The Bernardi formula for non-transitive deformations of the braid arrangement

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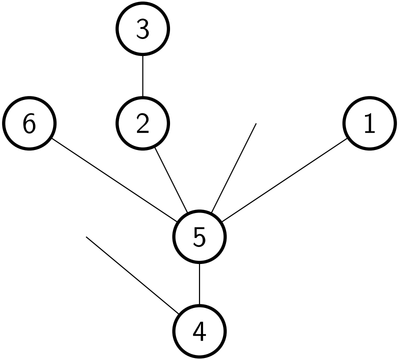
Outline

Rooted Labelled Plane Trees

- Consider trees on some labelled vertices (**nodes**) and some unlabelled vertices (**leaves**).



Rooted Labelled Plane Trees

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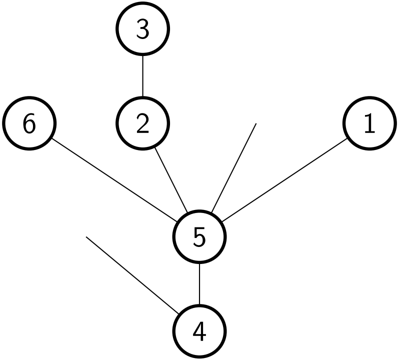
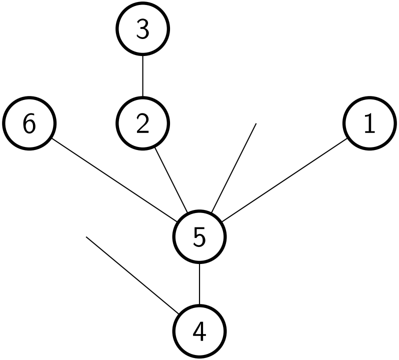
- Some node is a designated **root**.



Rooted Labelled Plane Trees

- Consider trees on some labelled vertices (**nodes**) and some unlabelled vertices (**leaves**).
- Some node is a designated **root**.
- The **parent** of a vertex v is next vertex on the path from v to the root, and v is a child of its parent.



Rooted Labelled Plane Trees

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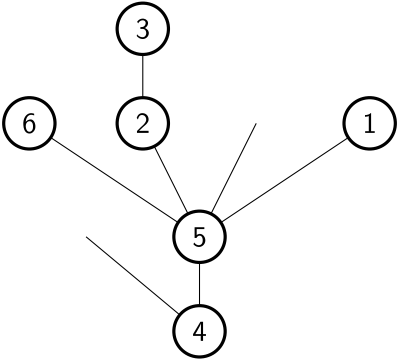
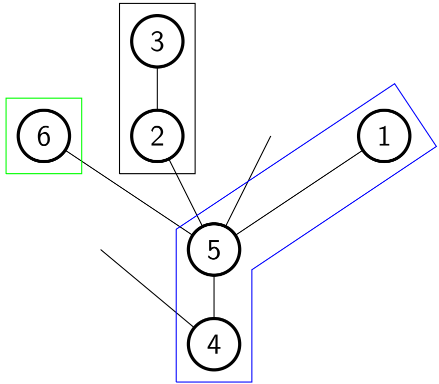
- Some node is a designated **root**.
- The **parent** of a vertex v is next vertex on the path from v to the root, and v is a child of its parent.
- The order in which vertices are drawn (from left to right) matters.



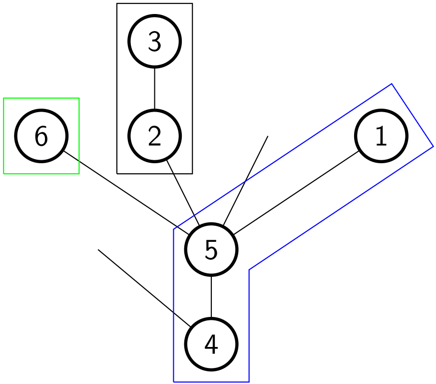
Node Relationships

- Let lsib(v) denote the number of left siblings of v (including leaves).



