Few Distance Sets in $\ell_p$ Spaces and $\ell_p$ Product Spaces

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Introduction

Definition

An **s-distance set** is a set of points such that the set of pairwise distances between the points has cardinality $s$. 

A regular pentagon is a 2-distance set in $\mathbb{R}^2$: A 1-distance set is also known as an **equilateral set**.
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![Diagram of a regular pentagon]
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![Diagram of a regular pentagon with distances between points]

Definition

A 1-distance set is also known as an **equilateral set**.
Given a metric space $X$ and a positive integer $s$, what is the maximum size of an $s$-distance set in $X$?
General Problem

Question
Given a metric space $X$ and a positive integer $s$, what is the maximum size of an $s$-distance set in $X$?

Definition
Let $e_s(X)$ denote the answer to this question, or just $e(X)$ if $s = 1$.

Some problems we can consider:
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Some problems we can consider:
- Different spaces: Euclidean space, sphere, hyperbolic space, etc.
- Different $s$ values: equilateral sets, 2-distance sets, general $s$-distance sets, etc.
- Almost-equilateral sets: all distances are within $\varepsilon$ of $d$, instead of being exactly $d$. 
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Example in Euclidean Space

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What is the maximum size of an equilateral set in $\mathbb{E}^n$?

The answer is $n+1$, so $e(\mathbb{E}^n) = n+1$. Equality holds with a standard simplex.
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**Figure:** An equilateral triangle in $\mathbb{E}^2$

**Figure:** A regular tetrahedron in $\mathbb{E}^3$
Linear Algebra Proof

Let the equilateral set consist of points $p_1, \ldots, p_m$. Assume that the common distance is 1. For each $1 \leq i \leq m$, define the polynomial $f_i(x) = 1 - \|x - p_i\|^2$, so that $f_i(p_j) = \delta_{ij}$. This implies the $f_i$ are linearly independent.

Expand $f_i$ as $f_i(x) = \|x\|^2 - \sum_{j=1}^{n} c_j x^j + C$. Thus, the $f_i$ live in a vector space of dimension $n + 2$ and $m \leq n + 2$.

[Blokhuis trick] The stronger claim that $f_1, \ldots, f_m, 1$ are linearly independent ends up giving the desired $m \leq n + 1$. 
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Our strategy for the example before was:

- Define an annihilating function, which happens to be a polynomial.
- Obtain bound from linear independence.

We can use this strategy whenever the annihilating function is a polynomial, e.g., with an $s$-distance set in $\mathbb{F}^n$. 
The $\ell_p$ norm

**Definition**

For any $p \geq 1$, the $\ell_p$ norm is defined by

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$  

Taking $p \to \infty$, we can define the $\ell_\infty$ norm by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$  

**Definition**

The space $\mathbb{R}^n$ equipped with the $\ell_p$ norm is denoted $\ell_p^n$. 
The $\ell_p$ sum

**Definition**

We define the $\ell_p$ sum $E^{a_1} \oplus_p \cdots \oplus_p E^{a_n}$ to be the product space $E^{a_1} \times \cdots \times E^{a_n}$ equipped with the norm

$$\|(x_1, \ldots, x_n)\|_p = \left( \sum_{i=1}^{n} \|x_i\|_p^2 \right)^{\frac{1}{p}}.$$

If we take $p \to \infty$, the norm becomes

$$\|(x_1, \ldots, x_n)\|_\infty = \max_{1 \leq i \leq n} \|x_i\|.$$
Our First Two Results

**Theorem (Chen et. al., 2020)**

Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces. Then,

$$e(\mathbb{E}^a \oplus \infty \mathbb{E}^b) \leq (a + 1)(b + 1) + 1.$$
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- Annihilating functions: $f_u(x) = \left(1 - \|\tilde{x}_1 - \tilde{u}_1\|^2\right)\left(1 - \|\tilde{x}_2 - \tilde{u}_2\|^2\right)$. 

