathbf{a}} := (0,a_2,\dots,a_n,-\sum_{j=2}^n a_j)$. Then for positive integers a,b,n and nonnegative integer c,*

$$K_{k_{n+2}^{1,b,c}}(\mathbf{a}) = K_{k_{n+1}^{c+1,b,c}}(\tilde{\mathbf{a}}). \tag{4.5}$$

Proof. Define the map

$$\varphi : \mathcal{F}^{\mathbb{Z}}_{k_{n+2}^{1,b,c}}(\mathbf{a}) \to \mathcal{F}^{\mathbb{Z}}_{k_{n+1}^{c+1,b,c}}(\tilde{\mathbf{a}}),$$

where $f \mapsto f'$, with $f'(i,j)$ given by

$$f'(i,j) = \begin{cases} 0, & i = 0 \\ f(i+1,j+1), & 1 \le i < j \le n. \end{cases}$$

This map contracts edge $(0,1)$ to create vertex 0 and relabels each vertex $i \in [2,n+1]$ by $i-1$. We see the c edges $(1,i)$ for $i \in [2,n]$ become identical with the edges of the form $(0,i), i \in [2,n]$. Hence, the graph transforms to become $k_{n+1}^{c+1,b,c}$. We now show that φ is a bijection. It is sufficient we show φ has a well-defined inverse function for all $f \in \mathcal{F}_{k_{n+1}^{c+1,b,c}}(\tilde{\mathbf{a}})$. Define the inverse map

$$\varphi^{-1} : \mathcal{F}^{\mathbb{Z}}_{k_{n+1}^{c+1,b,c}}(\tilde{\mathbf{a}}) \to \mathcal{F}^{\mathbb{Z}}_{k_{n+2}^{1,b,c}}(\mathbf{a})$$

where $f \mapsto f'$, with $f'(i,j)$ given by

$$f'(i,j) = \begin{cases} 0, & 0 \le i \le 1 \\ f(i-1,j-1), & 2 \le i < j \le n+1. \end{cases}$$

Note that for $(i,j) \in E(\mathcal{F}^{\mathbb{Z}}_{k_{n+2}^{1,b,c}}(\mathbf{a}))$, $f(0,j) = f(1,j) = 0$ as the net flows at vertices 0 and 1 are both zero, so we see φ^{-1} is indeed our desired inverse function. Thus φ is a bijection, and the result follows. □

We further strengthen this contraction identity to hold bijectively for $\Psi_n(k,1,b,c)$.

Lemma 4.6 (Contraction Lemma)**.** *For positive integers b and n and nonnegative integers c and $k \le n$,*

$$\Psi_n(k,1,b,c) = \Psi_{n-1}(k,c+1,b,c).$$

Proof. Recall $\Psi_n(k,1,b,c)$ is the sum of Kostant partition functions of the form $K_{k_{n+2}^{1,b,c}}(\mathbf{a})$, with $\mathbf{a} = (0,0,a_2,\dots,-\sum_{j=1}^n a_j)$ where for $i \in [2,n]$, $a_i < c(i-1)$ holds k times and $a_i = c(i-1)$ holds $n-1-k$ times (since the first internal vertex trivially satisfies this equality). Let A be the set of all such $\mathbf{a}$ satisfying these conditions. That is,

$$\Psi_n(k,1,b,c) = \sum_{\mathbf{a}\in A} K_{k_{n+2}^{1,b,c}}(\mathbf{a}).$$

Similarly, let A' be the set of all $\mathbf{a}' = (0,a_2,\dots,-\sum_{j=1}^n a_j)$ where for $i \in [2,n]$, $a_i < c(i-1)$ holds k times and $a_i = c(i-1)$ holds $n-1-k$ times. Then

$$\Psi_{n-1}(k,c+1,b,c) = \sum_{\mathbf{a}'\in A'} K_{k_{n+2}^{c+1,b,c}}(\mathbf{a}').$$

Noting the nearly identical on the restrictions on $\mathbf{a} \in A$ and $\mathbf{a}' \in A'$, we see that the map $\varphi : A \to A', \mathbf{a} \mapsto \tilde{\mathbf{a}}$ is a bijection. Equating the Kostant partition functions using equation (4.5), the result follows. $\square$

As a result of this lemma, the following two corollaries are immediate.

Corollary 4.7.

$$M_n(1, b, c) = M_{n-1}(c + 1, b, c).$$

Proof. This is a result of Lemma 4.6 when $k = 0$. $\square$

Corollary 4.8. *For positive integers a and n, it holds that*

$$M_n(a, 1, 1) = \sum_{k=0}^{n} \Psi_{n-1}(k, a, 1, 1).$$

Proof. By Corollary 4.7 and Proposition 2.2, we see that $M_n(a, 1, 1) = M_{n-1}(a, 2, 1)$. Hence applying Lemma 1.7, the result follows. $\square$

Following the approach of Baldoni-Vergne in their proof of Theorem 2.6, we now give relations of $\Psi_n(k, a, b, c)$ that we later show uniquely determine this function.

