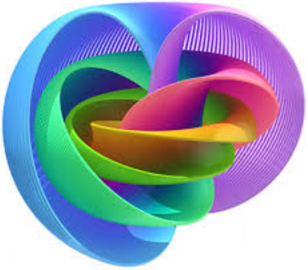
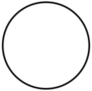
Borel cohomology of S^n mapping spaces

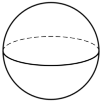


Justin Wu • October 17, 2020
Mentor: Ishan Levy
MIT PRIMES Conference

Natural Spaces

Spheres — S^n

S^1 : 

S^2 : 

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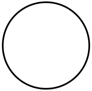
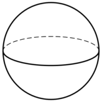
S^1 : 

S^2 : 

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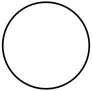
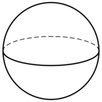
S^1 : 

S^2 : 

Unbased mapping spaces, denoted $\mathrm{Map}(Y, X)$ or X^Y — the space of continuous maps $Y \to X$.

X^{S^n} is well understood for $n = 1$ and the based case. We study the unbased case for general n.

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Unstable Steenrod Algebra: Maps

$$Sq^i : H^n(X) \to H^{n+i}(X)$$

satisfying

$$Sq^i(x) = 0 \quad i > |x| \quad \text{(instability condition)}$$

$$Sq^i(xy) = \sum_{a+b=i} Sq^a(x)Sq^b(y)$$

$$Sq^{|x|}x = x^2.$$

Examples

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n-dimensional space — cohomology only goes up to degree n.

Maps Between Spaces

Maps between spaces induces a map between Steenrod Algebras

$$f : X \to Y$$

$$f^* : H^*(Y) \to H^*(X).$$

f^* is compatible with addition, multiplication, degree, and Steenrod squares

$$f^*(x+y) = f^*(x) + f^*(y)$$
$$f^*(xy) = f^*(x)f^*(y)$$
$$|f^*(x)| = |x|$$
$$Sq^i f^*(x) = f^*(Sq^i x).$$

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For A an unstable Steenrod algebra, define $\Omega_n(A)$ as the free $\mathbb{F}_2$ algebra generated by $x, dx \in A$ with $|dx| = |x| - n$ modulo the following relations:

$$\begin{gathered} dx + dy = d(x+y) \\ d(xy) = d(x)y + d(y)x \\ d(Sq^n x) = (dx)^2 \\ d(Sq^i x) = 0 \qquad i < n. \end{gathered}$$

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The map $f : S^n \times X^{S^n} \to X$, $(p, f) \to f(p)$
induces the map

$$f^* : H^*(X) \to H^*(S^n \times X^{S^n}) = H^*(S^n) \otimes H^*(X^{S^n}),$$

$$x \to 1 \otimes x + c \otimes dx.$$

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The **Borel cohomology** or $G-$**equivariant cohomology** of a G-space X is

$$H^*(X_{hG})$$

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We wish to compute

$$H^*(X^{S^n}_{hSO(n+1)}).$$

Equivariant Approximation

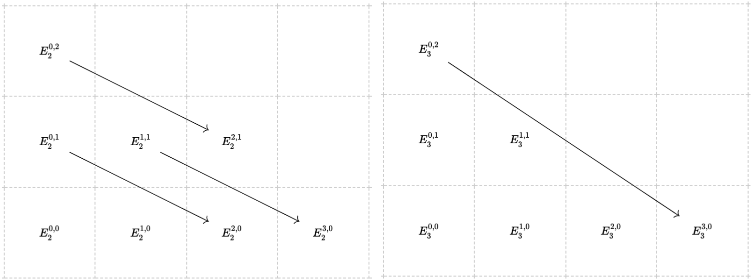
Define $\ell_n(A)$ to be the free algebra on generators $\phi_i(x), \delta(x) \quad (0 \leq i \leq n)$ for each homogeneous $x \in A$ of degree $2|x| - i$ and $|x| - n$ respectively, along with $w_2 \ldots w_n$ with $|w_i| = i$, modulo the following relations

- $\phi_i(x+y) = \phi_i(x) + \phi_i(y) + w_{n-i}\delta(xy)$ $(w_1 = 0, w_0 = 1)$
- $\delta(x+y) = \delta(x) + \delta(y)$
- $\delta(xy)\delta(z) + \delta(yz)\delta(x) + \delta(zx)\delta(y) + \sum c_I w_I = 0$
- $w_{n+1}\delta(a) = 0$
- $\delta(a)\phi_i(b) = \delta(aSq_i b) + \delta_{0n}\delta(ab)\delta(b) + \sum c'_I w_I$
- $\phi_k(xy) = \sum_{i+j=k} \phi_i(x)\phi_j(y) +$
  $$\sum_{\ell=n+1}^{2n} \sum_{i+j=\ell} \phi_i(x)\phi_j(y) \left(\sum_{\substack{2 \leq \alpha_1 \ldots \alpha_m \leq n+1 \\ \alpha_1 + \ldots \alpha_m = \ell - k \\ \alpha_m > n-k}} \prod_{f=1}^{m} w_{\alpha_f} \right)$$
- $Sq\phi_{n-k}(x) = (\sum w_i)^{-1} \sum_{j \geq 0} \sum_i \binom{k+|x|-j}{i-2j} \phi_{n-i-k+2j}(Sq^j x)$
- $Sq(\delta(x)) = (\sum w_i)^{-1}\delta((x))$.

ℓ_n comes with an approximation map $\ell_n(H^*(X)) \to H^*(X^{S^n}_{hSO(n+1)})$ which is an isomorphism for $X = K(\mathbb{Z}/2, m)$.

Spectral Sequence

Cohomology spectral sequence: $E_r^{p,q}$ an object in an abelian category on the rth "page" (typically) for $r \geq 2$ and a differential $d_r^{*,*} : E_r^{*,*} \to E_r^{*+r,*-r+1}$ such that $E_{r+1}^{*,*} = \ker(d_r^{*,*})/\mathrm{im}(d_r^{*,*})$.



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We want to compute $SO(n+1)$ Borel cohomology of X^{S^n} for $X = K(\mathbb{Z}/2, m)$ and $m > n$.

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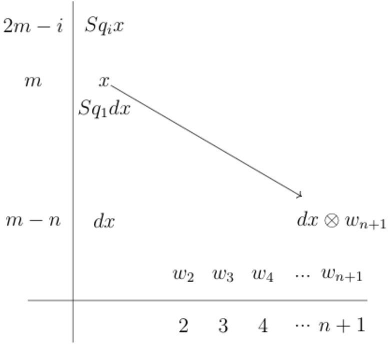
Use the Serre spectral sequence for

$$X^{S^n} \to X^{S^n}_{hSO(n+1)} \to BSO(n+1).$$

$$E_2^{p,q} = H^p(BSO(n+1)) \otimes H^q(X^{S^n}) \to H^*(X^{S^n}_{hSO(n+1)}).$$

$$H^*(BSO(n+1)) = \mathbb{F}_2[w_2 \dots w_{n+1}].$$



Acknowledgments

I would like to thank

- Ishan Levy
- Haynes Miller
- MIT PRIMES.

References

Marcel Bökstedt and Iver Ottosen. "Homotopy orbits of free loop spaces". In: Fundamenta Mathematicae 163 (1999), pp. 251–275.