BORELCOHOMOLOGY OF S^n MAPPING SPACES

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Abstract. We produce an algebraic approximation of the mod 2 cohomology of the homotopy quotient of mapping spaces from an odd dimensional sphere S^n to an arbitrary space Z by the action of the special orthogonal group $SO(n+1)$. Our approximation is constructed using the structure of the cohomology of Z as an algebra over the Steenrod algebra, and we prove that it agrees with the actual cohomology when Z is an Eilenberg Mac Lane space whose homotopy groups are finite type $\mathbb{F}_2$-modules. Our construction can be thought of as an analog of negative cyclic homology for higher dimensional spheres that takes into account an action of the Steenrod algebra, and it generalizes a construction of Ottosen and Bokstedt for $n = 1$. We also include an appendix where we give explicit formulas for computing a related algebraic approximation of the mod 2 cohomology of arbitrary mapping spaces out of spaces with finite type mod 2 cohomology.

Contents

1. Introduction

Given spaces Z, Y, there is an algebraic way to compute mod p cohomology of the mapping space Z^Y. As input, we take the cohomology of Y and Z as objects of the category $\mathcal{K}$ of algebras over the Steenrod algebra. This category encodes all of the primary algebraic structure on $H^*(Z)$, namely, the multiplication and the Steenrod operations.

Since the mapping space functor is the right adjoint of the product functor $(-)\times Y$, a first guess is that its cohomology of $(-)^Y$ is given by the left adjoint of the functor $(-) \otimes H^*(Y) : \mathcal{K} \to \mathcal{K}$. Such an adjoint exists if $H^*(Y)$ has finite dimension in each degree, and following Lannes it is denoted $H^*(Z) : H^*(Y)$. By its universal property, we obtain a natural map

$$(H^*(Z) : H^*(Y)) \to H^*(Z^Y)$$

which for formal reasons is an isomorphism when Z is a finite product of Eilenberg Mac Lane spaces of the form $K(\mathbb{Z}/p, m)$.[1] Moreover, $H^*(Z) : H^*(Y)$'s universal property allows it to be computed explicitly in terms of generators and relations (see Appendix A).

By resolving Z by Eilenberg Mac Lane spaces, one obtains a Bousfield-Kan homology spectral sequence whose E_2 term is the derived functors of $(-) : H^*(Y)$ conditionally converging to $H^*(Z^Y)$ [BO04].

One can wonder if there is a similar story equivariantly. For example if Y is a G-space, how can one compute the Borel cohomology $H^*(Z^Y_{hG})$ where hG denotes the homotopy quotient by G?

In [BO99; Ott03], Ottosen and Bokstedt gave an answer in the case that $Y = S^1$ equipped with the natural $G = SO(2)$ action by rotation. They produced an approximation functor ℓ of the S^1 equivariant cohomology of Z^{S^1} that depends on the structure of the cohomology of Z as an algebra over the Steenrod algebra. Their functor ℓ admits a natural approximation map $\eta : \ell(H^*(Z)) \to H^*(Z^Y_{hS^1})$ that is an isomorphism when $H^*(Z)$ is a polynomial algebra.

This project grew out of an attempt to understand their construction and the extent to which it generalizes. From now on, cohomology has coefficients in $\mathbb{F}_2$.

We consider $G = SO(n+1)$ acting on $Y = S^n$ in the standard way.

[1] In the case of S^1, one can do a bit better than in general. In [BO99], the approximation map is shown to be an isomorphism when $H^*(Z)$ is a polynomial algebra.

We show that this action induces a coaction of $H^*(S^n)$ on $H^*(Z) : H^*(Y)$ such that the map $H^*(Z) : H^*(Y) \to H^*(Y^Z)$ is a map of comodules in Proposition 4.7.

For odd n, we construct a functor $\ell_n : \mathcal{K} \to \mathcal{K}$ that approximates the cohomology of $Z^{S^n}_{hSO(n+1)}$ generalizing the Ottosen and Bokstedt's construction for $n = 1$. The space $Z^{S^n}_{hSO(n+1)}$ is part of a fibre sequence $Z^{S^n} \to Z^{S^n}_{hSO(n+1)} \to BSO(n+1)$, and as a consequence its cohomology is an algebra over $H^*BSO(n+1)$, and has a filtration coming from the Serre spectral sequence. We denote the category of objects with such structure $\mathcal{K}^n_f$. Now we state the main result.

Definition 1.1. A space Z is an **$\mathbb{F}_2$-EM space** if it is a generalized Eilenberg-Mac Lane space whose homotopy groups are finite dimensional $\mathbb{F}_2$-vector spaces.

Theorem A. *For odd n, there is a functor $\ell_n : \mathcal{K} \to \mathcal{K}^n_f$ and a natural transformation $\eta_n : \ell_n(H^*(Z)) \to H^*(Z^{S^n}_{hSO(n+1)})$, such that η_n is an isomorphism when Z is an $\mathbb{F}_2$-EM space.*

One can think of our functor ℓ_n as being built to make η_n an isomorphism for $\mathbb{F}_2$-EM spaces, and so reflects the fact that $H^*(Z^{S^n}_{hSO(n+1)})$ is understood in a functorial way for $\mathbb{F}_2$-EM spaces.

In order to understand the Borel cohomology of Z^{S^n} we study the Serre spectral sequence of the fibration

$$Z^{S^n} \to Z^{S^n}_{hSO(n+1)} \to BSO(n+1).$$

We show that the 'obvious' differentials on the E_{n+1}-page arising from the coaction of $H^*(SO(n+1))$ on $H^*(Z^{S^n})$ are the only ones when Z is an $\mathbb{F}_2$-EM space. To do this, we explicitly construct classes representing survivors. The interesting part of this construction uses the fact that the map $Z^{S^n} \to Z^2$ evaluating at antipodal points is $O(n)$-equivariant, where $SO(n) \subset O(n)$ acts trivially on Z^2. This equivariant evaluation map was studied in the case of S^1 in [Ott00], and they use it centrally in the construction of their approximation.

Since n is odd, the group $O(n)$ splits as a product of $SO(n)$ and C_2. Our classes are then constructed as the composite map

$$H^*(Z^2_{hC_2}) \otimes H^*(BSO(n)) \cong H^*(Z^2_{hO(n)}) \to H^*(Z^{S^n}_{hO(n)}) \to H^*(Z^{S^n}_{hSO(n+1)})$$

where the first map is the evaluation and the second is the pushforward in cohomology of the fibration $Z^{S^n}_{hO(n)} \to Z^{S^n}_{hSO(n+1)}$ with fibre $\mathbb{RP}^n = SO(n+1)/O(n)$.

Remark 1.1.1. It is easy to see that for any $SO(n+1)$-space, $H^*(X_{hO(n)})$ is a free module over $H^*(X_{hSO(n+1)})$ and the pushforward is a surjective module map (see Lemma 4.1). Thus the surprising step in the construction is that all of the needed classes in $H^*(Z^{S^n}_{SO(n+1)})$ can be constructed from the map $H^*(Z^{S^n}_{hO(n)}) \to Z^2_{hO(n)}$.

1.1. **Outline of paper.** This paper is organized as follows:

In section 2, we outline the proof of Theorem A, deferring several results to be proven later on.

In section 3, we set up notation and explain the nonequivariant version of the story, which is needed for the rest of the paper.

In section 4, we study some $SO(n+1)$ actions relevant to the main results of the paper.

In section 5, we set out to construct classes of $H^*(Z^{S^n}_{hSO(n+1)})$, and find relations among them.

In section 6, we study the spectral sequence $Z^{S^n} \to Z^{S^n}_{hSO(n+1)} \to BSO(n+1)$, showing that it degenerates at E_{n+2}, and giving a purely algebraic description of it.

In section 7 we give some concluding remarks and further directions. Additionally, we suggest approaches to finding a universal property of ℓ_n.

Finally section A is an appendix, where we explain in depth how to compute the functors $(-) : M$ in general, as well as examine what structure is actually required to compute the underlying $\mathbb{F}_2$-vector space of the functor $(-) : M$.

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2. Outline of the main argument

The goal of this section is to outline the proof of Theorem A, deferring much of the work to the rest of the paper. In particular, all of the Lemmas and Propositions are proven in later sections, and here they are just used to prove the main theorems.

2.1. **Notation.** Here we set up some notation that is used for the rest of the paper. More specific notation is introduced in later sections.

$\mathcal{A}$ denotes the mod 2 Steenrod algebra, and Sq^i the i^{th} Steenrod square. We use $Sq_i x$ to mean $Sq^{|x|-i}x$, where $|x|$ is the degree of x.

Our cohomology lives in $\mathcal{K}$, the category of unstable algebras over the Steenrod algebra. For more about this, see [Sch94]. We recall that objects in $\mathcal{K}$ are algebras equipped with a compatible action of $\mathcal{A}$. In particular, we require $Sq_0 a = a^2$ and $Sq_i a = 0$ for $i < 0$.

We say that a graded $\mathbb{F}_2$-vector space is **finite type** if it is finitely generated as an $\mathbb{F}_2$-vector space in each dimension. In particular this makes sense for objects in $\mathcal{K}$.

If an arbitrary space Y has a G-action, Y_{hG} denotes the homotopy quotient of Y by the action. Y_0 denotes Y except with the trivial action. The action map $\alpha : Y_0 \times G \to Y$ is G-equivariant.

Let $K(V_*)$ denote the generalized Eilenberg Mac Lane space with homotopy groups V_* where V_* is a nonnegatively graded elementary abelian 2-group. When V_* is $\mathbb{F}_2$ in dimension m and 0 otherwise, this is also be denoted $K(\mathbb{Z}/2, m)$.

When mixing homological/homotopical notation with cohomological notation, we view homology as having nonpositive degree. For example, in degree i, $\pi_*(Z) \otimes H^*(Y)$ is $(\oplus_{j\in\mathbb{Z}} \pi_j(Z) \otimes H^{i+j}(Y))$. For a graded vector space V, the dual V^* is taken nonpositively graded.

2.2. **Proof.** Throughout this section Z is an arbitrary space, and we let X be Z^{S^n}. Our main tool to understand it is the Serre spectral sequence of the Borel fibration $X \to X_{hSO(n+1)} \to BSO(n+1)$, which is referred to as the **main spectral sequence**.

We define $L_n(H^*(Z))$ to be $H^*(Z) : H^*(S^n)$. For formal reasons, it has an explicit presentation.

Proposition 3.3. *As an algebra over A, $L_n(A) = A :_{\mathcal{K}} H^*(S^n)$ is generated by generators da in degree $|a| - n$ for each $a \in A$, along with the relations:*

1. *$d(a + b) = da + db$.*
2. *$d(ab) = d(a)b + bd(a)$.*
3. *$(da)^2 = d(Sq_n a)$.*
4. *$d(Sq_i a) = 0$ for all $n > i \geq 0$.*

The action of the Steenrod algebra is determined by $Sq^i(da) = d(Sq^i a)$ and the Cartan formula, and the universal map $A \to L_n(A) \otimes H^*(S^n)$ sends $a \mapsto a \otimes 1 + da \otimes y$ where y is the nontrivial element of $H^n(S^n)$.

For formal reasons (Proposition 3.2), there is a natural transformation $L_n(H^*(Z)) \to H^*(X)$ that is an isomorphism for an $\mathbb{F}_2$-EM space.