Lemma 4.9. *For nonnegative integer c, positive integers a, b, n, and nonnegative integer $k \leq n$, the constant term $\Psi_n(k, a, b, c)$ satisfies the following identities:*

$$\Psi_n(n, a, b, c) = \Psi_n(0, a - 1, b + 1, c) \tag{4.6}$$

$$\Psi_n(n - 1, 1, b, c) = \Psi_{n-1}(0, c, b + 1, c) \tag{4.7}$$

$$\Psi_n(0, 1, b, 0) = 1 \tag{4.8}$$

$$k(b + (k - 1)c/2) \cdot \Psi_n(k, a, b, c) = (n - k + 1)(a - 1 + (n - k)c/2) \cdot \Psi_n(k - 1, a, b, c) \textit{ for } 1 \leq k \leq n. \tag{4.9}$$

Proof. The first relation follows immediately from the bijections in Proposition 4.4. The second relation follows by applying Lemma 4.6 to the left-hand side, which turns the equation into $\Psi_{n-1}(n-1, c+1, b, c) = \Psi_{n-1}(0, c, b + 1, c)$, which follows from the first relation. The third relation is immediate since the left-hand side implies $a_i = a - 1 + c(n - 1) = 0$ for all $i \in [n]$, giving only one possible flow where each edge has flow zero. Lastly we prove the fourth relation. This is the sole relation we prove algebraically.

Let $U := \prod_{i=1}^{n}(1 - x_i)^{-b} x_i^{-a} \prod_{1 \leq i < j \leq n}(x_j - x_i)^{-c}$, and let $P_k := k!(n - k)![t^k] \prod_{i=1}^{k}(1 + t\frac{x_i}{1-x_i})$, where $[t^k] \prod_{i=1}^{k}(1 + t\frac{x_i}{1-x_i}) = e_k(\frac{x_i}{1-x_i})$ is the kth elementary symmetric function in $\frac{x_i}{1-x_i}$. We have that

$$\begin{aligned}\frac{\partial}{\partial x_1}(1 - x_1)\frac{x_1}{1 - x_1} \cdots \frac{x_k}{1 - x_k}U \\ = \left(b\frac{x_1}{1 - x_1} \cdots \frac{x_k}{1 - x_k} + (1 - a)\frac{x_2}{1 - x_2} \cdots \frac{x_k}{1 - x_k} + c(1 - x_1)\frac{x_1}{1 - x_1} \cdots \frac{x_k}{1 - x_k} \sum_{j=2}^{n} \frac{1}{x_j - x_1}\right) U.\end{aligned} \tag{4.10}$$

If c is odd, then U is antisymmetric. Anti-symmetrizing over $\mathfrak{S}_n$ gives:

$$\sum_{w \in \mathfrak{S}_n} (-1)^w w \cdot (\frac{\partial}{\partial x_1}(1 - x_1)\frac{x_1}{1 - x_1}\frac{x_2}{1 - x_2} \cdots \frac{x_k}{1 - x_k}U) \tag{4.11}$$

$$= bP_kU + (1 - a)P_{k-1}U + c \sum_{w \in \mathfrak{S}_n} w \cdot \left((1 - x_1)\frac{x_1}{1 - x_1}\frac{x_2}{1 - x_2} \cdots \frac{x_k}{1 - x_k} \sum_{j=2}^{n} \frac{1}{x_j - x_1}\right) U. \tag{4.12}$$

To evaluate the sum, we seek pairings of summands that reduce easily. Consider when w is the identity permutation. Then for each summand in $\sum_{j=2}^{n} \frac{1}{x_1 - x_j}$, for $2 \leq j \leq k$, we see that

$$\frac{(1 - x_1)x_1x_j}{(x_j - x_1)(1 - x_1)(1 - x_j)} + \frac{(1 - x_j)x_1x_j}{(x_1 - x_j)(1 - x_1)(1 - x_j)} = \frac{x_1x_j}{(1 - x_1)(1 - x_j)}. \tag{4.13}$$

On the other hand for $j > k$,

$$\frac{(1 - x_1)x_1}{(x_j - x_1)(1 - x_1)} + \frac{(1 - x_j)x_j}{(x_1 - x_j)(1 - x_j)} = -1. \tag{4.14}$$

Thus for each $w \in \mathfrak{S}_n$ and $j \in [2, n]$, we pair the summand

$$w \cdot \left((1-x_1)\frac{x_1}{1-x_1}\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}\cdot\frac{1}{x_j-x_1} \right)$$

with the summand obtained by taking w and transposing $w(1)$ and $w(j)$. Hence we duplicate the sum and simplify with equations (4.13) and (4.14):

$$\begin{aligned}
&2\sum_{w\in\mathfrak{S}_n} w\cdot\left((1-x_1)\frac{x_1}{1-x_1}\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}\sum_{j=2}^{n}\frac{1}{x_j-x_1}\right)U\\
&=\sum_{w\in\mathfrak{S}_n} w\cdot\left((k-1)\frac{x_1}{1-x_1}\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}-(n-k)\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}\right)U\\
&=(k-1)P_kU-(n-k)P_{k-1}U.
\end{aligned}$$

