Ratios of Naruse-Newton Coefficients Obtained from Descent Polynomials

Andrew Cai
Mentor: Pakawut Jiradilok

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  - For example, if $I = \{1\}$, then the value $d_I(n)$ counts the number of permutations $w = (w_1, w_2, \ldots, w_n)$ of $[n] = \{1, 2, \ldots, n\}$ such that $w_1 > w_2 < w_3 < \cdots < w_n$. Then, $d_I(n) = n - 1$.

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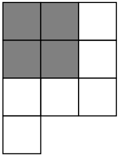
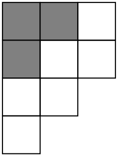
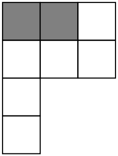

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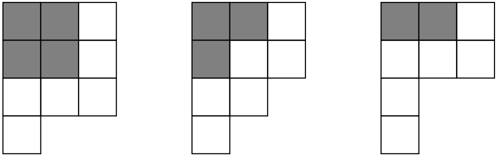

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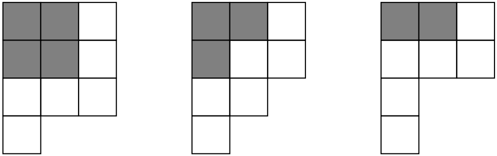

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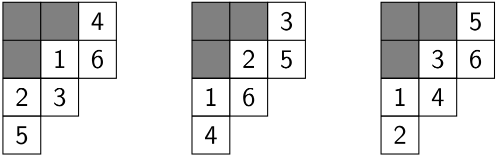
  - Three ribbon tableaux of the same ribbon shape:



Two Important Bijections

- There exists a bijection between ribbon tableaux and permutations, and a bijection between ribbon shapes of exactly n cells and descent sets $I \subseteq [n-1] = \{1, 2, \ldots, n-1\}$.

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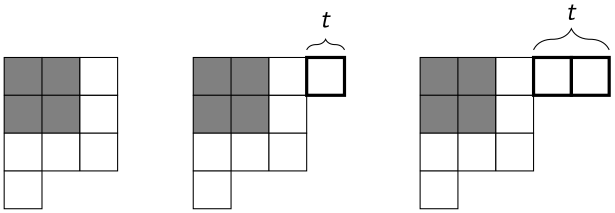
    - Examples of $\text{rib}_n(I)$ when $I = \{1, 4, 5\}$, where t is represented by the number of bolded cells:



$n = 6$ $n = 7$ $n = 8$

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- By the bijection described previously between descent sets $I \subseteq [n-1]$ and ribbons with n cells, we know $d_I(n) = f^{rib_n(I)}$.
- We have from Naruse's Hook Length Formula that

$$d_I(n) = f^{rib_n(I)} = f^{rib_{t+m}(I)} =$$

$$\underbrace{\frac{(m+t)!}{(t-1)!}\left(\prod \frac{1}{h(c)}\right)\left(\prod \frac{1}{t+\alpha_i}\right)}_{P(t)} \cdot E(t).$$

Expansion of $d_I(n)$

- Hence, we have

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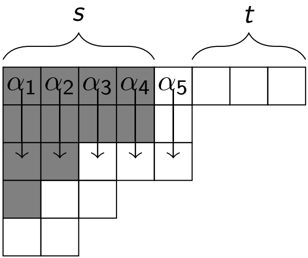

Figure: The hook length (defined as the number of cells weakly below or to the right of a cell) of a cell is denoted $h(c)$, while the hook length of the ith position of the first row is $t+\alpha_i-1$.

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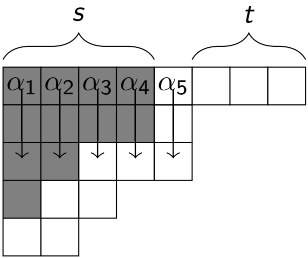

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- Naruse's Formula gives us the trivial product of monomials of $P(t)$.
- Meanwhile, $E(t)$ has more combinatorial significance given its mystery, so it is the subject of our interest.

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The *Naruse-Newton coefficients* $C_0, C_1, \ldots, C_s$ are positive integers defined such that

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- What are some properties of these coefficients?

