Lebesgue Measure Preserving Thompson’s Monoid

William Li
Mentor: Prof. Sergiy Merenkov

Delbarton School

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Continuous $h: [0, 1] \xrightarrow{\text{onto}} [0, 1]$. $\lambda$-preserving if $\forall A \in B, \lambda(A) = \lambda(h^{-1}(A))$.

- $\lambda$: Lebesgue measure on $[0, 1]$. $B$: Borel sets on $[0, 1]$.

The above definition does not imply $\lambda(A) = \lambda(h(A))$. In fact, if $h$ is $\lambda$-preserving, $\lambda(A) \leq \lambda(h(A))$ for any $A \in B$. 
Continuous $h: [0, 1] \xrightarrow{\text{onto}} [0, 1]$. $\lambda$-preserving if
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Dynamical System

Topological dynamical system: $h^n = \underbrace{h \circ h \circ \cdots \circ h}_{n \text{ times}}$.

Figure: Logistic map $x_{n+1} = rx_n(1 - x_n)$, which is NOT $\lambda$-preserving.
Continuous function $f: [0, 1] \mapsto [0, 1]$. Piecewise affine, dyadic breakpoints, derivative $= 2^k$ with integer $k$.

Any $f \in \mathbb{F}$ is generated by the above two generator maps.
\( \lambda \)-Preserving Thompson’s Monoid \( \mathbb{G} \)

- \( \mathbb{F} \) maps and \( \lambda \)-preserving maps do not naturally intersect.
  - Except for the identity map, any \( \mathbb{F} \) map does not preserve \( \lambda \) and any \( \lambda \)-preserving map does not preserve orientation and thus does not belong to \( \mathbb{F} \).

- We propose \( \lambda \)-preserving Thompson’s monoid, \( \mathbb{G} \), which is similar to \( \mathbb{F} \) except that the derivatives of piecewise affine maps can be negative to preserve \( \lambda \), i.e., \( \pm 2^k \) for integer \( k \).

- Monoids are semigroups with a single associative binary operation and an identity element.
- Unlike \( \mathbb{F} \), \( \mathbb{G} \) maps are non-invertible except for trivial maps.

- Monoid \( \mathbb{G} \) has not been proposed or studied in the literature and exhibits very different algebraic and dynamical properties from \( \mathbb{F} \) or \( \lambda \)-preserving interval maps in general.
λ-Preserving Thompson’s Monoid $\mathcal{G}$

- $\mathcal{F}$ maps and λ-preserving maps do not naturally intersect.
  - Except for the identity map, any $\mathcal{F}$ map does not preserve λ and any λ-preserving map does not preserve orientation and thus does not belong to $\mathcal{F}$.

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**λ-Preserving Thompson’s Monoid G**

- $F$ maps and $\lambda$-preserving maps do not naturally intersect.
  - Except for the identity map, any $F$ map does not preserve $\lambda$ and any $\lambda$-preserving map does not preserve orientation and thus does not belong to $F$.
  - We propose $\lambda$-preserving Thompson’s monoid, $G$, which is similar to $F$ except that the derivatives of piecewise affine maps can be negative to preserve $\lambda$, i.e., $\pm 2^k$ for integer $k$.

- Monoids are semigroups with a single associative binary operation and an identity element.
- Unlike $F$, $G$ maps are non-invertible except for trivial maps.

- Monoid $G$ has not been proposed or studied in the literature and exhibits very different *algebraic and dynamical properties* from $F$ or $\lambda$-preserving interval maps in general.
Properties of Monoid $\mathbb{G}$

In this project we have studied the following properties:

- **Algebraic properties**
  - Approximation
  - Entropy
  - Decomposition, equivalence classes and finitely generated monoid

- **Dynamical properties**
  - Mixing
  - Periodic points
  - Topological conjugacy

We will next focus on Mixing, Periodic points and Entropy. Unless explicitly mentioned, all the results presented in this talk are obtained by the research project.
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Mixing process: an example.

Repeated application of the baker’s map to points colored red and blue, initially separated. After several iterations, the red and blue points seem to be completely mixed.
Mixing — Theorems (1)

Definition (Topological Mixing (TM))

An interval map $h$ is TM if for all nonempty open sets $U, V$ in $[0, 1]$, $\exists N \geq 0$ such that $\forall n \geq N$, $f^n(U) \cap V \neq \emptyset$.

Definition (Locally Eventually Onto (LEO))

An interval map $h$ is LEO if for every nonempty open set $U$ in $[0, 1]$ there is an integer $N$ such that $h^N(U) = [0, 1]$.

In general, LEO implies TM and the converse does not hold. However, we prove that the two are equivalent for $g \in G$:

Theorem

*If* $g \in G$ *is TM, then* $g$ *is LEO.*
Mixing — Theorems (2)

**Definition**

$C(\lambda)$: set of continuous $\lambda$-preserving maps.

**Definition**

$\rho(h_1, h_2) = \sup_{x \in [0, 1]} |h_1(x) - h_2(x)|$. If $\rho(h_1, h_2) < \epsilon$, $h_2$ is said to be within $\epsilon$ neighborhood of $h_1$.

**Theorem**

Denote by $G_{LEO}$ the subset of $G$ whose elements are LEO. $G_{LEO}$ is dense in $C(\lambda)$.