Thus, the expression in equation (4.12) simplifies to:

$$\sum_{w\in\mathfrak{S}_n}(-1)^w w\cdot\left(\frac{\partial}{\partial x_1}(1-x_1)\frac{x_1}{1-x_1}\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}U\right)=bP_kU+(1-a)P_{k-1}U-\frac{c}{2}(n-k)P_{k-1}U+\frac{c}{2}(k-1)P_kU. \tag{4.15}$$

Since the residue of a partial derivative of an analytic function is always zero, taking the residues of the terms allows setting the equation to 0:

$$0 \;=\; b\mathrm{Res}_x P_kU+(1-a)\mathrm{Res}_x P_{k-1}U-\frac{c}{2}(n-k)\mathrm{Res}_x P_{k-1}U+\frac{c}{2}(k-1)\mathrm{Res}_x P_kU.$$

By definition of $\Psi(\cdot)$, we have that $\mathrm{Res}_x P_kU = k!(n-k)!\Psi_n(k,a,b,c)$, which gives:

$$(b+(k-1)\frac{c}{2})k!(n-k)!\Psi_n(k,a,b,c)=(a-1+(n-k)\frac{c}{2})(k-1)!(n-k+1)!\Psi_n(k-1,a,b,c).$$

Simplifying gives relation 5 for odd c. When c is even, U is symmetric, so symmetrizing over $\mathfrak{S}_n$ gives:

$$\begin{aligned}
&\sum_{w\in\mathfrak{S}_n} w\cdot\left(\frac{\partial}{\partial x_1}(1-x_1)\frac{x_1}{1-x_1}\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}U\right)\\
&=bP_kU+(1-a)P_{k-1}U+c\sum_{w\in\mathfrak{S}_n} w\cdot\left((1-x_1)\frac{x_1}{1-x_1}\frac{x_2}{1-x_2}\cdots\frac{x_k}{1-x_k}\sum_{j=2}^{n}\frac{1}{x_j-x_1}\right)U,
\end{aligned}$$

which is essentially identical to when c is odd, and the proof follows verbatim. □

4.3. Closed Formula for $\Psi_n(k,a,b,c)$.

Our proof for the closed formula of $\Psi_n(k,a,b,c)$ follows the recurrence approach used by Baldoni-Vergne [3, p. 10] (see also [14, Prop. 3.11]) using the recurrences proven in the previous section.

Lemma 4.10. *The relations* (4.6)-(4.9) *uniquely determine the function* $\Psi_n(k,a,b,c)$.

Proof. Case 1: Consider if $c = 0, n \geq 1$, and $a \geq 1$. To compute $\Psi_n(k,a,b,0)$, we repeatedly apply equation (4.9) to increment k until $k = n$, at which point we apply equation (4.6). Thus $\Psi_n(k,a,b,c)$ reduces to calculating $\Psi_n(0,a-1,b+1,c)$:

$$\Psi_n(k,a,b,0)\xrightarrow{(4.9)}\Psi_n(k+1,a,b,0)\overset{(4.9)^*}{\dashrightarrow}\Psi_n(n,a,b,0)\xrightarrow{(4.6)}\Psi_n(0,a-1,b+1,0).$$

By iterating this recursion, we see this is equivalent to calculating $\Psi_n(0,1,a+b-1,0)$. By equation (4.8), this is equal to 1.

Case 2: Consider if $c \geq 1, n = 1$, and $a \geq 1$. Since $\prod_{1\leq i<j\leq n}(x_j-x_i)^{-c}$ is the empty product, this is equivalent to when $c = 0$, which implies that

$$\Psi_1(k,c,a+b+c(n-2),c)=\Psi_1(k,c,a+b+c(n-2),0).$$

This reduces to Case 1.

Case 3: Consider if $c \geq 1, n \geq 2$, and $a \geq 1$. Similar to in Case 2, to compute $\Psi_n(k,a,b,c)$ we repeatedly apply equation (4.9) to increment k until $k = n$, at which point we apply equation (4.6). Thus $\Psi_n(k,a,b,c)$ reduces to calculating $\Psi_n(0, a-1, b+1, c)$.

$$\Psi_n(k,a,b,c) \xrightarrow{(4.9)} \Psi_n(k+1,a,b,c) \overset{(4.9)^*}{\dashrightarrow} \Psi_n(n,a,b,c) \xrightarrow{(4.6)} \Psi_n(0,a-1,b+1,c).$$

We iterate this recursion until $a = 1$, at which point we reduce the calculation to finding $\Psi_n(0,1,a+b-1,c)$. Now we again increment k by (4.9) until $k = n-1$. Applying equation (4.7) reduces the calculation to finding $\Psi_{n-1}(0,c,a+b,c)$.

$$\Psi_n(k,1,a+b-1,c) \xrightarrow{(4.9)} \Psi_n(k+1,1,a+b-1,c) \overset{(4.9)^*}{\dashrightarrow} \Psi_n(n-1,1,a+b-1,c) \xrightarrow{(4.7)} \Psi_{n-1}(0,c,a+b,c).$$

We now repeatedly apply the above two cycles until we reduce n to 1, in which case we reduce the computation to $\Psi_1(0,c,a+b+c(n-2),c)$. Since $n = 1$, this becomes Case 2.