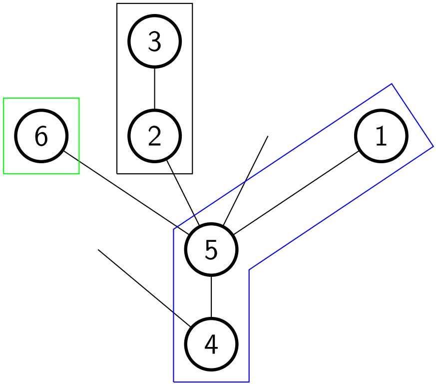
Node Relationships

- Let $\text{lsib}(v)$ denote the number of left siblings of v (including leaves).
- Define the **cadet** of a node to be its rightmost child that's also a node, denoted by $\text{cadet}(v)$.



Node Relationships

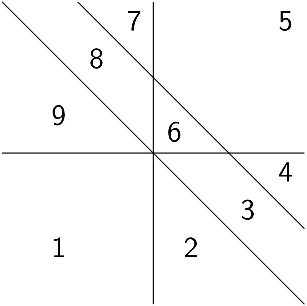
- Let $\text{lsib}(v)$ denote the number of left siblings of v (including leaves).
- Define the **cadet** of a node to be its rightmost child that's also a node, denoted by $\text{cadet}(v)$.
- A sequence $(v_1, v_2, \ldots, v_k)$ of nodes is a **cadet sequence** if $\text{cadet}(v_{i-1}) = v_i$ for all $i = 2, \ldots, k$.



Hyperplane Arrangements

Define a **hyperplane arrangement** to be a set of hyperplanes in $\mathbb{R}^n$.

We define the number of **regions** determined by a hyperplane arrangement to be the number of connected components of the complement of the hyperplane arrangement in $\mathbb{R}^n$. Less formally, the number of regions is the number of parts the hyperplane arrangement splits $\mathbb{R}^n$ into.



The hyperplane arrangement in $\mathbb{R}^2$ with hyperplanes (lines) $x = 0$, $y = 0$, $x + y = 0$, and $x + y = 1$.

Deformations of the Braid Arrangement

The **Braid arrangement** in $\mathbb{R}^n$ consists of the hyperplanes

$$x_i - x_j = 0 \qquad \forall\ 1 \le i < j \le n.$$

Deformations of the braid arrangement in $\mathbb{R}^n$ are formed by sets $S_{i,j}$ of integers, such that the hyperplanes are

$$x_i - x_j = s \qquad \forall\ 1 \le i < j \le n,\ \ s \in S_{i,j}.$$

For the remainder of the presentation, when we consider hyperplane arrangements, we only consider deformations of the Braid arrangement.

Bijection Outline

For this section, we work with a deformation of the Braid arrangement in $\mathbb{R}^n$ formed by fixed sets $S_{i,j}$ for $1 \leq i < j \leq n$.

For all distinct i, j, for a nonnegative a, we can define the ordered pair (i, j) to be a-**compatible** if and only if a is in some specific set determined by $S_{i,j}$ or $S_{j,i}$, whose definition we will omit.

Bernardi's formula gives a bijection between the number of regions of the hyperplanes determined by the $S_{i,j}$ and a signed sum over the set of trees on n nodes and m children per node for some fixed m.

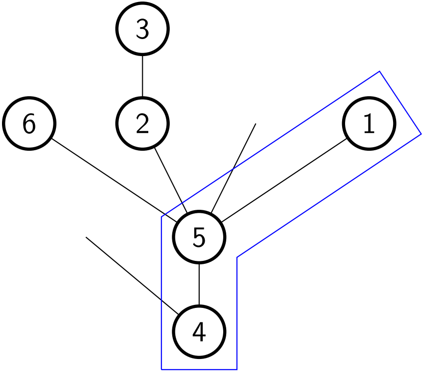
Tree Boxings (cont.)

In a tree, define a **box** to be a of nodes, and define a **boxing** to be a partition of nodes into boxes.

For any $i < j$ in cadet sequence $(v_1, \ldots, v_k)$, we say that v_i and v_j are **compatible** within the tree if (v_i, v_j) is

$$\left(\sum_{p=i+1}^{j} \text{lsib}(v_p) \right) \textbf{-compatible}$$

We call a box **valid** if it is a cadet sequence $(v_1, \ldots, v_k)$ where all v_i, v_j are compatible.

