The Sperner Property for 132-Avoiding Intervals in the Weak Order

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MIT PRIMES Conference
Joint work with Christian Gaetz
October 18, 2020

Weak order interval $[e, 4213]_R$
Let $S_n$ be the $n!$ permutations of \{1, 2, 3, ..., $n$\}.

Weak Bruhat order on $S_n$:

- Least element $e = [1 \ 2 \ ... \ n]$
- Covering: $\sigma \preceq \sigma(i \ i+1)$ if $\sigma_i < \sigma_{i+1}$.
  - $[15243] \preceq [15423]$. 

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\[
\begin{array}{c}
321 \\
231 \quad 312 \\
213 \quad 132 \\
123
\end{array}
\]
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Weak Bruhat order on $S_n$:

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- Covering: $\sigma \lessdot \sigma(i \ i+1)$ if $\sigma_i < \sigma_{i+1}$.
  - $[15243] \lessdot [15423]$.
- Rank function $\ell(\sigma) = \#$ inversions of $\sigma$
  - $\ell([312]) = 2$.
- Greatest element $w_0 = [n \ n-1 \ \ldots \ 2 \ 1]$ with rank $\binom{n}{2}$.
Multisets

We can similarly order the permutations of a set with repetitions such as the 60 orderings of 112333.

The latter set corresponds to the interval \([e, \pi = 456312]\), permutations less than or equal to 456312 in the weak order.

112333 $\longleftrightarrow$ 123456
313231 $\longleftrightarrow$ 415362
333211 $\longleftrightarrow$ 456312
A permutation $\pi$ avoids the pattern 132 if for no indices $i < j < k$ is $\pi_i < \pi_k < \pi_j$. So 4312 avoids 132, but 2143 does not avoid 132.

Any permutation corresponding to a greatest permutation of a multiset is 132-avoiding.

There are $2^{n-1}$ multisets and $C_n = \binom{2n}{n}/(n+1) \sim 4^n/(n^{3/2} \sqrt{\pi})$ 132-avoiding permutations.

We studied intervals $[e, \pi]_R$ with $\pi$ 132-avoiding, which generalizes the study of permutations of multisets.
Questions about \([e, \pi]_R\)

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  - Is there an order-reversing self-bijection?
- Are these posets rank unimodal?
  - Does the rank function increase up to a peak and then fall?
- Are these posets Sperner?
  - Is the size of the largest antichain (pairwise incomparable set) equal to the maximum number of elements with a particular rank?
Lie algebras

- Lie algebra $L$
  - Lie bracket $[x, y]$
Lie algebras

- Lie algebra $L$
  - Lie bracket $[x, y]$
    - Bilinear

\[ [x, y] = 0 \text{ for all } x \in L \]
\[ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in L \]
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The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

- Bracket operation $[a, b] = ab - ba$
- Basis elements
  \[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
- Relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$
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- Proctor, Stanley: if there is an $\mathfrak{sl}_2$ representation on $\mathbb{C}P$ respecting the order of $P$, then $P$ is
  - rank symmetric
  - rank unimodal
  - Sperner
\section*{$\mathfrak{sl}_2$ representations}

We want a lowering operator $F$ and a raising operator $E$ on $\mathbb{C}[e, \pi]_R$ so that

\begin{itemize}
  \item $F(\sigma)$ is a linear combination of permutations covered by $\sigma$
  \item $E(\sigma)$ is a linear combination of permutations of rank $\ell(\sigma) + 1$
  \item $[E, F] = H$ is diagonal with $H(\sigma) = (2\ell(\sigma) - \ell(\pi))\sigma$.
\end{itemize}

Given $F$, there is at most one $E$ that works (Jacobson and Morozov).
Another related order on $S_n$ is the strong order, which has the same rank function with more relations.