The cohomology of $BSO(n + 1)$ is a polynomial algebra on the Stiefel-Whitney classes $w_2, \ldots, w_{n+1}$, and we know that $H^*(X)$ receives a map from $L_n(H^*(Z))$, which has generators that are of the form a and da for $a \in H^*(Z)$. To compute the associated graded of $H^*(X_{hSO(n+1)})$, we study the differentials. To begin with, the E_2 page is $E_2^{p,q} = H^q(X) \otimes H^p(BSO(n + 1))$.

$$
\begin{tikzcd}[row sep=small]
2m-i & Sq_i a & & & & & \\
m & a \arrow[rrrrr] & & & & & {} \\
 & Sq_1 da & & & & & \\
m-n & da & & & & & da \otimes w_{n+1} \\
 & & w_2 & w_3 & w_4 & \cdots & w_{n+1} \\
 & & 2 & 3 & 4 & \cdots & n+1
\end{tikzcd}
$$

Fortunately, when Z is an $\mathbb{F}_2$-EM space, there is only one kind of differential, which happens on the E_{n+1}-page. This is the content of Proposition 5.7 and Corollary 2.1 below.

Proposition 5.7. *For Z an $\mathbb{F}_2$-EM space, there are no differentials in the main spectral sequence until the E_{n+1} page. On the E_{n+1} page, the differential is given by $d_{n+1}(a) = w_{n+1} da$ for $a \in H^*(Z^{S^n})$.*

Since $d(da) = 0$, $d(Sq_i(a)) = Sq_{i-n} da = 0$ for $i < n$, and $d(Sq_n(a) + ada) = Sq_0 da + (da)^2 = 0$, the classes $da, Sq_n a + ada$ and $Sq_i a$ for $i < n$ survive as a varies.

Proposition 6.1. *The classes $da, Sq_i a, Sq_n a + ada$, $a \in H^*(Z)$ along with the w_i generate the E_{n+2} page of the main spectral sequence for Z an $\mathbb{F}_2$-EM space.*

Thus to show that the spectral sequence has no more differentials, it suffices to show that the generators in Proposition 6.1 are realized by cohomology classes in $H^*(Z)$. By the Leibniz rule, this shows that all differentials are zero starting from the E_{n+2} page.

To see that the w_i survive, one notes that there is a section of the fibration $Z^{S^n}_{hSO(n+1)} \to BSO(n+1)$, given by choosing a fixed point of Z^{S^n} given by any constant map. Thus the map $H^*(BSO(n+1)) \to H^*(Z^{S^n}_{hSO(n+1)})$ is the inclusion of a retract, so in particular injective.

To see the rest of the classes survive, we construct cohomology classes that realize them. In section 5.2, we construct the classes $\delta(a)$, $\phi_i(a)$ for $0 \leq i \leq n$ in $H^*(X_{hSO(n+1)})$. These classes are actually constructed more generally: Z does not have to be an $\mathbb{F}_2$-EM space. Let Space be the category of spaces.

Proposition 5.3. *Consider the functors $H^j(Z), H^j(Z^{S^n}_{hSO(n+1)}), H^j(Z^{S^n}) : \mathrm{Space} \to \mathrm{Ab}$. There are natural transformations (of underlying abelian groups) $\delta : H^j(Z) \to H^{j-n}(Z^{S^n}_{hSO(n+1)}$, $\phi_i : H^j(Z) \to H^{2j-i}(Z)$ such that if $q^* : H^*(Z^{S^n}_{hSO(n+1)}) \to H^*(Z^{S^n})$ is the natural transformation induced from the quotient map, then $q^*(\delta(a)) = da$, $q^*(\phi_i(a)) = Sq_i a$ for $i < n$, and $q^*(\phi_n(a)) = Sq_n a + ada$, where a, da are the classes from $L_n(Z)$ in Proposition 3.2.*

Corollary 2.1. *For an $\mathbb{F}_2$-EM space, the main spectral sequence has no differentials starting from the E_{n+2}-page, so $E_{n+2} = E_\infty$.*

At this point we already know the associated graded of $H^*(X_{hSO(n+1)})$ for Z an $\mathbb{F}_2$-EM space: it is the the cohomology of an explicit fairly simple bigraded DGA given by the E_{n+1} differential. The rest of the proof amounts to lifting that to a statement about the actual cohomology.

Theorem A. *For odd n, there is a functor $\ell_n : \mathcal{K} \to \mathcal{K}^n_f$ and a natural transformation $\eta_n : \ell_n(H^*(Z)) \to H^*(Z^{S^n}_{hSO(n+1)})$, such that η_n is an isomorphism when Z is an $\mathbb{F}_2$-EM space.*

Proof of Theorem A. Our goal is to find all the universal relations that hold in $H^*(Z^{S^n}_{hSO(n+1)})$ among the classes $\delta(a), \phi_i(a), w_i$ for $a \in H^*(Z)$. Since the classes $\delta(a), \phi_i(a), w_i$ are natural with respect to Z, it suffices to prove any functorial relation among them in a universal space Z, which is an $\mathbb{F}_2$-EM space. In the case of an $\mathbb{F}_2$-EM space, we are not able to always obtain explicit relations, but we can nevertheless prove that many universal relations exist.

We know by Proposition 6.1 that the associated graded algebra is generated by the images of the w_i and $\delta(a)$ and $\phi_i(a)$ for $a \in H^*(Z)$. Let x_i be some variables, and let ξ be the map that takes a class in $H^*(X_{hSO(n+1)})$ and outputs what detects it in the main spectral sequence. Suppose that there is some expression $E(x_i, \xi(\delta)(-), \xi(\phi_i)(-), \xi(w_i)$ built using products, sums and Steenrod operations that is equalt to 0 in the spectral sequence. Then $E(x_i, \delta(-), \phi_i(-), w_i)$ is of higher filtration, so by Proposition 6.1 is of the form $E'(x_i, \delta(-), \phi_i(-), w_i)$ where E' is an expression that has higher filtration than the lowest filtration appearing in E.

The next step is then to observe some relations that hold in the main spectral sequence. Consider the relations below

(1) $\phi_i(a+b) = \phi_i(a) + \phi_i(b)$
(2) $\delta(a+b) = \delta(a) + \delta(b)$
(3) $\delta(ab)\delta(c) + \delta(bc)\delta(a) + \delta(ca)\delta(b) = 0$
(4) $w_{n+1}\delta(a) = 0$
(5) $\delta(a)\phi_i(b) = \delta(aSq_i b) + \delta_{0n}\delta(ab)\delta(b)$
(6) $\phi_k(ab) = \sum_{i+j=k} \phi_i(a)\phi_j(b)$

(7) $Sq\phi_{n-k}(a) = \sum_{j\geq 0}\sum_i \binom{k+|a|-j}{i-2j}\phi_{n-i-k+2j}(Sq^i a) + \delta_{k0}\sum_{2j<i}\delta(Sq^j a \times Sq^{i-j}a)$
(8) $Sq^i\delta(a) = \delta(Sq^i a)$
(9) $Sq_i(\delta(a)) = Sq_i(\phi_j(a)) = 0$ for $i < 0$
(10) $Sq_0(\delta(a)) = \delta(a)^2$, $Sq_0(\phi_j(a)) = \phi_j(a)^2$
(11) $\phi_k(Sq_i a) = \sum_{j=0}^{(2|a|-i-k)/2}\binom{|a|-i-j-1}{2|a|-i-k-2j}\phi_{2j-2|a|+2i+k}(Sq^j a) + \delta_{k,n}\phi_i(a)\delta(Sq_i a)$ for $k > i$
(12) $\phi_0(Sq_j a) = \phi_j(x)^2 + \delta_{j,n}\delta(a^2 Sq_n a)$.

We would like to verify that they hold in the spectral sequence after applying ξ. We emphasize that many of these do **not** hold among the classes, but only hold in the associated graded algebra.

Relation (7) is the only one that is difficult to verify.

Lemma 5.6. *The relations*

$$Sq\phi_{n-k}(a) = \sum_{j\geq 0}\sum_i \binom{k+|a|-j}{i-2j}\phi_{n-i-k+2j}(Sq^i a) + \delta_{k0}\sum_{2j<i}\delta(Sq^j a \times Sq^{i-j}a)$$

$$Sq^i\delta(a) = \delta(Sq^i a)$$

hold.

We now explain the rest of the relations:

(1) This is essentially linearity of Sq_i.
(2) This is linearity of d.
(3) This follows from the Liebniz rule for d.
(4) This follows since the d_{n+1} differential applied to a is $w_{n+1}da$.
(5) We can calculate $d(a)Sq_i(b) = d(aSq_i b)$ for $i < n$ and $d(a)(Sq_n(b) - bdb) = d(ab)db$
(6) This is essentially the Cartan formula.
(7) Lemma 5.6.
(8) Lemma 5.6, or alternatively the fact that Sq_i commutes with d.
(9) This is the instability condition.
(10) This is the compatibility of Sq_0 with the algebra structure.
(11) This is the Adem relations.
(12) This follows from the fact that $Sq_0(Sq_i a) = (Sq_i a)^2$.

Having verified these relations, there are expressions lifting the relations $(1)-(12)$ holding among the real classes $\phi_i(-), \delta(-), w_i$. Call these relations E_i for $1 \leq i \leq 12$.

Definition 2.2. For an object $A_* \in \mathcal{K}_f^n$, $\ell_n(A_*)$ is defined to be the tensor product of $H^*(BSO(n+1))$ with the free object in $\mathcal{K}$ on the following generators for each $a \in A_m$:

- $\delta(a)$ with degree $m - n$,

- $\phi_i(a)$ with degree $2a - i$ for $0 \leq i \leq n$.

We give ℓ_n has a multiplicative filtration by declaring $\phi_i(a)$ and $\delta(a)$ to be in filtration 0, and w_i in filtration i, and we impose the relations $E_1, \ldots, E_{12}$.

We have abused notation by using the same symbols to represent classes in $\ell_n(A)$ and classes in $H^*(X_{hSO(n+1)})$. The natural transformation η_n justifies this since it identifies the two uses.

Definition 2.3. $\eta_n : \ell_n(H^*(Z)) \to H^*(X_{hSO(n+1)})$ is defined as the map in $\mathcal{K}_f^n$ sending $\phi_i(a)$ to $\phi_i(a)$, $\delta(a)$ to $\delta(a)$ and w_i to w_i.

It is well defined: the relations $E_1 \ldots E_{12}$ are constructed so that they hold in $H^*(X_{hSO(n+1)})$, so since ℓ_n is given by a presentation, the map on tbe generators induces one from ℓ_n. The map preserves filtrations since it does on generators, and is clearly a map of $H^*(BSO(n+1))$ algebras.

To see that η_n is an isomorphism, since it is a filtered map, and the filtration is finite for an $\mathbb{F}_2$-EM space, it suffices to check that the associated graded map is an isomorphism. Let

$$\hat{\eta}_n : \hat{\ell}_n(H^*(Z)) \to E_\infty(Z),$$

be the associated graded map, where $\hat{\ell}$ is the associated graded of ℓ and $E_\infty(Z)$ denotes the associated graded of $H^*(Z^{S^n}_{hSO(n+1)})$ from the filtration coming from the map $Z^{S^n}_{hSO(n+1)} \to BSO(n+1)$ which is the E_∞-page of the main spectral sequence.

The proof then reduces to the result below.

□

Proposition 6.13. *$\hat{\eta}_n$ is an isomorphism when Z is an $\mathbb{F}_2$-EM space.*

3. Ordinary Cohomology

The content of this section is mostly unoriginal, but rather put here to set up notation, for convenience, and to outline the perspective from which we intend to generalize to an equivariant setting. The goal of this subsection is to understand $H^*(Z^Y)$ in terms of $H^*(Z)$.

The main observation, as outlined in the introduction, is to define our approximation $H^*(Z) : H^*(Y)$ by interpreting the universal property of the space Z^Y on the level of cohomology. The mapping space functor $(-)^Y$ is the right adjoint in spaces of $(-) \times Y$, taking the product with Y. Thus we might guess that $H^*((-)^Y)$ is the left adjoint in $\mathcal{K}$ of the functor $H^*(Y) \otimes (-)$ in $\mathcal{K}$. When $H^*(Y)$ is finite dimensional in each degree, then this adjoint is known to exist [Sch94], and is denoted $(-) : H^*(Y)$ or $(-) :_{\mathcal{K}} H^*(Y)$. This functor is computable, and we give explicit formulas for computing it in Section A.