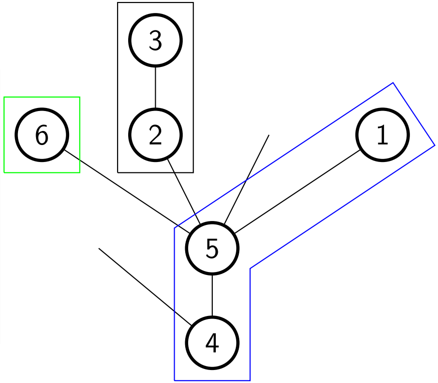


$$\text{lsib}(5) = 1$$
$$\text{lsib}(1) = 3$$
$$\text{lsib}(5) + \text{lsib}(1) = 4$$

Bernardi's Formula

Theorem (Bernardi's Formula (Bernardi 2018))

The number of regions in a hyperplane arrangement is equal to the sum of $(-1)^{n-b}$ over all trees on n nodes with m children per node and boxings of the tree into b valid boxes.



Transitivity

Transitive hyperplane arrangements are defined such that

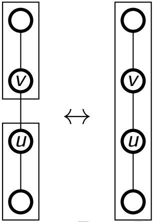
$$(a, b) \text{ and } (b, c) \text{ are compatible} \implies (a, c) \text{ is compatible.}$$

Theorem (Transitive Reduction (Bernardi 2018))

The number of regions in a transitive hyperplane arrangement is equal to the number of trees for which the only valid boxing is the case where every element is in its own box.

Proof.

Take two consecutive compatible vertices (u, v), and pair the cases where they are in the same box with the cases where they are in different boxes. □



The Nontransitive Ish Arrangement

These are the Braid, Catalan, Shi (Shi 1986), and Ish (Armstrong 2013) arrangements, in that order, and the trees bijected to them by Bernardi's Formula.

Hyperplanes	# Regions	Corresponding Trees
$x_i - x_j = 0$	$n!$	Path of n nodes
$x_i - x_j \in \{-1, 0, 1\}$	$n! C_n$	Binary trees on n nodes
$x_i - x_j \in \{0, 1\}$	$(n+1)^{n-1}$	Binary trees on n nodes $\text{lsib}(v) \neq 0 \Rightarrow \text{parent}(v) > v$
$x_i - x_j = 0$ $x_1 - x_j \in [j-1]$	$(n+1)^{n-1}$	???????

This project was inspired by the following question — How do we reduce the bijection given by Bernardi in the case of the nontransitive Ish Arrangement?

Contributions of Trees

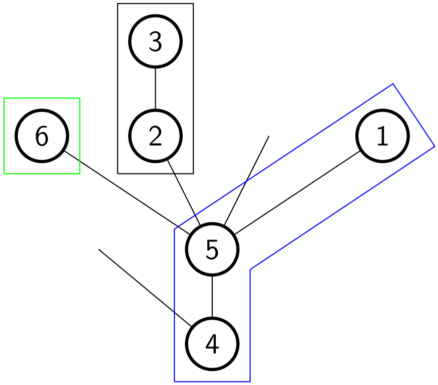
For this section, we'll work with an arbitrary hyperplane arrangement.

Define the **contribution** of a tree to be the sum of $(-1)^{n-b}$ over all valid boxings of the tree into b boxes. By Bernardi's Formula, the sum of the contributions of all of the trees is equal to the number of regions in the hyperplane arrangement.

Main Result 1: We show that the contribution of any tree is -1, 0, or 1, and find a way to determine which it is.

Splitting up Trees

We can partition the nodes into **connected cadet sequences**, which are the smallest cadet sequences such that each valid box is contained in one of the connected cadet sequences.



Lemma (B.-Hanson)

The contribution of a tree is equal to the product of the contributions the boxings of its connected cadet sequences in this partition.

Contributions are $-1, 0, 1$

Theorem (B.-Hanson)

We provide an algorithm to compute the contribution of a connected cadet sequence $(v_1, v_2, \ldots v_k)$*:*

- *Find the valid boxes not contained in any other valid boxes with nodes in the cadet sequence* $X_1, X_2, \ldots, X_m$*, in increasing order of lowest indexed vertex, and let* X_0 *be an empty box.*
- *Draw a directed graph on the* X_i*, where for* $i < j$*, there is an edge from* X_i *to* X_j *if and only if the parent of the highest indexed node of* $X_j \setminus X_{j+1}$ *is in* X_i*. When* $i = 0$*, we need the parent to not exist.*
- *Construct a path* $X_{i_0}, X_{i_1}, \ldots, X_{i_{m'}}$ *such that* $i_0 = 0$*,* $i_{m'} = m$*, and* $X_{i_{j+1}}$ *be the first maximal box reached by* X_{i_j} *but not reached by* X_{i_p} *for* $p < j$*.*

Now, the contribution is $(-1)^{k-m'}$*, unless this algorithm fails, in which case it's* 0*.*

Nested Ish Arrangements

Recall that transitive arrangements, by definition, have the property that within any tree, if (a, b) is compatible, and (b, c) is compatible, then (a, c) is compatible.