The theorem states that $\forall h \in C(\lambda)$ and $\epsilon > 0$, there exists $g \in G_{LEO}$ such that $\rho(h, g) < \epsilon$. 
**Definition (Preperiodic and Periodic Points)**

Point $x$ is *preperiodic* if $\exists n > m > 0$ such that $h^n(x) = h^m(x)$. If $m = 0$, then $x$ is *periodic*.

**Theorem**

*On any* $g \in \mathbb{G}$*, if* $c$ is dyadic, *then point* $(c, g(c))$ *is preperiodic.*
**Markov Maps — Definition and Theorem**

**Definition (Markov Map)**

A piecewise affine map is Markov if all breakpoints are preperiodic.

**Theorem**

*Any* $g \in \mathcal{G}$ *is a Markov map.*
**Definition (Period of a Periodic Point)**

The period of periodic point $x$ is the least positive integer $p$ such that $h^p(x) = x$.

**Definition (Chaotic Function)**

Map $h$ is chaotic if for any $k > 0$, point $x$ of period $k$ exists.

- Li-Yorke theorem (1975) states that if a periodic point $x$ of period 3 exists, then $h$ is chaotic.

We characterize periods of periodic points of all maps in $G$:

- Maps in one specific subset of $G$ always have periodic points with period 3.
- For any remaining map, $\exists$ odd $n_0$ such that there exist any odd period $n \geq n_0$, period $n = 1$ and any even period $n$. 

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Lebesgue Measure Preserving Thompson’s Monoid
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Examples of Periodicity of $G$ Maps

- (Left) Periodic points of period 3 do not exist but periodic points of periods 5 and 7 exist.
- (Right) Periodic points of period 3, 5, 7 all exist.
Definition (Entropy)

\[ c_\lambda(h) = \int_0^1 \log_2 |h'(x)| \, d\lambda(x) \]
**Definition**

\( PA(\lambda) \): set of piecewise affine \( \lambda \)-preserving maps.

Bobok and Troubetzkoy (2019) showed that \( \forall c \in (0, \infty) \), Markov LEO \( PA(\lambda) \) is dense in \( C(\lambda) \) with \( c_\lambda(h) = c \).

*What entropy range can \( G \) achieve?*
Suppose that $g$ on $g^{-1}(Y)$ is $m$ affine legs with absolute values of the derivatives equal to $\{2^{k_i}\}$. To minimize $c_\lambda(g)$ with $g \in \mathbb{G}$,

$$\min_{k_1, \ldots, k_m} \sum_{i=1}^{m} k_i 2^{-k_i}, \quad \text{s.t.} \quad \sum_{i=1}^{m} 2^{-k_i} = 1 \quad \Rightarrow \quad k_i^* = \begin{cases} i, & i = 1, 2, \ldots, m - 1 \\ m - 1, & i = m. \end{cases}$$

Key idea: Any set of $m$ affine legs of $\{2^{k_i}\}$ can be replaced by another set of $\{2^{k_i^*}\}$ within $\epsilon$ neighborhood; converse is not true.
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*Key idea:* Any set of $m$ affine legs of $\{2^{k_i}\}$ can be replaced by another set of $\{2^{k_i^*}\}$ within $\epsilon$ neighborhood; converse is not true.
Entropy — Theorem

For any $c \in [2, \infty)$ and $\varepsilon > 0$, the set of Markov LEO maps in $G$ whose entropy is within $\varepsilon$ of $c$ is dense in $C(\lambda)$.

- With $\{2^{k_i^*}\}$, minimum $c_\lambda(g)$ is given by
  $\sum_{i=1}^{m-1} i2^{-i} + (m - 1)2^{-(m-1)} < 2$ for any $m$.
- Maximum $c_\lambda(g)$ is unbounded.

Compared with $c_\lambda(h)$, the constraints on $G$ lead to:

- $c_\lambda(g)$ can only be within $\varepsilon$ of, but may not be exactly equal to, target $c$.
- Minimum $c_\lambda(g)$ is greater than minimum $c_\lambda(h)$. 

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Entropy — Theorem

**Theorem**

For any \( c \in [2, \infty) \) and \( \epsilon > 0 \), the set of Markov LEO maps in \( \mathbb{G} \) whose entropy is within \( \epsilon \) of \( c \) is dense in \( C(\lambda) \).

- With \( \{2^{k_i^*}\} \), minimum \( c_\lambda(g) \) is given by
  \[
  \sum_{i=1}^{m-1} i2^{-i} + (m - 1)2^{-(m-1)} < 2 \quad \text{for any } m.
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Compared with \( c_\lambda(h) \), the constraints on \( \mathbb{G} \) lead to

- \( c_\lambda(g) \) can only be within \( \epsilon \) of, but may not be exactly equal to, target \( c \).
- Minimum \( c_\lambda(g) \) is greater than minimum \( c_\lambda(h) \).
I would like to thank my mentor, Professor Sergiy Merenkov, for his continuous and insightful guidance and advice throughout the entire research process. He introduced me to the general fields of Lebesgue measure preserving interval maps and Thompson’s groups, provided direction in my research, and informed me of the connection between my work and other results in the literature.

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