Since all cases eventually reduce to case 1, the result follows. □

Using the fact that the relations (4.6)-(4.9) uniquely define $\Psi_n(k,a,b,c)$, we now prove our explicit product formula.

Proof of Theorem 1.8. By Lemma 4.10, it is sufficient to show a formula of $\Psi_n(k,a,b,c)$ that satisfies the relations (4.6)-(4.9). To show equation (4.6), recall that:

$$M_n(a-1,b+1,c) = \prod_{j=0}^{n-1} \frac{\Gamma(a+b-1+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2}+1)}{\Gamma(a-1+j\frac{c}{2})\Gamma(b+1+j\frac{c}{2})\Gamma((j+1)\frac{c}{2}+1)}.$$

Recall that $\Gamma(x+1) = x\Gamma(x)$. Hence, $\Gamma(a-1+j\frac{c}{2}) = \frac{\Gamma(a+j\frac{c}{2})}{a-1+j\frac{c}{2}}$, and $\Gamma(b+1+j\frac{c}{2}) = (b+j\frac{c}{2})\Gamma(b+j\frac{c}{2})$. Substituting gives

$$M_n(a-1,b+1,c) = M_n(a,b,c)\prod_{j=0}^{n-1}\frac{a-1+j\frac{c}{2}}{b+j\frac{c}{2}} = M_n(a,b,c)\prod_{j=1}^{n}\frac{a-1+(n-j)\frac{c}{2}}{b+(j-1)\frac{c}{2}}.$$

To show equation (4.7) from (4.6), it is sufficient to show the product formula satisfies Lemma 4.6.

First we show $M_n(1,b,c) = M_{n-1}(c+1,b,c)$. To do so, consider the ratio

$$\frac{M_n(1,b,c)}{M_{n-1}(c+1,b,c)} = \frac{1}{n}\cdot\frac{\Gamma(b+(n-1)\frac{c}{2})\Gamma(\frac{c}{2})}{\Gamma(b+(n-1)\frac{c}{2})\Gamma(\frac{c}{2}n)}\cdot\frac{\Gamma(\frac{c}{2}n+1)}{\Gamma(\frac{c}{2}+1)\Gamma(1)}$$

Since $\Gamma(x+1) = x\Gamma(x)$, the above ratio simplifies to 1, and the result follows.

Using the above equality, it is sufficient to show that $\binom{n}{k}\prod_{j=1}^{k}(n-j) = \binom{n-1}{k}\prod_{j=1}^{k}(n+1-j)$. Both sides of the equation simplify to $\prod_{j=1}^{k}(n+1-j)(n-j)/j$, thus proving relation the product formula satisfies Lemma 4.6 and hence equation (4.7).

To show equation (4.8), recall that since $k = 0$, we have that $\binom{n}{k}\prod_{j=1}^{k}\frac{a-1+(n-j)\frac{c}{2}}{b+(j-1)\frac{c}{2}} = 1$. Then

$$\Psi_n(0,1,b,0) = M_n(1,b,0) = 1.$$

To show equation (4.9), note that

$$\frac{\Psi_n(k,a,b,c)}{\Psi_n(k-1,a,b,c)} = \frac{\binom{n}{k}}{\binom{n}{k-1}}\cdot\frac{a-1+(n-k)\frac{c}{2}}{b+(k-1)\frac{c}{2}} = \frac{n-k+1}{k}\cdot\frac{a-1+(n-k)\frac{c}{2}}{b+(k-1)\frac{c}{2}}.$$

Rearranging gives the desired recurrence relation, and the result follows. □

We also compute the following special cases of $\Psi_n(k,a,b,c)$, which generalize the special cases of $M_n(a,b,c)$ computed in Section 2.4.

Corollary 4.11. *The constant term $\Psi_n(k,a,b,c)$ satisfies the following:*

$$\Psi_n(k,a,1,1) = \frac{1}{n+2(a-1)}\binom{n}{k}\binom{n+2(a-1)}{k+1}M_n(a,1,1) \tag{4.16}$$

$$\Psi_n(k,1,b,1) = \binom{n-1}{k}\binom{n}{k}\binom{k+2b-1}{k}^{-1}M_n(1,b,1) \tag{4.17}$$

$$\Psi_n(k,1,1,c) = N(n,k+1)\prod_{j=1}^{k}\frac{c(j+1)}{c(j-1)+2}M_n(1,1,c). \tag{4.18}$$