The covering relation is that $\sigma \preccurlyeq \tau$ if $\tau = \sigma(i \; j)$ and $\ell(\tau) = \ell(\sigma) + 1$. 

\[
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\uparrow \\
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213 \\
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\end{array}
\]
\[ F_\sigma = \sum_{i: \sigma(i, i+1) \leq \sigma} i\sigma(i, i+1). \]

\[ E_\sigma = \sum_{\sigma \prec \sigma(i, j)} \text{wt} (\sigma, \sigma(i, j)) \sigma(i, j) \]

\[ H_\sigma = (2\ell(\sigma) - \ell(w_0)) \sigma \]

where

\[ \text{wt}(\sigma, \sigma(i, j)) := 1 + 2|\{k > j \mid \sigma_i < \sigma_k < \sigma_j\}|. \]
Above are edge weights for order raising operator $E$ (left) and lowering operator $F$ (right). Example: $E[132] = [231] + 3[312]$ and $F[132] = 2[123]$. 

$\mathfrak{sl}_2$ repr. on $S_n$, weak order (Gaetz and Gao)
We can construct an \( \mathfrak{sl}_2 \) representation on \([e, \pi]_R\) by

\[
F_\sigma = \sum_{i : \sigma(i+1) < \sigma} i\sigma(i,i+1).
\]

\[
E_\sigma = \sum_{\sigma \prec \sigma(i,j) \leq \pi} \text{wt}_\pi(\sigma, \sigma(i,j)) \sigma(i,j)
\]

\[
H_\sigma = (2\ell(\sigma) - \ell(\pi)) \sigma
\]

where

\[
\text{wt}_\pi(\sigma, \sigma(i,j)) := 1 + \left| \{ k > j \mid \sigma_i < \sigma_k < \sigma_j \} \right|
\]

\[
+ \left| \{ k > j \mid \pi^{-1}(\sigma_j) < \pi^{-1}(\sigma_k) < \pi^{-1}(\sigma_i) \} \right|.
\]
Above are edge weights for order raising operator $E$ (left) and lowering operator $F$ (right).
Schubert polynomials

\[ S_{w_0} = x_1^{n-1}x_2^{n-2}\ldots x_{n-1}^1 \]

Let \( N_i \) act on a polynomial \( f \) by:

\[ N_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}} \]

We have the recursive relation \( S_{s_i\sigma} = N_i S_\sigma \) if \( \ell(s_i\sigma) = \ell(\sigma) - 1 \).
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Examples:

- \( S_{3412} = x_1^2 x_2^2 \).
- \( S_{1432} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 \).
Corollary

If $\sigma \in [e, \pi]_R$ with $\pi$ 132-avoiding, then

$$S_\sigma(1, 1, 1, \ldots, 1) = \frac{1}{(\ell(\pi) - \ell(\sigma))!} \sum_{\sigma \prec \sigma^1 \prec \ldots \prec \pi} \prod_{i} \text{wt}^\pi(\sigma^i, \sigma^{i+1}).$$

If $\sigma$ is 132-avoiding, we can use $\pi = \sigma$ which makes the product empty, so $S_\sigma(1, 1, 1, \ldots, 1) = 1$. 
Corollary

If $\sigma \in [e, \pi]_R$ with $\pi$ 132-avoiding, then

$$\mathcal{G}_\sigma(1, 1, 1, ..., 1) = \frac{1}{(\ell(\pi) - \ell(\sigma))!} \sum_{\sigma \ll \sigma^1 \ll ... \ll \pi} \prod_i \text{wt}^\pi(\sigma^i, \sigma^{i+1}).$$

If $\sigma$ is 132-avoiding, we can use $\pi = \sigma$ which makes the product empty, so $\mathcal{G}_\sigma(1, 1, 1, ..., 1) = 1$.

Examples:

- $\mathcal{G}_{3412}(1, 1, 1, 1) = 1.$
- $\mathcal{G}_{1432}(1, 1, 1, 1) = 5.$
Acknowledgements

- MIT PRIMES USA
- My mentor Christian Gaetz
- My parents