Start with $(a + 2)(b + 2)$ element spanning set of the $f_i$. We eliminate $a + b + 2$ elements from the spanning set via a Blokhuis trick technique.
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Theorem (Chen et. al., 2020)

Let $E^a$ and $E^b$ be Euclidean spaces, and let $p$ be an even integer. Then,

$$e(E^a \oplus_p E^b) \leq \left(\frac{a + p/2}{a}\right) + \left(\frac{b + p/2}{b}\right).$$

- Annihilating functions:
  $$f_u(x) = 1 - \left(\sum_{t=1}^{a} (\tilde{x}_1t - \tilde{u}_1t)^2\right)^{p/2} - \left(\sum_{t=1}^{b} (\tilde{x}_2t - \tilde{u}_2t)^2\right)^{p/2}.$$
- We use the same strategy as the previous result.
Conjectures and Known Results

**Conjecture (Kusner, 1983)**

For $p \in (1, \infty)$, we have $e(\ell_p^n) = n + 1$.

Currently, the best bound is

**Theorem (Alon-Pudlák, 2003)**

For every $p \geq 1$, we have $e(\ell_p^n) \leq c_p n^{(2p+2)/(2p-1)}$, where one may take $c_p = cp$ for an absolute $c > 0$. 
Rank Lemma

Definition
The **rank** of a matrix is the dimension of the space spanned by its column (or row) vectors.

Lemma (Rank Lemma)
For a real symmetric $n \times n$ nonzero matrix $A$,

$$\text{rank } A \geq \frac{(\sum_{i=1}^{n} a_{ii})^2}{\sum_{i,j=1}^{n} a_{ij}^2}.$$
If the polynomials \( f_i \) are not linearly independent, we can take the matrix \( A \) with \( a_{ij} = f_i(p_j) \) and hope that it looks something like

\[
\begin{pmatrix}
1 & \\
1 & \varepsilon \\
1 & \\
\varepsilon & \cdots \\
\varepsilon & \cdots \\
1 & 
\end{pmatrix}
\]

By the Rank Lemma, this matrix has large rank in terms of \( n \). We can then upper bound the rank using generators of the polynomials, like in the Euclidean Distance example.
Jackson’s Theorem

Jackson’s Theorem allows us to approximate a function by a polynomial with sufficiently small error.

**Lemma (Jackson’s Theorem on $|x|^p$)**

For any $p \geq 1$ and $d \geq \lceil p \rceil$, there exists a polynomial $P$ with degree at most $d$ such that

$$|P(x) - |x|^p| \leq \frac{B(p)}{d^p}$$

for all $x \in [-1, 1]$, where $B(p) < (cp)^p$ for an absolute $c > 0$. 
Recall our strategy from the example in Euclidean space:

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Strategy

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**Theorem (Alon-Pudlák, 2003)**

For every $p \geq 1$, we have $e(\ell^n_p) \leq c_p n^{(2p+2)/(2p-1)}$, where one may take $c_p = c p$ for an absolute $c > 0$.

In Alon and Pudlák’s case, the annihilating function isn’t a polynomial, so the strategy is now:
- Define an annihilating function.
- Approximate this function with a polynomial.
- Obtain bound from rank argument.
Our Results

When $p$ is large in terms of $n$, we have the stronger result:

**Theorem (Chen et. al., 2020)**

If $n > 1$ and $p \geq c(n \log n)^2$ for an absolute $c > 0$, then $e(\ell_p^n) \leq 2(p + 1)n$. 

Consider the space $\ell_n^k$, where $k$ is the closest even number to $p$. If $p$ is large enough, we can approximate the $\ell_p^n$ norm with the $\ell_k^n$ norm. We instead bound the size of an almost-equilateral set in $\ell_n^k$, which can be done with the Rank Lemma.
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We can also generalise Alon and Pudlák’s theorem to $s$-distance sets.

**Theorem (Chen et. al., 2020)**

If $s$ is a positive integer and $p$ is a real number satisfying $2p > s$, then

$$e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$$

for a constant $c_{p,s}$ depending on $p$ and $s$. If the ratio between the largest and smallest distance is bounded, we use an approximation argument with Jackson's Theorem. If this ratio is unbounded, our set will look like “clusters” of points. We can use this to induct downwards.
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Our Results

We also use approximation techniques to bound the size of an equilateral set in an $\ell_p$ sum.

**Theorem (Chen et. al., 2020)**

Let $E^{a_1}, \ldots, E^{a_n}$ be Euclidean spaces and set $a = \max_{1 \leq i \leq n} a_i$. If $2p > a$, then

$$e\left(\bigoplus_{i=1}^n E^{a_i}\right) \leq c_{p,a} n^{\frac{2p+2a}{2p-a}}$$

for a constant $c_{p,a}$ depending on $p$ and $a$. 
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