Proof. We manipulate the product formula given in Theorem (1.8) to obtain the desired relations. Note that

$$\prod_{j=1}^{k}\frac{2(a-1)+(n-j)}{j+1} = \frac{1}{n+2(a-1)}\binom{n+2(a-1)}{k+1}$$

$$\prod_{j=1}^{k}\frac{n-j}{j+2b-1} = \frac{\binom{n-1}{k}}{\binom{k+2b-1}{k}}$$

$$\prod_{j=1}^{k}\frac{(n-j)c}{2+(j-1)c} = \frac{1}{n}\binom{n}{k+1}\prod_{j=1}^{k}\frac{c(j-1)+2c}{c(j-1)+2}.$$

The results follows from substitution. □

Remark 4.12. Note that unlike $M_n(a,b,c)$ (see Proposition 2.2), in most cases $\Psi_n(k,a,b,c) \neq \Psi_n(k,b,a,c)$. Instead, we have the following symmetry.

Proposition 4.13. *For positive integers a,b,n and nonnegative integers c and $k \leq n$,*

$$\Psi_n(k,a,b,c) = \Psi_n(n-k,b+1,a-1,c).$$

Proof. Reversing flows in the flow polytope interpretation of $\Psi_n(k,a,b,c)$ gives the polytope interpretation of $\Psi_n(n-k,b+1,a-1,c)$. Since this map is an involution, it must be volume-preserving, so we see the two collections of polytopes have equal volume, and the result follows. □

As a corollary, we also have the following special case when $a=b=c=1$.

Corollary 4.14.

$$\Psi_n(k,1,1,1) = N(n,k+1)C_{n-1}\cdots C_1.$$

Proof. This follows from (4.16) when $a=1$. □

Corollary 4.15. *Theorem 1.6 and Theorem 1.8 imply Theorem 1.5.*

Proof. Theorem 1.8 implies Corollary 4.14. Applying Corollary 4.8 and the Kostant partition function interpretation in Theorem 1.6 thus gives Theorem 1.5. □

Remark 4.16. Corollary 4.14 and Lemma 4.3 in the case that $a=b=c=1$ give a coarser version of the refinement provided by Mészáros [11, Thm. 13].

5. The Baldoni-Vergne refinement of $M_n(a,b,c)$

In this section, we modify the Baldoni-Vergne refinement in [3] of the Morris identity to give a Kostant partition function interpretation, and we combinatorialize the proof of some of the recurrence relations used to define $\Phi'_n(k,a,b,c)$. To more naturally interpret this constant term with Kostant partition functions, we scale $\Phi'_n(k,a,b,c)$.

Definition 5.1. We define the following modification of the Baldoni-Vergne constant term:

$$\Phi_n(k,a,b,c) := \mathrm{CT}_x[t^k]\prod_{i=1}^{n}(1-x_i)^{-b}x_i^{-a+1}(1+tx_i)\prod_{1\leq i<j\leq n}(x_j-x_i)^{-c}. \tag{5.1}$$

Equivalently, we have $\Phi_n(k,a,b,c) := \frac{\Phi'_n(k,a,b,c)}{k!(n-k)!}$. We now prove Theorem 5.2.

Theorem 5.2. *$\Phi_n(k,a,b,c)$ is the sum of Kostant partition functions $K_{k_{n+2}^{a,b,c}}(0,a_1,\ldots,a_n,-\sum_{j=1}^n a_j)$ such that for $i\in[n]$, $a-2+c(i-1)\le a_i\le a-1+c(i-1)$, with $a_i=a-1+c(i-1)$ for exactly $n-k$ values of i. $\Phi'_n(k,a,b,c)$ can then be interpreted identically to $\Phi_n(k,a,b,c)$, but with the internal vertices of the graph distinguished.*

Proof. Recall that specializing the generating function of equation (2.3) gives the Morris constant term

$$M_n(a,b,c)=\mathrm{CT}_x\prod_{i=1}^n(1-x_i)^{-b}x_i^{-a+1}\prod_{1\le i<j\le n}(x_j-x_i)^{-c}.$$

We see that $\Phi_n(k,a,b,c)$ has an additional term $[t^k]\prod_{i=1}^n(1+tx_i)$, which replaces CT_{x_i} with $[x_i^{-1}]$ for exactly k values of i, so this is equivalent to decreasing the net flow at these vertices by 1.

Recall from equation (1.3) that $M_n(a,b,c)=K_{k_{n+2}^{a,b,c}}(0,a_1,a_2,\ldots,a_n,-\sum_{j=1}^n a_j)$, with $a_i=a-1+c(i-1)$ for all $i\in[n]$. Decreasing the net flow by one at exactly k vertices gives the desired Kostant partition function interpretation.