The following results are formal and well known. By convention here, V_i is in cohomological degree $-i$.

Lemma 3.1. *Suppose V_* is a graded $\mathbb{F}_2$-vector space and let $K(V_*)$ denote the Eilenberg-Mac Lane space with homotopy groups V_* (truncated to be in nonnegative degrees). There is a natural homotopy equivalence $K(V_*)^Y \simeq K(V_* \otimes H^*(Y))$.*

Proposition 3.2. *Suppose that Y has finite type cohomology. There is a natural map $H^*(Z) : H^*(Y) \to H^*(Z^Y)$ that is an isomorphism when $Z = K(V_*)$ and either $H^*(Y)$ or V_* is finite.*

All the finiteness hypotheses are only needed because we are working with cohomology. Two ways they can be removed are by either working with homology, or by working in a profinite setting.

Next, we explain how to compute $A : H^*(S^n)$, which we denote $L_n(A)$ for $A \in \mathcal{K}$. The result is a special case of a more general computation of division functors in the appendix.

Proposition 3.3. *As an algebra over A, $L_n(A) = A :_{\mathcal{K}} H^*(S^n)$ is generated by generators da in degree $|a| - n$ for each $a \in A$, along with the relations:*

(1) $d(a + b) = da + db$
(2) $d(ab) = d(a)b + ad(b)$
(3) $(da)^2 = d(Sq_n a)$
(4) $d(Sq_i a) = 0$ *for all* $n > i \geq 0$.

The action of the Steenrod algebra is determined by $Sq^i(da) = d(Sq^i a)$ and the Cartan formula, and the universal map $A \to L_n(A) \otimes H^*(S^n)$ sends $a \mapsto a \otimes 1 + da \otimes y$ where y is the nontrivial element of $H^n(S^n)$.

Proof. Consider the functor UF taking a graded $\mathbb{F}_2$-vector space V to the free object on V in $\mathcal{K}$. Let $N = H^*(S^n)$. Then $\mathrm{Hom}(UF(V_*) : N, M) = \mathrm{Hom}(UF(V_*), N \otimes M) = \mathrm{Hom}_{\mathbb{F}_2}(UV_*, N \otimes M) = \mathrm{Hom}_{\mathbb{F}_2}(UV_* \otimes N^*, M) = \mathrm{Hom}(UF(V_* \otimes N^*), M)$, so by the Yoneda Lemma, there is an isomorphism $UF(V_*) : N \cong UF(V_* \otimes N^*)$.

There is a unique graded basis $1, d$ for N^*, and so the generators in $UF(V_* \otimes N^*)$ of the form $v \otimes 1$ we call v, and the ones of the form $v \otimes d$ we call dv. Moreover, following the isomorphisms show that the coevaluation is given by $v \mapsto v \otimes 1 + dv \otimes y$. More generally, given $a \in A \in \mathcal{K}$, giving a class $a \in A_n$ is the same as a map $a : UF(\Sigma^n \mathbb{F}_2) \to A_n$, and we define a and da to be the images of the corresponding classes after applying $(-) : N$. Clearly relation (1) holds.

Applying Sq^i on the coevaluation map, we see $Sq^i a \mapsto Sq^i a \otimes 1 + Sq^i(da) \otimes y$ so we must have $dSq^i a = Sq^i da$. From this it follows that relations (3) and (4) hold. Furthermore, applying the formula for the coevaluation on a product, we find $ac \mapsto (a \otimes 1 + da \otimes y)(c \otimes 1 + dc \otimes y)$, yielding relation (2).

Now an arbitrary algebra A can be presented canonically as a pushout

$$\begin{tikzcd} UF(A \otimes A) \otimes UF(\bigoplus_0^\infty \Sigma^i A) \arrow[r] \arrow[d] & UF(A) \arrow[d] \\ 0 \arrow[r] & M \end{tikzcd}$$

The vertical nonzero map is given by $[a] \mapsto a$ and the horizontal one sends $[a \otimes b]$ to $[a][b] - [ab]$, and $a \in \Sigma^i M$ to $Sq^i[a] - [Sq^i a]$.

Applying $(-) : N$, since $(-) : N$ preserves pushouts, we get a pushout square

$$\begin{tikzcd} UF(A \otimes A \otimes N^*) \otimes UF(\bigoplus_0^\infty \Sigma^i A \otimes N^*) \arrow[r] \arrow[d] & UF(A \otimes N^*) \arrow[d] \\ 0 \arrow[r] & A : N \end{tikzcd}$$

This gives a presentation for $A : N$ as the quotient of $UF(A \otimes N^*)$ by relation (2), and the additional relation $Sq^i(da) = dSq^i(a)$.

From this we can extract a presentation for $A : N$ as an algebra. The fact that $Sq^i da = dSq^i a$ simplifies things: The fact that $Sq_i a = 0$ for $i < 0$ and $Sq_0 a = a^2$ reduces to relations (3) and (4). This presentation then reduces to the presentation of an algebra: a and da are generators for $a \in A$ with relations $(1) - (4)$. □

4. $SO(n+1)$ actions

In this section, we collect relevant facts about spaces with $SO(n+1)$-actions, to set up our understanding of the $SO(n+1)$ action on the mapping space from S^n. The eager or familiar reader may wish to read later sections first. Throughout this section, X denotes a left or right $SO(n+1)$ space.

4.1. **Pushforward.** An important construction we use later is that of the pushforward on cohomology. It makes sense in great generality, but we are interested in it for a fibration $f : E \to B$ whose fibre F is a compact manifold of dimension n. The pushforward, denoted $f_* : H^*(E; H^n(F)) \to H^{*-n}(B)$ can be defined in two equivalent ways [CK07].

The first description we need is via the normal bundle ν of the fibration $E \to B$. This normal bundle ν is a virtual bundle of dimension $-n$ on E, whose fibre is the negative of the vectors tangent to the fibre. Then there is a Gysin map $\Sigma^\infty_+ B \to E^\nu$, where E^ν is the Thom spectrum of ν. We can use the composite $H^*(E) \cong H^{*-n}(E^\nu) \to H^{*-n}(B)$, where the first map is the Thom isomorphism, to obtain the pushforward.

The second description works when the fibre F is connected, and is defined via the Serre spectral sequence of the fibration. The E_2 term is $H^p(B; H^q(F))$. $H^n(F)$ is $\mathbb{Z}/2\mathbb{Z}$ (with trivial coefficients) by our connectedness assumption, and $H^i(F) = 0$ for $i > n$, so the q^{th} column of the E_∞ page of the spectral sequence is a quotient of $H^*(E)$. The E_∞ page injects into the E_2 page, so the composite $H^*(E) \to E_\infty^{*-n,n} \to E_2^{*-n,n} = H^{*-n}(B)$ gives the desired map.

The pushforward satisfies some nice properties. What we need are

(1) f_* is a map of $H^*(B)$ modules.
(2) f_* is natural with respect to pullback fibrations.
(3) f_* is functorial.

We are primarily interested in the following situation. Suppose G, H are compact Lie groups. If X is a G-space and $H \subset G$ is a subgroup, then there is a fibration $X_{hH} \to X_{hG}$ with fibre G/H. The pushforward of this fibration is be denoted τ_H^G.

4.2. **Coactions and comparisons.** The group $O(n)$ sits inside $SO(n+1)$ as the subgroup fixing (as a set) any given 1-dimensional subspace of the $n+1$-dimensional real vector space $SO(n+1)$ naturally acts on. First we compare the $O(n)$ quotient and the $SO(n+1)$ quotient.

Lemma 4.1. *The map $X_{hO(n)} \to X_{SO(n+1)}$ realizes $H^*(X_{hO(n)})$ as*

$$H^*(BO(n)) \otimes_{H^*BSO(n+1)} H^*(X_{hSO(n+1)}),$$

which is a free module over $H^*(X_{hSO(n+1)})$ generated by the classes $1, w_1, \ldots, w_n$ (alternatively $1, w_1, \ldots, w_1^n$). Moreover, w_i in $H^*(X_{hSO(n+1)})$ is sent to $w_i + w_{i-1}w_1$ in $H^*(X_{hO(n)})$.

Proof. First consider the case when X is a point. In this case, the map is the fibration $BO(n) \to BSO(n+1)$, given by taking an n-plane bundle and adding a copy of its determinant bundle. By the Whitney sum formula, we get that $\sum_0^{n+1} w_i$ in $H^*(BSO(n+1))$ pulls back to $(\sum_0^n w_i)(1 + w_1)$. Thus w_i pulls back to $w_i + w_1 w_{i-1}$, where we interpret w_i to be 0 when it is 0 for the universal bundle. The fibre of the map is $\mathbb{RP}^n$, where the map to $BO(n)$ is the orthogonal complement of the tautological bundle. Thus by the Whitney sum formula, its total Stiefel-Whitney class is $\frac{1}{t+1} = 1 + t + \cdots + t^n$. Thus w_i get sent to a basis of $H^*(\mathbb{RP}^n)$. Now considering the Serre spectral sequence of the fibration $\mathbb{RP}^n \to BO(n) \to BSO(n+1)$, and comparing Poincare polynomials, there is no room for differentials, so the spectral sequence degenerates at E_2, and the w_i generate $H^*(BO(n))$ freely. The basis w_i is related to the classes w_1^i via a triangular matrix with diagonal entries 1, so both families work as a basis.

For the general case, consider the Eilenberg-Moore spectral sequence of the pullback square

$$\begin{tikzcd} X_{hO(n)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_{hSO(n+1)} \arrow[d] \\ BO(n) \arrow[r] & BSO(n+1) \end{tikzcd}$$

Since $H^*(BO(n))$ is a free module, the E_2-term is concentrated in one line and is $H^*(BO(n)) \otimes_{H^*BSO(n+1)} H^*(X_{hSO(n+1)})$. Thus the spectral sequence collapses at E_2, and gives the result we want. □

Next, we study the left action of $SO(n+1)$ on S^n. Let y denote the generator of $H^*(S^n)$, and x_i the odd degree generators of $H^*(SO(n+1))$.

Lemma 4.2. *The Serre spectral sequence of the fibration $SO(n) \to SO(n+1) \to S^n$ has no differentials. Moreover the class y pulls back to $x_i^{2^p}$ where $i2^p = n$.*

Proof. The first statement follows from comparing Poincaré polynomials. Since the Serre spectral sequence degenerates, y must generate the kernel of the map $H^*(SO(n+1)) \to H^*(SO(n))$, but this kernel has a unique generator which is the class indicated in the statement of the lemma. □

Lemma 4.3. *The coaction of $H^*(SO(n+1))$ on $H^*(S^n)$ is given by $y \mapsto y \otimes 1 + 1 \otimes x_i^{2^p}$ where $i2^p = n$.*

Proof. Since the action map $SO(n+1) \times S^n \to S^n$ is unital and $H^*(S^n)$ is 2-dimensional, y has to get sent to $y \otimes 1 + 1 \otimes c$ for some class c. The class c is then the image of y in the composite $SO(n+1) \to SO(n+1) \times S^n \to S^n$. But the composite map is the fibration in Lemma 4.2, so we are done by that lemma. □

Lemma 4.4. *In the Serre spectral sequence of the path space fibration of $SO(n+1) \to \cdot \to BSO(n+1)$, the class $x_i^{2^p}$ transgresses to hit w_{i2^p+1}.*

Proof. By induction on n using the map $BSO(k) \to BSO(k+1)$ induced from the inclusion and comparing Serre spectral sequences for the path space fibrations, we only need to show this for $x_i^{2^p}$ where $2^p = n$. This class pulls back to 0 in $H^*(SO(n))$, so by comparing the path space fibrations, it cannot have any differential other than a transgression. Via the transgression it must hit the class w_{n+1} as it is the only class left on the $E_{n+1}^{n+1,0}$ and the E_∞ page just has $\mathbb{F}_2$ in bidegree $0,0$.