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- Jiradilok and McConville later examined ratios between Naruse-Newton coefficients (constructed through Naruse's extension of the Hook Length Formula). They used analytic properties to prove the aforementioned conjecture.
- Our research seeks to determine more properties about these Naruse-Newton coefficients.

Ratios of Naruse-Newton Coefficients

Proposition 2.4 (Jiradilok, McConville 2019)

For a fixed non-empty descent set I with the width of $\mathrm{rib}(I)$ *equal to* $s+1$,

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- Natural next question: What are the equality cases?

Equality Cases

Theorem (C.)

Let I be a non-empty descent set of positive integers, and let k be the number of columns of $\mathrm{rib}(I)$ *with height 2. Then, for positive integers $i < j$, we have $C_{i,j} = \frac{i!}{j!}$ if and only if $i, j \leq k$.*

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- This theorem implies that

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- Sketch of Proof:

Adding Cells

Proposition (C.)

Let I be a non-empty set of positive integers, and let $\text{rib}(J)$ *be* $\text{rib}(I)$ *with a cell appended to the left of its lower left cell. Then, if positive integers a and b are defined such that $s \geq b > k$ and $b > a \geq 0$, and k is the number of columns of* $\text{rib}(I)$ *with height* 2, *then $C_{a,b}(I) > C_{a,b}(J)$.*

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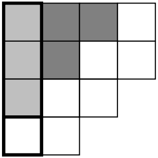

Figure: Appending a cell to the lower left of a ribbon.

- From this proposition, we have proven that $C_{a,b} > \frac{a!}{b!}$ if $b > k$. Proving equality for $k \geq b$ comes down to simple computation. □

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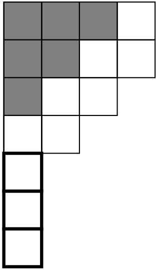

Figure: Adding three cells to the leftmost column of a ribbon.

- Define ϕ to be the function that adds one cell to the bottom of the leftmost column of rib(I). Define ψ to be the function that removes all cells in the leftmost column of rib(I). For example, the figure to the left illustrates $\phi^3(I)$ and the figure to the right illustrates $\psi^2(I)$.

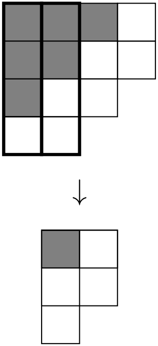

Figure: Removing two leftmost columns of a ribbon.

Construction of Maxima

Theorem (C.)

Take nonnegative integers a and b such that $b > a \geq 0$, and let λ_1 denote the width of the first row of $\mathrm{rib}(I)$. *Set either $s = b$ if $\lambda_1 = 2$ or $s > b$ if $\lambda_1 > 2$. Then,*

$$\lim_{n\to\infty} C_{a,b}(\phi^n(I)) = \begin{cases} \infty, & \lambda_1 = 2, \\ C_{a,b}(\psi(I)), & \lambda_1 > 2. \end{cases}$$

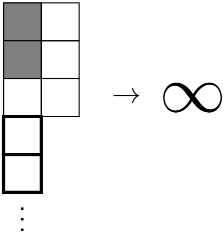

Figure: If $\lambda_1 = 2$.

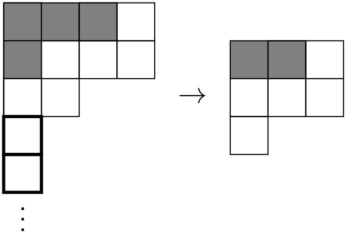

Figure: If $\lambda_1 > 2$.

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Corollary (C.)

For any integers a and b such that $a < b$ and subset $R' \subseteq R_{a,b}$ such that $|R_{a,b} - R'|$ is finite, the closure $\overline{R'}$ coincides with the closure $\overline{R_{a,b}}$ in the Euclidean topology.

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- Study the asymptotic growth of $C_{a,b}$ upon having specific operations conducted on I.

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- Prof. Etingof, Dr. Gerovitch, Dr. Khovanova, and everyone else who made this program possible.
- My family, and especially my parents.

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