There are $k!(n-k)!$ ways of distinguishing the vertices based on their net flow, we also obtain the combinatorial interpretation for $\Phi'_n(k,a,b,c)$, and the result follows. □

Remark 5.3. We note that $\Phi_n(k,a,b,c)$ does not seem to have a refinement similar to Corollary 1.7. Summing $\Phi(\cdot)$ over k removes all restrictions on t terms from the expression, giving:

$$\sum_{k=0}^n\Phi_n(k,a,b,c)\ :=\ \mathrm{CT}_x\prod_{i=1}^n(1-x_i)^{-b}x_i^{-a+1}(1+x_i)\prod_{1\le i<j\le n}(x_j-x_i)^{-c},$$

for which a simplified expression is not immediate.

We now give the recurrence relations used by Baldoni-Vergne [3] to prove Theorem 2.6. These relations served as the inspiration for the relations for $\Psi_n(k,a,b,c)$ in Section 4.

Proposition 5.4 (Baldoni-Vergne)**.** *The constant term $\Phi'_n(k,a,b,c)$ is uniquely determined by the following relations:*

(1) $\Phi'_n(n,a,b,c)=\Phi'_n(0,a-1,b,c)$
(2) $\Phi'_n(n-1,1,b,c)=\Phi'_{n-1}(0,c,b,c)$
(3) $\Phi'_n(0,1,b,0)=r!$
(4) $\Phi'_1(k,0,b,c)=0$
(5) $(a+b-2+\frac{c}{2}(2n-k-1))\Phi'_n(k,a,b,c)=(a-1+\frac{c}{2}(n-k))\Phi'_n(k-1,a,b,c)$.

Remark 5.5. One can give combinatorial proofs for all but the last relation in a nearly identical manner to our combinatorial proofs for $\Psi_n(k,a,b,c)$ in Lemma 4.9.

6. Final remarks and Future Work

In this paper we investigated a refinement of the Morris identity with several combinatorial interpretations, including a certain sum of Kostant partition functions and the volume of a collection of polytopes. We demonstrated how these collections of polytopes subdivide the graph $k_{n+2}^{a,b,c}$, and proved a product formula for our refinement. We now give some possible avenues for future exploration.

6.1. **The recurring appearance of Aomoto's integral.** The Morris constant term identity strongly resembles the Selberg integral, and the two identities are known to be equivalent. Interestingly, the product formula for the Baldoni-Vergne refinement of the Morris identity greatly resembles Aomoto's integral. However, the relationship between these two seemingly related identities is as of yet unclear. Intriguingly, Zeilberger also cites Aomoto's integral in his proof of Conjecture 2 of Chan-Robbins-Yuen, and while we did not see an immediate application of Aomoto's integral in our proof of the product formula of $\Psi_n(k,a,b,c)$, this seems to suggest these refinements of the Morris identity are in some way related to Aomoto's generalization of the Selberg integral.[1]

[1]Interestingly, note that recently the Selberg integral was proven bijectively [8].

6.2. **Towards a combinatorial proof of the Morris identity.** This paper provides multiple combinatorial proofs of recurrence relations for $\Psi_n(k,a,b,c)$ that could contribute to a combinatorial proof of the Morris constant term identity, and therefore, the volume formula for the Chan-Robbins-Yuen polytope. With the approach of this paper, the only remaining step is to give a combinatorial proof for equation (4.9). A combinatorialization of our algebraic proof of (4.9), or a new combinatorial proof altogether, would certainly be interesting. We also note that equation (4.9) can be written as

$$(kb+\binom{k}{2}c)\cdot\Psi_n(k,a,b,c)=((n-k+1)(a-1)+\binom{n-k+1}{2}c)\cdot\Psi_n(k-1,a,b,c).$$

where the extra factors on the left-hand and right-hand sides appear to be selecting certain edges of the graph $k_{n+2}^{a,b,c}$. Applying the identity in Proposition 4.13 to the right-hand side, the expression becomes

$$(kb+\binom{k}{2}c)\cdot\Psi_n(k,a,b,c)=((n-k+1)(a-1)+\binom{n-k+1}{2}c)\cdot\Psi_n(n-k+1,b+1,a-1,c).$$

where both sides have very similar structures. Given that a combinatorial proof of the Morris identity has been elusive and would serve immediately as a combinatorial proof for the volume formula of the Chan-Robbins-Yuen polytope, this is an intriguing avenue for future work.

6.3. **Volume of polytopes with different net flow vectors.** In Section 3, we presented a new recursive proof of Theorem 1.1. Generalizing Theorem 1.1 is the following theorem of Baldoni-Vergne-Lidskii.

Theorem 6.1 (Baldoni-Vergne-Lidskii [2])**.** *Let G be a connected digraph on vertex set $\{0,1,\dots,n,n+1\}$ with m edges directed $i\to j$ if $i<j$ and such that for $i\in\{0,1,\dots n\}$, there is at least one outgoing edge at vertex i. Then for a fixed net flow vector $\mathbf{a}=(a_0,a_1,\dots,a_n,-\sum_{j=0}^n a_j)$, $a_i\in\mathbb{Z}_{\geq 0}$, it holds that*

$$\operatorname{vol}\mathcal{F}_G(\mathbf{a})=\sum_{\mathbf{j}}\binom{m-n-1}{j_0,\dots,j_n}a_0^{j_0}\cdots a_n^{j_n}\cdot K_G(j_0-\operatorname{outdeg}_G(0),\dots,j_n-\operatorname{outdeg}_G(n),0).$$

In our proof in Section 3, the map φ on Kostant partition functions that we introduced is not specific to flow polytopes with net flow vector $(1,0,\dots,0,-1)$. This means that the inductive step will not change significantly for a different net flow vector, and as such, it is worth investigating whether there is a simple recursive proof for Theorem 6.1 considering new base cases with net flow vector $\mathbf{a}$. Such a proof would provide a better understanding of how volumes of flow polytope and Kostant partition functions are refined by the subdivision lemma.