However this is the only class in dimension n whose differential has not inductively been computed, and w_{n+1} is the only class in dimension $n+1$ that has not been hit by a differential. □

Here is another related spectral sequence we use later. It can also be used to give an alternate proof that y pulls back to x^{2^p} in Lemma 4.2.

Lemma 4.5. *The Serre spectral sequence of the fibration $SO(n) \to S^{n-1} \to BSO(n-1)$ has differentials sending $x_i^{2^p}$ to w_{i2^p+1} for $i < n$.*

Proof. Consider the comparison map from this spectral sequence to the one in Lemma 4.4. ☐

4.3. **The mapping space.** We now compute the coaction and evaluation map relevant to the mapping space Z^{S^n} where Z is a space. Recall that given a class a in $H^*(Z)$, there are corresponding classes a, da in $H^*(Z^{S^n})$.

Lemma 4.6. *The coevaluation map $H^*(Z) \to H^*(S^n \times Z^{S^n})$ on cohomology sends the class a to $a \otimes 1 + da \otimes y$.*

Proof. The cohomology of Z sits in the cohomology of Z^{S^n} by the evaluation map $Z^{S^n} \to Z$ at a fixed basepoint of S^n, with section given by the constant map from S^n to a point of Z. In particular this implies that the $x \otimes 1$ term of the image of a in $H^*(S^n \times Z^{S^n})$ is $a \otimes 1$. da is defined to be the class of Z^{S^n} such that a is sent to $a \otimes 1 + da \otimes y$.

$$\begin{tikzcd} H^*(S^n \times Z^{S^n}) \arrow[r] & H^*(Z^{S^n}) \\ H^*(Z) \arrow[u] \arrow[ur] & \end{tikzcd}$$

☐

Proposition 4.7. *The coaction $H^*(Z^{S^n}) \to H^*(Z^{S^n}) \otimes H^*(SO(n+1))$ sends $da \mapsto da \otimes 1, a \mapsto a \otimes 1 + da \otimes x_i^{2^p}$ where $i2^p = n$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} Z^{S^n} \times S^n \arrow[rr, "ev"] & & Z \\ Z^{S^n} \times SO(n+1) \times S^n \arrow[u, "\alpha_L"] \arrow[r, "\alpha_R"] & Z^{S^n} \times S^n \arrow[ur, "ev"'] & \end{tikzcd}$$

Where α_L is the left action on S^n and α_R is the right action on Z^{S^n}. We can pull back the class $a \in H^*(Z)$ to $H^*(Z^{S^n} \times SO(n+1) \times S^n$ along the top corner, which by Lemma 4.6 and Lemma 4.3 is given by $a \mapsto a \otimes 1 + da \otimes y \mapsto a \otimes 1 \otimes 1 + da \otimes y \otimes 1 + da \otimes 1 \otimes x_i^{2^p}$. Thus by commutativity, this is where α_R^* sends the class $a \otimes 1 + da \otimes y$, giving the result. ☐

Next we end with some computations that are needed later. $SO(n)^2_{hSO(n-1)}$ denotes the quotient by the diagonal action, and $SO(n)^2_{hSO(n-1)\times C_2}$ denotes the further quotient by the swap action.

Lemma 4.8. *The Serre spectral sequence of $SO(n) \to SO(n)^2_{hSO(n-1)} \to S^{n-1}$ degenerates at E_2, as does the Eilenberg Moore spectral sequence of the product of $SO(n)/SO(n-1)$ over $BSO(n-1)$. Moreover, the map $SO(n)^2 \to SO(n)^2_{hSO(n-1)}$ is injective on cohomology.*

Proof. For the first and last statement, one considers the map of spectral sequences coming from the following map of fibrations:

$$\begin{tikzcd} SO(n) \arrow[r] & SO(n)^2_{hSO(n-1)} \arrow[r] & S^{n-1} \\ SO(n) \arrow[r] \arrow[u, equal] & SO(n)^2 \arrow[r] \arrow[u] & SO(n) \arrow[u] \end{tikzcd}$$

Then one uses the injectivity of the map on cohomology and the lack of differentials in the target to conclude. For the second statement, since the map $S^{n-1} \to BSO(n-1)$ is injective on trivial on cohomology by Lemma 4.4, the E_2 term is $H^*(S^n) \otimes H^*(S^n) \otimes \mathrm{Tor}^{*,*}(H^*(BSO(n)))$, which cannot have any differentials, as it is the same dimension as the cohomology of $SO(n)^2_{hSO(n-1)}$. ☐

5. Equivariant evaluations

Throughout this section, $n \in \mathbb{N}$ is *odd*.

5.1. **Construction of the classes.** Here for an arbitrary space Z, we study classes in the cohomology of $Z^{S^n}_{hSO(n+1)}$. From the discussion in Section 2, we should expect that for $\mathbb{F}_2$-EM spaces, the cohomology should be generated by the classes w_i coming from $BSO(n+1)$, along with classes that under the map $Z^{S^n} \to Z^{S^n}_{hSO(n+1)}$ pullback to $\mathrm{Sq}^i a, i < n, \mathrm{Sq}^n a + ada, da$ where a is an arbitrary cohomology class in $H^*(Z)$ (see Section 3 for notation). Our goal is to construct classes $\phi_i(a), \delta(a)$ in $Z^{S^n}_{hSO(n+1)}$ that pull back to these (Proposition 5.3).

Our construction uses the **equivariant evaluation** maps. The first of these, ev_0 is the map $Z^{S^n} \to Z$ given by evaluating at a basepoint of S^n. Since $SO(n) \subset SO(n+1)$ is the stabilizer of a point in S^n, this map is $SO(n)$-equivariant, where we have given Z a trivial action.

The second is the map $ev_1 : Z^{S^n} \to Z^2$ given by evaluating at two antipodal points. The stabilizer of these points on S^n is $O(n) \subset SO(n+1)$, so this map is $SO(n+1)$-equivariant, where we equip Z^2 with the 'swap' $O(n)$ action that factors through the quotient $O(n)/SO(n)$.

The notation ev_0, ev_1 refers both to these maps as well as the maps on the homotopy quotients, and hopefully is not confusing.

Let π denote the projection map $Z^2_{hO(n)} = (Z^2 \times BSO(n))_{hO(n)/SO(n)} \to Z^2_{hO(n)/SO(n)}$

Recall that for $a \in H^*(Z)$, there is a unique class $P(a)$ in $H^*(Z^2_{hC_2})$ with the property that its pullback to Z^2 is $a \otimes a$, and its pullback to the fixed points $Z \times \mathbb{RP}^\infty$ is $\sum_i t^i \operatorname{Sq}_i(a)$.

Construction 5.0.1. Given a class $a \in H^*(Z)$, we define $\delta(a)$ to be the image of a in the composite

$$H^*(Z \times BSO(n)) = H^*(Z_{hSO(n)}) \xrightarrow{ev_0^*} H^*(Z^{S^n}_{hSO(n)}) \xrightarrow{\tau^{SO(n+1)}_{SO(n)}} H^{*-n}(Z^{S^n}_{hSO(n+1)}).$$

For $0 \leq i \leq n$ we define ϕ_i to be the image of $t^{n-i}P(a)$ in the composite

$$H^*(Z^2_{hC_2}) \xrightarrow{\pi^*} H^*(Z^2_{hO(n)}) \xrightarrow{ev_1^*} H^*(Z^{S^n}_{hO(n)}) \xrightarrow{\tau^{SO(n+1)}_{O(n)}} H^*(Z^{S^n}_{hSO(n+1)}).$$

First we show that $\delta(a)$ has the desired image in $H^*(Z^{S^n})$. Our approach for this kind of argument is via descent.

Lemma 5.1. *The image of $\delta(a)$ via the map $H^*(Z^{S^n}_{hSO(n+1)}) \to H^*(Z^{S^n})$ is the class da.*

Proof. The key observation is that the homotopy quotient of the action map $\alpha : Z_0^{S^n} \times SO(n+1) \to Z^{S^n}$ by $SO(n+1)$ is exactly the quotient map we want to understand $Z^{S^n} \to Z^{S^n}_{hSO(n+1)}$. Then we can consider the commutative diagram

$$\begin{array}{ccccc} H^*(Z_{hSO(n)}) & \xrightarrow{ev_0^*} & H^*(Z^{S^n}_{hSO(n)}) & \xrightarrow{\tau^{SO(n+1)}_{SO(n)}} & H^*(Z^{S^n}_{hSO(n+1)}) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\ H^*(Z) & \xrightarrow{ev^*} & H^*(Z_0^{S^n} \times S^n) & \xrightarrow{\tau^{SO(n+1)}_{SO(n)}} & H^*(Z_0^{S^n}). \end{array}$$

By the definition of $\delta(a)$, we are trying to understand the image of a from the top left corner to the bottom right corner. But following the lower part of the diagram, and using Lemma 4.6 this gives $a \mapsto a \otimes 1 + da \otimes y \mapsto da$. ☐

We use the same basic idea as in the previous lemma to show that $\phi_i(a)$ pull back to the classes we want, except there is more difficulty in making the argument succeed. In particular, the analog of the commutative square on the left requires more effort to produce.

We now construct a map involved in the analogous diagram, which uses the fact that n is odd. Since n is odd, $O(n) = SO(n) \times C_2$, where the generator of C_2 is the negative of the identity in $SO(n+1)$.

Construction 5.1.1. Suppose that Y carries an $SO(n+1)$-action. Then we can put a $SO(n)\times C_2 = O(n)$ action on Y^2, where $SO(n)$ acts diagonally on each factor, and C_2 swaps the two factors. Then there is a natural $O(n)$-equivariant map

$$f_Y : SO(n+1)\times Y_0 \to Y^2$$

given by $(\alpha, -\alpha)$ where α denotes the action map, and $-\alpha$ denotes the action but composed with the action of $-1 \subset SO(n+1)$. The corresponding map after applying $hO(n)$ is given the same name.

Now we explain why the $\phi_i(a)$ have the correct images in $H^*(Z^{S^n})$. The argument is similar to Lemma 5.1 in principle, except we have to use descent twice, and computing the maps involved are significantly more difficult.

Let Z' temporarily denote $Z_0^{S^n}\times SO(n+1)$. There is a commutative diagram as below where we have identified $SO(n+1)/O(n) = \mathbb{RP}^n$.

(1)
$$\begin{tikzcd}
H^*(Z^2_{hO(n)}) \arrow[r, "ev_1^*"] \arrow[d, "(ev_0^2)^*"] & H^*(Z^{S^n}_{hO(n)}) \arrow[r, "\tau^{SO(n+1)}_{O(n)}"] \arrow[d, "\alpha^*"] & H^*(Z^{S^n}_{hSO(n+1)}) \arrow[d, "\alpha^*"] \\
H^*((Z^{S^n})^2_{hO(n)}) \arrow[r, "f^*_{Z^{S^n}}"] \arrow[d, "(\alpha^2)^*"] & H^*(Z_0^{S^n}\times \mathbb{RP}^n) \arrow[r, "\tau^{SO(n+1)}_{O(n)}"] \arrow[d, hook, "1\times\alpha^*"] & H^*(Z_0^{S^n}) \\
H^*((Z')^2_{hO(n)}) \arrow[r, "f^*_{Z'}"] & H^*(Z'_0\times\mathbb{RP}^n) &
\end{tikzcd}$$

Computing the image of $\phi_i(a)$ in $H^*(Z^{S^n})$ is image of the class $t^{n-i}P(a)$ in the diagram above from $H^*(Z^2)_{hO(n)}$ to $H^*(Z_0^{S^n})$. The pushforward from $H^*(Z_0^{S^n}\times\mathbb{RP}^n)$ to $H^*(Z_0^{S^n})$ extracts the coefficient of t^n so is easy to understand, and we can focus our efforts on computing the map from $H^*(Z^2_{hO(n)})$ to $H^*(Z_0^{S^n}\otimes\mathbb{RP}^n)$.