We define **nested Ish** arrangements to have the property that within any tree,

- The transitivity condition holds for $c \neq 1$.
- If $a, b \neq 1$, (a, b) is k-compatible if and only if $k > 0$
- $S_{1,i} \subseteq S_{1,j}$ for all $i < j$

Main Result 2: We reduce Bernardi's Formula to counting the size of a set of trees in the case of **nested Ish** hyperplane arrangements, which happen to include the Ish arrangement.

Characterizing Contributions

We can characterize the contribution of trees as follows:

1. If there exists a cadet sequence $(v_1, v_2, \ldots, v_\ell, 1)$ (possibly with $\ell = 1$) such that
   - $\{v_1, v_2, v_3 \ldots, v_\ell\}$ is a valid box
   - $\{v_2, v_3, \ldots, v_\ell, 1\}$ is a valid box
   - every valid box with more than one node is contained in one of the previously listed boxes

   then, the contribution is $(-1)^{\ell-1}$.
2. If 1 is the root and no valid box has size greater than 1, the contribution is 1. We let $\ell = 1$ in this case.
3. If neither of the previous cases are true, the contribution is 0.

E and Length

Now, we focus on the trees with nonzero contribution. For each of these, define:

ℓ to be the parameter defined in the previous slide.

E to be the number of left siblings v of 1 such that v and 1 are lsib(v)-compatible.

Let $f(E_0, \ell_0)$ denote the number of trees with $E = E_0$ and $\ell = \ell_0$. Recall that the contribution of these trees is $(-1)^{\ell-1}$, so the total number of regions is

$$\sum_{E,\ell} (-1)^{\ell-1} f(E, \ell).$$

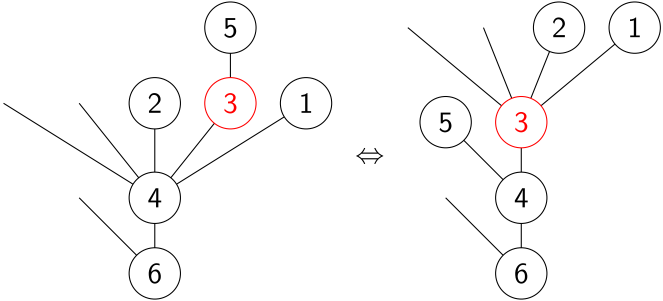
In this section, we show that the number of regions of the hyperplane arrangement is equal to $f(0, 1)$, thus constructing a set of trees that it is equinumerous with.

Bijections with ℓ and E

Lemma (B.-Hanson)

For all $E_0 \geq 0$ and $\ell_0 \geq 1$,

$$f(E_0, \ell_0) = f(E_0+1, \ell_0-1) + f(E_0+2, \ell_0-1) + f(E_0+3, \ell_0-1) + \cdots .$$



$\{6,4\},\{4,1\},\ \ell = 2$ $\qquad$ $\{6,4,3\},\{4,3,1\},\ \ell = 3$

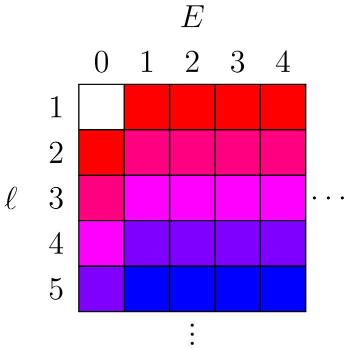
Reducing to $E = 0$ and $\ell = 1$

Theorem (B.-Hanson)

The number of regions of the hyperplane arrangement is equal to $f(0, 1)$.

Proof.

Draw a grid of the values of $(-1)^{\ell-1} f(E, \ell)$ at each value of E and ℓ, as shown on the right. By the previous lemma in the $\ell = 1$ case, every color other than white contributes zero, giving the desired result. □



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