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Appendix

In this section, we give some computational proofs for Section 2. In multiple of the proofs below, we use the Legendre duplication formula:

$$\Gamma(x+\frac{1}{2})\Gamma(x) = 2^{1-2x}\sqrt{\pi}\Gamma(2x). \tag{6.1}$$

We also use the following expression deducible from the Legendre duplication formula. For positive integers x and k,

$$\Gamma(x+k+1/2)\Gamma(x) = 2^{1-2x}\sqrt{\pi}\cdot\Gamma(2x)\prod_{j=0}^{k-1}(x+j+\frac{1}{2}). \tag{6.2}$$

Proof of Corollary 2.4. First, consider the ratio $M_n(a,b,1)/M_{n-1}(a,b,1)$. By (6.1),

$$\begin{aligned}\frac{M_n(a,b,1)}{M_{n-1}(a,b,1)} &= \frac{1}{n}\cdot\frac{\Gamma(a+b+n-\frac{5}{2})\Gamma(a+b+n-2)}{\Gamma(a+b-2+\frac{1}{2}n)}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(a+\frac{n-1}{2})\Gamma(b+\frac{n-1}{2})\Gamma(\frac{n}{2})}\\ &= \frac{1}{n}\cdot\frac{2^{6-2(a+b+n)}\Gamma(2(a+b+n)-5)\pi}{\Gamma(a+b-2+\frac{1}{2}n)\Gamma(a+\frac{n-1}{2})\Gamma(b+\frac{n-1}{2})\Gamma(\frac{n}{2})}.\end{aligned}$$

Substitution with (6.2) then gives

$$\begin{aligned}\frac{M_n(a,b,1)}{M_{n-1}(a,b,1)} &= \frac{(2(a+b+n)-6)!}{n!(2a+n-2)!\prod_{j=0}^{b-3}(2a+n+2j)\prod_{j=0}^{b-2}(n+2j+1)}\\ &= \frac{(2(a+b+n)-6)!}{n!!(2a+n-3)!!(2b+n-3)!!(2a+2b+n-6)!!}.\end{aligned}$$

To cancel the double factorials, we instead consider the ratio $M_{n+1}(1,1,1)/M_{n-1}(1,1,1)$:

$$\begin{aligned}\frac{M_{n+1}(a,b,1)}{M_{n-1}(a,b,1)} &= \frac{(2(a+b+n)-4)!(2(a+b+n)-6)!}{(n+1)!(2a+n-2)!(2b+n-2)!(2a+2b+n-5)!}\\ &= \frac{\binom{2a+2b+2n-4}{2a+n-2}}{\binom{2a+2b+2n-4}{n}}C_{n-1}C_n\prod_{i=1}^{n-1}\frac{2(a+b-2)+n+i-1}{n+i-1}\prod_{i=1}^{n}\frac{2(a+b-2)+n+i}{n+i}.\end{aligned}\tag{6.3}$$

We establish the base cases $M_0(a,b,1)=1$, $M_1(a,b,1)=\binom{a+b-2}{a-1}$, so by telescoping, we see that:

$$\begin{aligned}M_{2n}(a,b,1) &= \prod_{i=1}^{n}\frac{M_{2i}(a,b,1)}{M_{2i-2}(a,b,1)}\\ M_{2n-1}(a,b,1) &= \binom{a+b-2}{a-1}\prod_{i=1}^{n-1}\frac{M_{2i+1}(a,b,1)}{M_{2i-1}(a,b,1)}.\end{aligned}$$

Plugging in with (6.3) gives the desired result. □

The remaining proofs in this section follow a similar scheme, using $M_0(a,b,c)=1$.