However, the arrow indicated $1\otimes\alpha$ is an injection, so it really suffices to compute the map to $H^*(Z'_0\times\mathbb{RP}^n)$.

To do this, we need to understand the cohomology of $(Z')^2_{hO(n)}$. It admits a projection map to $(Z^{S^n})^2_{hC_2}$, and the pullback of a class $P(a)$ along this map is given the same name.

Furthermore there is a projection map to $H^*(SO(n+1)^2_{hO(n)})$ (where -1 acts by swapping the factors). We construct a class $P(y)\in H^*(SO(n+1)^2_{hO(n)})$ whose pullback to $(Z')^2_{hO(n)}$ is given the same name.

Lemma 5.2. *There is a unique class $P(y)\in H^*(SO(n+1)^2_{hO(n)})$ that pulls back to the class $y\otimes y$ in $SO(n+1)^2_{hSO(n)}$, and to 0 in $SO(n)^2_{hO(n)}$.*

Proof. Recall that y usually refers to class in the mod 2 cohomology of $H^*(S^n)$. Here, we have two projections π_1, π_2 from $SO(n+1)^2_{hSO(n)}$ to S^n, and the pullback of y along these projections are $y \otimes 1$ and $1 \otimes y$ respectively.

Now consider the following map of fibrations:

$$\begin{tikzcd} SO(n+1)^2 \arrow[r] \arrow[d] & SO(n+1)^2_{hC_2} \arrow[r] \arrow[d] & BC_2 \arrow[d] \\ SO(n+1)^2_{hSO(n)} \arrow[r] & SO(n+1)^2_{hO(n)} \arrow[r] & BC_2. \end{tikzcd}$$

The left and right vertical maps are injective on cohomology, so since all the classes of the form $a \otimes a$ of $H^*(SO(n+1)^2 \otimes H^0(BC_2)$ survive to E_∞ (because of $P(a)$), the same is true of classes in $H^*(SO(n+1)^2)$ that pullback to ones of the form $a \otimes a$, for example $y \otimes y$. By looking at the relative Serre spectral sequences with respect to $SO(n) \subset SO(n+1)$, this class is seen to be unique as described. ☐

Theorem 5.3. *Under the map $H^*(Z^{S^n}) \to H^*(Z^{S^n}_{hSO(n+1)})$, $\delta(a)$ is sent to da, $\phi_i(a)$ is sent to $Sq_i a$ for $i < n$ and $Sq_n a + ada$ for $i = n$.*

Proof. The case of $\delta(a)$ was treated in Lemma 5.1.

For $\phi_i(a)$, we first claim that the class $P(a) \in H^*(Z^2_{hC_2})$ is sent to the sum of the three terms $P(a) + \tau^{O(n)}_{SO(n)} a \otimes da \otimes 1 \otimes y + P(da)P(y)$ in $(Z')^2_{hO(n)}$.

To do this, we observe the map $Z'^2_{hO(n)} \to Z^2_{hC_2}$ factors through the map $(Z^{S^n} \times S^n)^2_{hC_2} \to Z^2_{hC_2}$ given by evaluation on each factor. Via the latter map, by Lemma 4.6 $P(a)$ pulls back to $P(a \otimes 1 + da \otimes y)$. $P(a \otimes 1 + da \otimes y)$ is equal to $P(a) + P(da)P(y) + \tau^{C_2}_*(a \otimes da \otimes 1 \otimes y)$, and so pulling back to $Z'^2_{hO(n)}$ yields the claim.

Next we study each of the three terms. We can compute the image of $P(a)$ in $Z_0^{S^n} \otimes \mathbb{RP}^n$ by examining the commutative square:

$$\begin{tikzcd} Z' \times \mathbb{RP}^n \arrow[r] \arrow[d] & Z'^2_{hO(n)} \arrow[d] \\ Z_0^{S^n} \times \mathbb{RP}^n \arrow[r] & (Z_0^{S^n})^2_{hC_2} \times BSO(n) \end{tikzcd}$$

$P(a)$ is pulled back from the class with the same name in $H^*((Z_0^{S^n})_{hC_2} \times BSO(n))$. That class pulls back to $\sum_i \mathrm{Sq}_i\, at^i$ in $H^*(Z_0^{S^n} \times \mathbb{RP}^n)$ essentially by definition of the Steenrod operations.

For $\tau^{O(n)}_{SO(n)} a \otimes da \otimes 1 \otimes y$, by the naturality of the pushforward, its image in $H^*(Z^{S^n} \times \mathbb{RP}^n)$ can be obtained by pulling back the class $a \otimes da \otimes 1 \otimes y \in H^*((Z^{S^n} \times S^n)^2)$ along the diagonal to $Z^{S^n} \times S^n$, and then pushing forward to $Z^{S^n} \times \mathbb{RP}^n$. Pulling back yields $ada \otimes y$, and the pushing forward is a product with the pushforward map $S^n \to \mathbb{RP}^n$, giving the class $ada \otimes t^i$.

Finally, we examine the class $P(y)$ using the fact that it is pulled back from $(SO(n))^2_{hO(n)}$ and the commutative diagram

$$\begin{tikzcd} Z' \times \mathbb{RP}^n \arrow[r] \arrow[d] & Z'^2_{hO(n)} \arrow[d] \\ SO(n+1) \times \mathbb{RP}^n \arrow[r] & SO(n+1)^2_{hO(n)} \end{tikzcd}$$

Then, for degree reasons, the class $P(y)$ pulls back along the bottom map to $\sum c_i \otimes t^i$, where c_i are classes of degree $i > 1$.

Putting it all together, the class $P(a) \in H^*(Z^2_{hC_2})$ is sent to $\sum c'_i \otimes t^i + \sum_i Sq_i a \otimes t^i + ada \otimes t^n$ in $H^*(Z' \times \mathbb{RP}^n)$ where c'_i are classes of positive degree. Because in the commutative diagram above (1), the map indicated by $\hookrightarrow$ is injective, this means that $P(a)$ is sent to $\sum_0^n Sq_i a t^i + adat^n$ in $H^*(Z^{S^n}) \times \mathbb{RP}^n)$. Since the pushforward map $\tau^{SO(n+1)}_{O(n)} : H^*(Z_0^{S^n} \times \mathbb{RP}^n) \to H^*(Z_0^{S^n})$ extracts the power of t^n, $t^i P(a)$ (and thus $\phi_i(a)$) is sent to $Sq_{n-i} a$ for $i \neq 0$ and $Sq_n a - ada$ for $i = 0$. $\square$

5.2. **Verification of the relations.** Now that we have constructed the classes that survive to E_∞, it remains to verify a few crucial relations among the real classes. Specifically, $w_{n+1}\delta(a) = 0$ and the Steenrod action for ϕ_i and δ.

To begin verifying the relations, we study the following diagram, where $SO(n)$ acts trivially on the spaces in the left side of the diagram:

$$\begin{tikzcd} H^*(Z_{hSO(n)}) \arrow[r, "ev_0^*"] & H^*(Z^{S^n}_{hSO(n)}) \arrow[r, "\tau^{SO(n+1)}_{SO(n)}"] \arrow[d, equal] & H^*(Z^{S^n}_{hSO(n+1)}) \\ H^*(Z^2_{hSO(n)}) \arrow[u, "\Delta^*"] \arrow[r, "ev_1^*"] \arrow[d, "\tau^{O(n)}_{SO(n)}"'] & H^*(Z^{S^n}_{hSO(n)}) \arrow[d, "\tau^{O(n)}_{SO(n)}"'] & \\ H^*(Z^2_{hO(n)}) \arrow[r, "ev_1^*"] & H^*(Z^{S^n}_{hO(n)}) \arrow[uur, "\tau^{SO(n+1)}_{O(n)}"'] & \end{tikzcd} \tag{2}$$

The only square that does not obviously commute is the top left one. But the corresponding maps of spaces commutes up to homotopy since evaluation at any two points give homotopic maps as S^n is connected.

Lemma 5.4. *The class $a \otimes b \in H^*(Z^2) \to H^*(Z^2_{hSO(n)})$ maps to $\sum_0^n t^i q^*(w_i \delta(ab))$ in $H^*(Z^{S^n}_{hO(n)})$ in diagram [2], where $q : Z^{S^n}_{hO(n)} \to Z^{S^n}_{hSO(n+1)}$ is the quotient map, $q^*(w_1) = 0, q^*(w_0) = 1$. This implies $w_{n+1}\delta(a) = 0$.*

Proof. Moving the class $a \otimes b$ through the top row of the diagram gives $\delta(ab)$. $H^*(Z^{S^n}_{hO(n)})$ is a free module over $H^*(Z^{S^n}_{hSO(n+1)})$ with basis t^i, $0 \leq i \leq n$ (see Lemma 4.1), and $\tau^{SO(n+1)}_{O(n)}$ extracts the coefficient of t^n. Thus the t^n coefficient of the image of $a \otimes b$ is $\delta(ab)$. Now in $H^*(Z^2_{hO(n)})$, $\tau^{O(n)}_{SO(n)} a \otimes b$ is killed bb multiplication bb t, so the same is true of the image of $a \otimes b$. But if q is the map $Z^{S^n}_{hO(n)} \to Z^{S^n}_{hSO(n+1)}$, then $t^{n+1} = t^{n-1}q^*w_2 + t^{n-2}q^*w_3 + \cdots + q^*w_{n+1}$. Thus if $t(t^n\delta(ab) + \sum_0^{n-1} c_i t^i) = 0$, we must have that $c_i = \delta(ab)q^*w_{n-i}$ ($c_{n-1} = 0$), and $q^*w_{n+1}\delta(ab) = 0$, giving the result. □

Next, we look at the Steenrod action on $\delta(a), \phi_i(a)$. There is only one step in the construction of $\delta(a), \phi_i(a)$ that does not necessarily commute with the Steenrod action, namely the pushforward map. Nevertheless, the interaction of the pushforward with the Steenrod squares is understandable via the Stiefel-Whitney classes of the normal bundle of the fibration.

Lemma 5.5. *Let $f : A \to B$ be a fibration whose fibres are manifolds of dimension n, let ν denote the normal bundle of f, w_ν its total Stiefel-Whitney class, and τ_f the pushforward map of f. Then $Sq(\tau_f(a)) = w_\nu Sq(a)$.*

Proof. Recall that the pushforward is the composite of the Thom isomorphism of ν with the the Gysin map $H^*(A^\nu) \to H^*(B)$ in cohomology. Since the latter map comes from a map of spectra, it commutes with Steenrod operations, and the former is given by multiplication by the Thom class u. $Squ = w_\nu u$ essentially by definition of the w_i [MS74], so $Sq(ua) = SquSqa = w_\nu uSqa$. Then composing with the Gysin map gives the result. □

Lemma 5.6. *The relations*

$$Sq\phi_{n-k}(a) = \sum_{j \geq 0} \sum_i \binom{k + |a| - j}{i - 2j} \phi_{n-i-k+2j}(Sq^i a) + \delta_{k0} \sum_{2j<i} \delta(Sq^j a \times Sq^{i-j} a)$$

$$Sq^i \delta(a) = \delta(Sq^i a)$$

hold.

Proof. For $\delta(a)$, by Lemma 5.5, we have that $\mathrm{Sq}\,\delta(a) = w_\nu \delta(\mathrm{Sq}(a))$. However, w_ν is easy to compute: ν is the pullback of the normal bundle of the fibration $BSO(n) \to BSO(n+1)$. But $-\nu$ is the bundle that gives the tangent of the fibre, which is the tautological bundle with Stiefel Whitney classes w_i. Thus $w_\nu = (\sum w_i)^{-1}$. Putting this together gives relation (7).

For $\phi_i(a)$, because we started with the class $t^i P(a)$, it suffices to understand the action of the Steenrod squares on that. This is given by the Nishida relations (see [Mil]):

$$\mathrm{Sq}(t^k P(a)) = \sum_{i>0} \left(\sum_{j\geq 0} \binom{k+|a|-j}{i-2j} t^{i+k-2j} P(\mathrm{Sq}^j a) + \delta_{k0} \sum_{2j<i} \tau_*^{C_2}(Sq^j a \otimes Sq^{i-j} a) \right)$$

which gives the desired result. □

We now examine the main spectral sequence.

Proposition 5.7. *For Z an $\mathbb{F}_2$-EM space, there are no differentials in the main spectral sequence until the E_{n+1} page. On the E_{n+1} page, the differential is given by $d_{n+1}(a) = w_{n+1} da$ for $a \in H^*(Z^{S^n})$.*

Proof. The cohomology of $H^*(Z^{S^n})$ is generated by classes of the form a and da, where a is a class in $H^*(Z)$ pulled back via evaluation at a point. By the Liebniz rule, we can just understand differentials on a, and da for these classes. But by naturality of the differential and the fact that a is pulled back from an Eilenberg-Mac Lane space, we just need to prove the assertions when $Z = K(\mathbb{Z}/2, m)$ and the cohomology of the fibre is a free unstable algebra on a and da. By Lemma 5.1, the class da survives to E_∞, so the unstable algebra generated by da cannot have any differentials by the universality argument along with the Liebniz rule. We know that $w_{n+1} da = 0$, so that class must get killed in the spectral sequence, and so for degree reasons, the only class that can kill $w_{n+1} da$ is a, giving the desired E_{n+1} differential. This also shows that a cannot have any differentials before the E_{n+1} page, and hence that nothing else can either. □

Remark 5.7.1. Note that the E_{n+1} differential commutes with the Steenrod algebra action on the columns of the E_2-page of the sequence.

6. Algebraic Computation

The goal of this section is to prove that for an $\mathbb{F}_2$-EM space and n odd, the functor $\ell_n(H^*(Z))$ describes the cohomology of $Z^{S^n}_{hSO(n+1)}$. The work done here is completely algebraic: the inputs are the functor ℓ_n, and the E_{n+2}-page of the main spectral sequence. Our goal is to identify the associated graded objects by breaking it up into more manageable pieces, and doing an explicit computation.

We begin by proving a proposition needed earlier. Recall that $X := Z^{S^n}$ where Z is an $\mathbb{F}_2$-EM space.

Proposition 6.1. *The classes $da, Sq_i a, Sq_n a + ada$, $a \in H^*(Z)$ along with the w_i generate the E_{n+2} page of the main spectral sequence for Z an $\mathbb{F}_2$-EM space.*

Proof. Since the differential for the E_{n+1} page is $d_{n+1}(a) = w_{n+1}da$, it suffices to show that $\ker(d) \subset H^*(X)$ is generated by $Sq_i a, Sq_n a + ada, da$. This in turn is true if and only if the cohomology of d acting on $H^*(X)$ is generated by $Sq_i a$, $Sq_n a + ada$ for $i < n$. $H^*(X)$ with its differential is a tensor product of $H^*(K(\mathbb{Z}/2, m))$ for various m, so it is sufficient to prove it for $Z = K(\mathbb{Z}/2, m)$. Denote $Sq_I = Sq_{a_1} Sq_{a_2} \ldots Sq_{a_k}$ an admissible sequence of excess $< m$ and ι the generator in degree m. These $Sq_I \iota$ generate $H^*(Z)$ as a polynomial algebra.

Assume $m \neq n, 0$. Then note that $H^*(Z)$ is also generated as a polynomial algebra by $Sq_I\iota$ with leading term Sq_i for $i > n$, $Sq_I\iota$ with leading term Sq_i for $i < n$, and $Sq_n Sq_I \iota + Sq_I \iota d Sq_I \iota$. Indeed, these generators are obtained by adding products of generators of lower degrees to the original set of generators, which is an invertible morphism of polynomial algebras.

$d(Sq_I\iota)$ is 0 when the first term is Sq_i with $i < n$, $d(Sq_I\iota)$ is $Sq_{I-n} d Sq_{I-n}\iota$ where $I - n$ is the sequence where n is subtracted from every term in I. Thus $d(Sq_n Sq_I \iota + Sq_I \iota d Sq_I)\iota = 0$ and $d(Sq_I\iota) = Sq_{I-n} d\iota$. Thus by pairing up the generators $Sq_I(\iota)$ with $Sq_{I-n}d\iota$, we have decomposed $H^*(X)$ as a tensor product of differential graded algebras, so by the Kunneth formula, we can write its cohomology as

$$\mathbb{F}_2[(Sq_I\iota)^2] \otimes \mathbb{F}_2[Sq_i Sq_I \iota] \otimes \mathbb{F}_2[Sq_n Sq_I \iota + Sq_I \iota d Sq_I \iota].$$

Now if $m = n$, we can decompose

$$H^*(K(\mathbb{Z}/2, n)) = \mathbb{F}_2[\iota, d\iota]/(d\iota^2 = d\iota) \otimes \mathbb{F}_2[Sq_I\iota]$$

as algebras with a differential. By the Kunneth formula, the cohomology is

$$\mathbb{F}_2[Sq_n\iota + \iota d\iota] \otimes \mathbb{F}_2[Sq_I\iota]$$

,showing that the result is also true in this case.

For $m = 0$, we have

$$H^*(K(\mathbb{Z}/2, n)) = \mathbb{F}_2[\iota]/(\iota^2 = \iota)$$

and the differential is trivial, implying the cohomology is also

$$\mathbb{F}_2[\iota]/(\iota^2 = \iota).$$

□

Recall the multiplicative filtration $F_k(\ell_n)$ on ℓ_n where we declare ϕ_i, δ to have filtration 0, and w_i to have filtration i.

We denote the associated graded map $\hat{\eta}_n$. We next make clear the category which we are thinking of $\hat{\eta}_n$ living in.

Definition 6.2. $G\mathcal{K}$ as the category of bigraded commutative algebras $A^{*,*}$, where both indices are in $\mathbb{Z}_{\geq 0}$, each $A^{p,*}$ is in $\mathcal{U}$, the translation maps $A^{p,*} \otimes A^{0,q} \to A^{p,*+q}$ are maps in $\mathcal{U}$, and $A^{0,*} \in \mathcal{K}$. Let $G_n\mathcal{K}$ denote the category of differential bigraded algebras whose underlying is in $G\mathcal{K}$, and whose differential is of bidegree $(n+1, -n)$, satisfies the Liebniz rule, and commutes with the Steenrod action.

A standard way to view objects in $G\mathcal{K}$ is as having the p grading be in the horizontal direction,and the q grading as well as Steenrod action be in the vertical direction.

Lemma 6.3. *$\hat{\ell}_n(A)$ is generated by classes $\hat{\phi}_i(a), \delta(a)$ in bidegrees $(0, 2|a| - i)$ and $(0, |a| - n)$ for $a \in A$ and w_i in bidegree $(i, 0)$, and the relations*

(1) $\hat{\phi}_i(a + b) = \hat{\phi}_i(a) + \hat{\phi}_i(b)$
(2) $\delta(a + b) = \delta(a) + \delta(b)$
(3) $\delta(ab)\delta(c) + \delta(bc)\delta(a) + \delta(ca)\delta(b) = 0$
(4) $w_{n+1}\delta(a) = 0$
(5) $\delta(a)\hat{\phi}_i(b) = \delta(aSq_ib) + \delta_{0n}\delta(ab)\delta(b)$
(6) $\hat{\phi}_k(ab) = \sum_{i+j=k} \hat{\phi}_i(a)\hat{\phi}_j(b)$
(7) $Sq\hat{\phi}_{n-k}(a) = \sum_{j\geq 0} \sum_i \binom{k+|a|-j}{i-2j} \hat{\phi}_{n-i-k+2j}(Sq^ia) + \delta_{k0} \sum_{2j<i} \delta(Sq^ja \times Sq^{i-j}a)$
(8) $Sq^i\delta(a) = \delta(Sq^ia)$
(9) $Sq_i(\delta(a)) = Sq_i(\hat{\phi}_j(a)) = 0$ *for* $i < 0$
(10) $Sq_0(\delta(a)) = \delta(a)^2$, $Sq_0(\hat{\phi}_j(a)) = \hat{\phi}_j(a)^2$
(11) $\hat{\phi}_k(Sq_ia) = \sum_{j=0}^{(2|a|-i-k)/2} \binom{|a|-i-j-1}{2|a|-i-k-2j} \hat{\phi}_{2j-2|a|+2i+k}(Sq^ja) + \delta_{k,n}\hat{\phi}_i(a)\delta(Sq_ia)$ *for* $k > i$
(12) $\hat{\phi}_0(Sq_ja) = \hat{\phi}_j(a)^2 + \delta_{j,n}\delta(a^2Sq_na)$

hold where $\hat{\phi}_m(a) = 0$ for $m > n$.

Proof. ℓ_n was defined to have these as associated graded relations. □

We prove that for $Z = K(\mathbb{Z}/2, m)$ the relations above are actually all the relations in $\hat{\ell}_n(A)$, and that these map isomorphically onto the associated graded of Z.

Definition 6.4. $\bar{\ell}_n(A)$ is defined as the free object in $G\mathcal{K}$ with the generators and relations in Lemma 6.3.

Next we describe functors that give what we expect associated graded of $H^*(Z^{S^n})_{hSO(n+1)}$ to look like in the case of $K(\mathbb{Z}/2, m)$.

Definition 6.5. Define $\overline{E}_{n+1}(A)$ to be $L_n(A)^{*,0} \otimes H^*(BSO(n+1))$ where the differential sends $a \in L_n(A)$ to $da \otimes w_{n+1}$. Let its cohomology, which is in $G\mathcal{K}$, be denoted $\overline{E}_\infty$.

$\overline{E}_\infty$ of course depends on n, but we supress the dependence in the notation.

Lemma 6.6. *Suppose that the natural map $L_n(H^*(Z)) \to H^*(Z^{S^n})$ is an isomorphism. Then there is a natural map $\overline{\eta}_n : \overline{\ell}_n \to \overline{E}_{n+1}$ in the category $G\mathcal{K}$, such that for such Z, there is a commutative diagram*

$$\begin{tikzcd} \bar{\ell}_n(H^*(Z)) \arrow[r, "\bar{\eta}_n"] \arrow[d, two heads] & \overline{E}_\infty(H^*(Z)) \arrow[d, equal] \\ \hat{\ell}_n(H^*(Z)) \arrow[r, "\hat{\eta}_n"] & E_\infty(Z) \end{tikzcd}$$

where E_∞ is the E_∞-page of the main spectral sequence.

Proof. The map from $\overline{\ell}_n$ to $\overline{E}_\infty$ well defined, as all the relations in 6.3 hold. To see this, by naturality, it suffices to prove the relations hold in the universal cases. But the universal cases are $\mathbb{F}_2$-EM spaces, where we have an identification of $\overline{E}_\infty$ with E_∞ which agrees with our defined map.