Corollary 6.2. *For c even, we have that:*

$$M_n(1,1,c) = \prod_{i=1}^{n} \binom{(2i-3)\frac{c}{2}}{(i-1)\frac{c}{2}} \binom{(2i-2)\frac{c}{2}}{(i-1)\frac{c}{2}} \binom{i\frac{c}{2}}{(i-1)\frac{c}{2}}^{-1}.$$

Proof. Again, consider $M_n(1,1,c)/M_{n-1}(1,1,c)$. We see that

$$\begin{aligned}\frac{M_n(1,1,c)}{M_{n-1}(1,1,c)} &= \frac{1}{n} \cdot \frac{((2n-3)\frac{c}{2})!((2n-2)\frac{c}{2})!}{((n-2)\frac{c}{2})!} \cdot \frac{(\frac{c}{2}-1)!}{((n-1)\frac{c}{2})!^2(\frac{c}{2}n-1)!} \\ &= \frac{((2n-3)\frac{c}{2})!((2n-2)\frac{c}{2})!}{((n-2)\frac{c}{2})!} \cdot \frac{(\frac{c}{2})!}{((n-1)\frac{c}{2})!^2(\frac{c}{2}n)!}.\end{aligned}$$

With some rearrangement, we get the equation

$$\frac{M_n(1,1,c)}{M_{n-1}(1,1,c)} = \binom{n\frac{c}{2}}{(n-1)\frac{c}{2}}^{-1} \binom{(2n-3)\frac{c}{2}}{(n-1)\frac{c}{2}} \binom{(2n-2)\frac{c}{2}}{(n-1)\frac{c}{2}}.$$

Since $M_n(1,1,c) = \prod_{i=1}^{n} M_i(1,1,c)/M_{i-1}(1,1,c)$, the result follows. □

Corollary 6.3. *For c odd, we have that:*

$$M_n(1,1,c) = \prod_{i=1}^{n} \frac{(1+(2i-3)c)!((i-1)c)!c!!}{((i-2)c)!!((i-1)c)!!^2(ic)!!((2i-3)\frac{c}{2}+\frac{1}{2})!}.$$

Proof. Let $k = \lfloor \frac{c}{2} \rfloor$ or $\frac{c}{2} = k + \frac{1}{2}$. We then simplify the ratio $M_n(1,1,c)/M_{n-1}(1,1,c)$ using (6.2).

$$\begin{aligned}\frac{M_n(1,1,c)}{M_{n-1}(1,1,c)} &= \frac{1}{n} \cdot \frac{\Gamma(1+(2n-3)\frac{c}{2})\Gamma(1+(2n-2)\frac{c}{2})}{\Gamma(1+(n-2)\frac{c}{2})} \cdot \frac{\Gamma(\frac{c}{2})}{\Gamma(1+(n-1)\frac{c}{2})^2\Gamma(\frac{c}{2}n)} \\ &= \frac{1}{n} \cdot \frac{2^{c(3-2n)-1}\Gamma(2+(2n-3)c)\prod_{j=0}^{k-1}(1+(2n-3)\frac{c}{2}+j+\frac{1}{2})}{2^{c(2-n)-1}\Gamma(2+(n-2)c)\prod_{j=0}^{k-1}(1+(n-2)\frac{c}{2}+j+\frac{1}{2})} \cdot \frac{\Gamma(\frac{c}{2})}{\Gamma(1+(n-1)\frac{c}{2})\Gamma(\frac{c}{2}n)} \\ &= 2^{1-c} \cdot \frac{(1+(2n-3)c)!c!!}{(1+(n-2)c)!(1+(n-1)c)!} \prod_{j=0}^{k-1} \frac{1+(2n-3)\frac{c}{2}+j+\frac{1}{2}}{(1+(n-2)\frac{c}{2}+j+\frac{1}{2})(1+(n-1)\frac{c}{2}+j+\frac{1}{2}))} \\ &= 2^{1-c+k} \cdot \frac{((2n-3)c)!!c!!((2n-2)c)!!}{((n-2)c)!!((n-1)c)!!^2(nc)!!} = \frac{(1+(2n-3)c)!((n-1)c)!c!!}{((n-2)c)!!((n-1)c)!!^2(nc)!!((2n-3)\frac{c}{2}+\frac{1}{2})!}.\end{aligned}$$

We have $M_n(1,1,c) = \prod_{i=1}^{n} M_i(1,1,c)/M_{i-1}(1,1,c)$, and the result follows. □

Proof of Corollary 2.5. We again compute the ratio $M_n(a,b,2k)/M_{n-1}(a,b,2k)$:

$$\begin{aligned}\frac{M_n(a,b,2k)}{M_{n-1}(a,b,2k)} &= \frac{1}{n} \cdot \frac{\Gamma(a+b-1+(2n-3)k)\Gamma(a+b-1+(2n-2)k)}{\Gamma(1+(n-2)k)} \cdot \frac{\Gamma(k)}{\Gamma(a+(n-1)k)\Gamma(b+(n-1)k)\Gamma(kn)} \\ &= \frac{(a+b-2+(2n-3)k)!(a+b-2+(2n-2)k)!}{((n-2)k)!} \cdot \frac{k!}{((a-1)+(n-1)k)!((b-1)+(n-1)k)!(kn)!} \\ &= \frac{(a+b-2+(2n-3)k)!k!}{((n-2)k)!(nk)!} \cdot \binom{a+b-2+(2n-2)k}{a-1+(n-1)k}.\end{aligned}$$

As with the above proofs, $M_n(a,b,2k) = \prod_{i=1}^{n} M_i(a,b,2k)/M_{i-1}(a,b,2k)$, so the result follows. □