The identification $\overline{E}_\infty(H^*(Z)) \to E_\infty(Z)$ happens because our choice of Z tells us that $H^*(Z^{S^n}) = L_n(H^*(Z))$ and Proposition 5.7 and Corollary 2.1 describe completely the E_∞-term of the spectral sequence to be what we have defined $\overline{E}_\infty$ as. ☐

By definition, there is a natural surjective map $\overline{\ell}_n \to \hat{\ell}_n$. We show that the composite $\overline{\eta}_n : \overline{\eta}_n \to \eta_n$ is an isomorphism on free algebras. This implies that in this case, the map $\overline{\ell}_n \to \hat{\ell}_n$ is an isomorphism.

In order to study $\overline{\ell}_n$, we break it up into smaller pieces.

Define ℓ'_n to be the quotient of $\bar{\ell}_n$ by $w_i, i \leq n+1$. Define ℓ'_n/δ to be the quotient of ℓ'_n by $\delta(a)$ for all a.

Lemma 6.7. *Lemma 6.3 gives a presentation for ℓ'_n in $\mathcal{K}$ if the w_i generators as well as relation (4) are removed. Furthermore, $\ell'_n/\delta(A)$ is generated by $\phi_i(a)$ for $a \in A$ modulo the following relations:*

(1) $\phi_i(a+b) = \phi_i(a) + \phi_i(b)$
(2) $\phi_k(ab) = \sum_{i+j=k} \phi_i(a)\phi_j(b)$
(3) $Sq^i\phi_{n-k}(a) = \sum_j \binom{k+|a|-j}{i-2j}\phi_{n-i-k+2j}(Sq^j a)$
(4) $Sq_i(\phi_j(a)) = 0$ *for* $i < 0$
(5) $Sq_0(\phi_j(a)) = \phi_j(a)^2$.
(6) $\phi_k(Sq_i a) = \sum_{j=0}^{(2|a|-i-k)/2} \binom{|a|-i-j-1}{2|a|-i-k-2j}\phi_{2j-2|a|+2i+k}(Sq^j a)$ *for* $k > i$.
(7) $\phi_0(Sq_j a) = \phi_j(a)^2$

with $\phi_m(a) = 0$ for $m > n$.

Proof. The first statement is clear since the w_i, $i \leq n$ are not involved in the relations, and the only relation on w_{n+1} is $w_{n+1}\delta(a) = 0$.

The second statement is also clear since ℓ'_n/δ is obtained from ℓ'_n by adding more relations, namely killing the δ classes. ☐

We break up $\overline{E}_\infty$ into smaller pieces.

Definition 6.8. E'_∞ is the quotient of E'_∞ by the w_i. E'_∞/d is the quotient of E'_∞ by elements of the form da.

The natural map $\overline{\ell}_n \to \overline{E}_\infty \to E'_\infty$ sends the w_i to 0, so factors as a map $\eta'_n : \ell'_n \to E'_\infty$. Further quotienting the map to E'_∞/d factors makes the map factor through ℓ'_n/δ, giving a map we can call η'_n/d

Note that $E'_\infty(A), E'_\infty(A)/d$ can also be described as follows: Consider $L_n(A)$ as a differential graded algebra, where the differential is $a \mapsto da$. Then $E'_\infty(A)$ is the kernel of d and $E'_\infty(A)/\delta$ is the cohomology of d.

Finally, we need an algebraic version of the pushforward map $\tau^{SO(n+1)}_{SO(n)}$, and a ring map from ℓ'_n into a subalgebra of L_n that is an algebraic version of the quotient map in cohomology.

Definition 6.9. The map $\tau : L_n(A) \to \ell'_n(A)$ is an $\mathbb{F}_2$-linear map that sends $adb_1 \dots db_n$ to $\delta(a)\delta(b_1)\dots\delta(b_n)$. The map $i : \ell'_n(A) \to L_n(A)$ is a ring map sending $\delta(a)$ to da, $\phi_i(a)$ to $Sq_i a$ for $i < n$ and $\phi_n(a)$ to $Sq_n a + ada$.

Proposition 6.10. *The maps i, τ are well-defined, and the differential d on L_n factors as $i \circ \eta'_n \circ \tau$. Moreover, $\tau \circ \eta' = 0$ and τ fits in an eaact sequence $L_n(A) \xrightarrow{\tau} \ell'_n(A) \to \ell'_n/\delta(A) \to 0$.*

Proof. To check τ is well defined, it suffices to check the ideal generated by relations $(1), (2), (3), (4)$ from Proposition 3.3 is sent to 0. Relation (1) is easy and relation $(3), (4)$ follows from the fact that $\ell'_n \in \mathcal{K}$. To check (2), if we have an element of the form $adb_1 \dots db_n(d(xy) - d(x)y + d(y)x)$, it is sent to $\delta(b_1)\dots\delta(b_n)(\delta(ay)\delta(x) + \delta(xy)\delta(a) + \delta(ax)\delta(y)) = 0$.

To check i is well defined, we must check that the relations for ℓ'_n hold in $L_n(A)$. To see this, by naturality, it suffices to prove the relations hold in the universal cases. But the universal cases are $\mathbb{F}_2$-EM spaces, where we have an identification of $L_n(H^*(Z))$ with $H^*(Z^{S^n})$ which agrees with our defined map.

∎

Lemma 6.11. *The natural map $\overline{\ell}_n(A) \to \overline{E}_\infty(A)$ is an isomorphism iff $\eta'_n/d(A)$ is isomorphisms.*

Proof. We observe from the presentation that $\overline{\ell}_n(A)$ decomposes as a direct sum

$$\bigoplus_{\alpha_i \geq 0} w_2^{\alpha_2} \cdots w_n^{\alpha_n} \ell'_n(A) \oplus \bigoplus_{\beta_i \geq 0, \beta_{n+1} > 0} w_2^{\beta_2} \dots w_{n+1}^{\beta_{n+1}} \ell'_n/\delta(A)$$

Similarly, $\overline{E}_\infty$ decomposes as

$$\bigoplus_{\alpha_i \geq 0} w_2^{\alpha_2} \cdots w_n^{\alpha_n} E'_\infty \oplus \bigoplus_{\beta_i \geq 0, \beta_{n+1} > 0} w_2^{\beta_2} \ldots w_{n+1}^{\beta_{n+1}} E'_\infty / d$$

via this decomposition the map $\overline{\ell}_n(A) \to \overline{E}_\infty(A)$ splits as a sum of copies of $\eta'_n/d(A)$ and $\eta'_n(A)$ accordingly. Thus it suffices to prove that $\eta'_n/d(A)$ and $\eta'_n(A)$ are isomorphism.

We now show that η'_n/d being an isomorphism implies η'_n is too. E'_∞ is generated by lifts of E'_∞/d as well as the elements da. Because η'_n/d is surjective, and da is hit by $\delta(a)$, the map is surjective. For injectivity, suppose a is in the kernel of η'_n. Then because η'_n/d is injective, we can assume a is in the kernel of d. But by Proposition 6.10, this means $a = \tau(b)$ where $db = 0$. By surjectivity of η'_n/d this means that b is the sum of dc and a class in the image of η'_n. But $\tau(dc) = 0$ and $\tau \circ \eta'_n = 0$, which shows $a = 0$. □

The next lemma is the key to proving that η is an isomorphism.

Lemma 6.12. *For $Z = K(\mathbb{Z}/2, m)$, η'/d is an isomorphism.*

Proof. First assume $m \neq n, 0$.

From the proof of Proposition 6.1 we have a decomposition of $E'_\infty/d(H^*(Z))$ of the form

$$\mathbb{F}_2[(Sq_I\iota)^2] \otimes \mathbb{F}_2[Sq_iSq_I\iota] \otimes \mathbb{F}_2[Sq_nSq_I\iota + Sq_I\iota dSq_I\iota].$$

Since this is a polynomial algebra, we can define a map $\zeta : E'_\infty(A)/d \to \ell'_n(A)/d$ by sending the generators of the cohomology of d to $\phi_0(Sq_I\iota)$, $\phi_i(Sq_I\iota)$, and $\phi_n(Sq_I\iota)$ respectively. Note that $\eta'_n \circ \zeta = id$, implying that ζ is injective, so it suffices to check that ζ is surjective. Due to the multiplicative and additive relations for ϕ_i, it suffices to check that $\phi_i(g)$ is in the image for each multiplicative generator g of $H^*(Z)$.

(1) $\phi_0(Sq_I\iota)$ is automatically in the image for Sq_I leading with Sq_i for $i > n$. For $i \leq n$, relation (7): $(\phi_i(a))^2 = \phi_0(Sq_ia)$ shows that it is in the image.

(2) $\phi_i(Sq_I\iota)$ is automatically in the image for Sq_I leading with Sq_j for $j \geq i$. For $j < i$, the Adem relation (6) lets us write $\phi_i(Sq_I\iota)$ in terms of expressions involving

$$\sum \phi_\alpha(Sq_\beta\iota)$$

with $\alpha \leq \beta < j$ which is in the image.

(3) $\phi_n(Sq_I\iota)$ is automatically in the image for Sq_I leading with Sq_i for $i \geq n$, and for $i < n$, the Adem relations (6) similarly show that the generator is in the image of ζ.

For $m = 0$, It is easy to identify ℓ'/δ with the free Boolean algebra generated by $\phi_0(\iota)$, and E'_∞/d with the free Boolean algebra generated by ι, and these generators are identified.

For $m = n$, we obtained a similar expression in Proposition 6.1 for the cohomology of d, and the proof proceeds similarly. The only difference is that we use relation (6) to show that $\phi_n(\iota)^2 = \phi_0(\iota)$ in ℓ'/δ because $E'_\infty/d(H^*(Z))$ is a polynomial algebra on the same generators except for that relation.

□

Proposition 6.13. *$\hat{\eta}_n$ is an isomorphism when Z is an $\mathbb{F}_2$-EM space.*

Proof. We can compose $\hat{\eta}_n$ with the natural surjection $\overline{\ell}_n \to \ell_n$, and if the composite is an isomorphism, $\hat{\eta}$ is as well. For Z an $\mathbb{F}_2$-EM space, $H^*(Z^{S^n}) = L_n(H^*(Z))$, thus by Lemma 6.6, it suffices to show that the map $\overline{\ell}_n \to \overline{E}_\infty$ is an isomorphism, which by Lemma 6.11 reduces to showing that η'/d is an isomorphism for Z. However, ℓ'_n/δ preserves coproducts as does E'_∞/d, so we are reduced to the case when $Z = K(\mathbb{Z}/2, m)$, which is Lemma 6.12.

□

7. Questions and Further Directions

The functor ℓ_n constructed in this paper should be thought of as the closest approximation to the Borel cohomology of the free loop space, as evidenced by the fact that η_n in Theorem A is often an isomorphism. However a lot more than this theorem should be true. For example, one should be able to resolve an arbitrary space Z by Eilenberg-Mac Lane spaces of type $\mathbb{Z}/2\mathbb{Z}, m$, yielding a Bousfield homology spectral sequence whose E_2 term is the derived functors of $\ell_n(H^*(Z))$ and who converges to $H^*(Z^{S^n}_{hSO(n+1)})$, similar to what was done in [BO04]. This spectral sequence should converge under favorable conditions (including n-connectivity of the space Z). In the case of S^1, and $Z = \mathbb{CP}^n$, Ottosen and Bokstedt find that it degenerates at E_2 and $H^*(\mathbb{CP}^n)$ is not acyclic with respect to the derived functors of ℓ_1, because they are able to compute the cohomology via other means.

The ability to construct the functor ℓ_n indicates the possibility of more general constructions, as well as a more elegant conceptual description for functors approximating equivariant cohomology of mapping spaces, as we had in the nonequivariant setting. Here are some questions we offer to the reader along these lines:

- How dependent is our construction on the group $G = SO(n+1)$, and its representation on S^n? We suspect that our techniques with modification should extend to many other groups. Namely, for the standard representations of $U(n)$ and $Sp(n)$ on odd dimensional spheres, essentially the same construction should allow the construction of an approximation functor (we have privately checked that this is the case for $Sp(1)$). It would be interesting to find general conditions on a representation which would allow for the construction of an approximation functor using similar techniques. One possibly necessary condition is that the coaction of G on S^n is nontrivial on cohomology.

- Can the techniques of this paper be modified to work for even-dimensional spheres? The immediate issue that arises is that the group $O(n)$ does not split as the product of $SO(n)$ and C_2 for n even, but rather is a nontrivial extension. This prevents Construction 5.1.1 from working. We would be surprised however, if the corresponding result for even dimensional spheres was false.
- Is there an analog of ℓ_n for odd primes? Ottosen has constructed in [Ott03] an approximation for the Borel cohomology of the free loop space at odd primes, but our construction seems to only work in generality at the prime 2.
- Our construction treats the Borel homotopy type of the mapping space, but is it possible to understand the cohomology of a genuine equivariant homotopy type? However, the correct setting in which to do such a construction might not involve the Steenrod algebra, or Borel cohomology.

One of the most interesting questions is if there is a universal characterization of the functor ℓ_n, and furthermore if it is possible to construct ℓ_n without doing any serious computations (as can be done nonequivariantly). The universal property that was used to define $L_n(H^*(Z))$ was that it has the universal coevaluation map $L_n(H^*(Z)) \to H^*(Z^{S^n}) \otimes H^*(S^n)$. We can instead ask for a functor that has a universal coevaluation map that equalizes the $H^*(SO(n+1))$ actions on both factors on the right. This unfortunately does not yield ℓ_n (the w_i are missing for example), but it yields something close. We wonder if there is a modification of this that yields ℓ_n. Along the same lines, we wonder if ℓ_n has a right adjoint if its target category is appropriately specified.

Finally, there are some facts we do not yet know about ℓ_n.

- What are explicit formulae for the expressions that were used to define ℓ? To answer this, one could try to prove relations among the real classes, or one could attempt to use Theorem A to deduce what the relations must be. We are able to prove some relations in Section B.
- In [BO99], the underlying algebra of ℓ_1 could be computed in the category $\mathcal{K}_1$ (see Section A). Is there an m such that the underlying algebra of ℓ_n can be computed in $\mathcal{K}_m$? It is expected that if such an m exists, it should be atleast n.

Appendix A. The Top k Squares and Computing Division

Here we explain how to explicitly compute the division functor, as well as what structure is necessary to compute the division functor if we ignore the action of the Steenrod squares. Everything in this section generalizes to odd primes without significant change.

In [Ott03], the division functor is only constructed using the Sq_1-operations. In order to take this into account, the category $\mathscr{F}$ is introduced in which their approximation functors can be defined. $\mathscr{F}$ only encodes the data of Sq_1, which for the cohomology of a space can be thought of as only using the power operations coming from the E_2-structure on the cochains.

A similar category $\mathcal{U}_n$ has been studied in [Li20], where only the operations Sq_i for $i \leq n$ are remembered. It is significantly simpler than modules over the Steenrod algebra: it is proven there for example that $\mathcal{U}_n$ has homological dimension n. Thus it would be useful to know if our approximation functors L_n and ℓ_n can be constructed after forgetting from $\mathcal{U} \to \mathcal{U}_n$.

We want a version of $\mathcal{U}_n$ that remembers the algebra structure as well as the Sq_i for $i \leq n$. We define an analagous category $\mathcal{K}_n$ which only encodes the top n Steenrod squares, generalizing $\mathscr{F}$ which is the case $n = 1$.

Definition A.1. The objects of $\mathcal{K}_n$ are nonnegatively graded commutative $\mathbb{F}_2$-algebras A_* together with for $0 \leq i \leq n$, linear maps $Sq_i : A_m \to A_{2m-i}$ satisfying the following requirements:

(1) $Sq_0 a = a^2$
(2) $Sq_{|a|} a = a$
(3) $Sq_i a = 0$ for $i > |a|$
(4) $Sq_i(ab) = \sum_{j=0}^{i} Sq_j(a) Sq_{i-j}(b)$
(5) $Sq_i Sq_j a = \sum_k \binom{k-j-1}{2k-i-j} Sq_{i+2j-k} Sq_k a$ for $i > j$.

The morphisms are those preserving all present structure. Except for (2), (3), the relations are essentially the relations of the Dyer-Lashof operations on E_{n+1} $\mathbb{F}_2$-algebras with vanishing Browder bracket [Law20]. $\mathcal{K}_0$ is the category of graded $\mathbb{F}_2$-algebras with the degree 0 subalgebra a Boolean algebra, and $\mathcal{K}_1$ is the category $\mathscr{F}$ used in [Ott03; BO04].

There is a natural functor $\mathcal{K} \to \mathcal{K}_n$ forgetting some of the operations, as well as a functor $\mathcal{K}_n \to \mathcal{U}_n$ forgetting multiplicative structure. We investigate to what extent the Steenrod operations are necessary to compute the division functor. In otherwords, we would like to ask for which m there is a factorization in the diagram:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "(-):N"] \arrow[d] & \mathcal{K} \arrow[d] \\ \mathcal{K}_m \arrow[r, dashed] & \mathcal{K}_n. \end{tikzcd}$$

To do this, we give an explicit description of the division functor on both $\mathcal{K}$ and $\mathcal{U}$. For another account of the division functor see [Sch94].

Recall that $(-) :_{\mathcal{U}} N$ is the left adjoint of the functor $(-)\otimes N$ from $\mathcal{U}$ to itself. This left adjoint exists when N is finite type, which is the condition for $(-)\otimes N$ to preserve limits, at which point one can apply an adjoint functor theorem for existence. From now on, N is finite type.

As a left adjoint, $(-) : N$ preserves colimits. Since any module can be presented as a cokernel of free modules, in order to compute this functor, we can restrict our attention to free modules first. There is a forgetful functor from $\mathcal{U}$ to graded $\mathbb{F}_2$-vector spaces, let F be the left adjoint, taking a graded vector space V_* to $F(V_*)$, the free unstable module on V_*. An example of a graded vector space is $\Sigma^n\mathbb{F}_2$, which is $\mathbb{F}_2$ in degree n and 0 elsewhere. Then $F(\Sigma^n\mathbb{F}_2)$ is a vector space with basis given by $Sq_I x$ where x is in degree n and I is an admissable sequence of excess $\leq n$.

Lemma A.2. *$F(V_*) : N$ is canonically isomorphic to $F(V_* \otimes N^*)$. The adjunction isomorphism $\nu : \mathrm{Hom}(F(V_*\otimes N^*), M) \to \mathrm{Hom}(F(V_*), N\otimes M)$ is given by the formula $f(\nu(h)(v)) = h(v \otimes f)$ where $f \in N^*, v \in M$.*

Proof. We can compute $\mathrm{Hom}(F(V_*) : N, M) = \mathrm{Hom}(F(V_*), N\otimes M) = \mathrm{Hom}_{\mathbb{F}_2}(V_*, N\otimes M) = \mathrm{Hom}_{\mathbb{F}_2}(V_* \otimes N^*, M) = \mathrm{Hom}(F(V_* \otimes N^*), M)$, and use the Yoneda lemma to conclude. The explicit formula is given by following the isomorphisms. □

We use the notation $d_f(v)$ to denote the element $v \otimes f$ in $F(V_*) : N$. Note that if v is an arbitrary element in M of degree m, $d_f(v)$ makes sense: we can think of v as a map from $F(\Sigma^m\mathbb{F}_2)$ to M, and by applying $: N$, $d_f(v)$ is the image of the corresponding class in $F(\Sigma^m\mathbb{F}_2) : N$. The class $d_f(v)$ is clearly natural with respect to maps of modules: given a map $g : M \to M'$, $(g : N)(d_f(v)) = d_f(g(v))$

Recall that by convention N^* is nonzero in nonpositive degree. It has a right action of the Steenrod algebra given by precomposition with the action on N.

Lemma A.3. *$d_f(Sq^i(x)) = \sum_0^i Sq^k d_{fSq^{i-k}}(x)$.*

Proof. Let ν be as in Lemma A.2, $f \in N^*$ and $h : (M' : N) \to M$ a map. Then we have $h(d_f(Sq^i(a))) = f(\eta(h)(Sq^i a)) = f(Sq^i\eta(h)(a))$. Choose a basis x_α of N with dual basis x^*_α. Then $\eta(h)(a)$ is $\sum_\alpha h(d_{x^*_\alpha}(a)) \otimes x_\alpha)$, so

$$\begin{aligned}
f(Sq^i\eta(h)(a)) &= f(\sum_0^i \sum_\alpha Sq^k h(d_{x^*_\alpha}(a)) \otimes Sq^{i-k}x_\alpha) \\
&= \sum_0^i \sum_\alpha Sq^k h(d_{x^*_\alpha}(a)) f(Sq^{i-k}x_\alpha) \\
&= \sum_0^i Sq^k h(d_{fSq^{i-k}}(a))
\end{aligned}$$

which completes the proof, since h is an arbitrary map (it can be taken to be the identity, for example). □

Proposition A.4. *$M :_{\mathcal{U}} N$ is presented by the generators $d_f(v)$ for $f \in N^*, v \in M$ along with the following relations for $v, v' \in M$:*

(1) $d_f(v + v') = d_f(v) + d_f(v')$
(2) $d_{f+f'}(v) = d_f(v) + d_{f'}(v)$
(3) $Sq^i(d_f(v)) = d_f Sq^i(v) - \sum_1^i Sq^{i-j} d_{fSq^j} v$
(4) $Sq_i d_f(v) = 0$ for $i < 0$.

Proof. Any module M in $\mathcal{U}$ can be presented via the exact sequence

$$F(\bigoplus_0^\infty \Sigma^i M) \to F(M) \to M \to 0$$

where the second map sends the generator $[v]$ to v, and for the first map, we send the generators in $F(\Sigma^i M)$ to $[Sq^i v] - Sq^i[v]$. Now apply the functor $: N$ to this exact sequence and Lemma A.2 to obtain

$$F(\bigoplus_0^\infty \Sigma^i M \otimes N^*) \to F(M \otimes N^*) \to M : N \to 0.$$

Thus we have a presentation of $M : N$, and we just need to read off the presentation. The surjectivity of second map shows that $d_f(v)$ generates $M : N$, and the relations $(1), (2)$ are obvious. (4) comes from the instability condition which says that $Sq^I x = 0$ when I is an admissable sequence of excess $> |x|$, but this is implied in general by $Sq^i x = 0$ for $i > |x|$ and repeatedly applying relation (3). We get the relation that $d_f([Sq^i x]) = d_f(Sq^i[x])$, and the right hand side is given by Lemma A.3, giving relation (3). □

This description of the division functor is quite useful. For example, given a module N, we can figure out how many operations Sq_i we need to remember in order to produce the underlying $\mathbb{F}_2$-vector space of the functor $M : N$. When computing the underlying $\mathbb{F}_2$-vector space, relation (3) should be thought of as just giving a way of computing the Steenrod action, as opposed to imposing more relations. Rather, it is the instability condition (4) that imposes relations. Thus by putting together (4) and (3), we can deduce how much structure is needed to compute the underlying vector space. We sketch how to do this, but do not give all the details for the proofs.

Fix a finite type module N. For each x in the Steenrod algebra, we define $\langle x, N\rangle \in \mathbb{N} \cup \infty$ to be the largest k such that there exists a $y \in N$ of degree k with $xy \neq 0$.

Example A.4.1. $\langle 1, N\rangle$ is infinite unless N has its degree bounded.

Now define $\beta_N(x)$ inductively by:

- $\beta_N(x) = -\infty$ if $\langle x, N\rangle = 0$
- $\beta_N(x) = \sup_k(\langle x, N\rangle, 2k + \beta_N(xSq^k))$ otherwise.

Example A.4.2. Let $N = \Sigma^n \mathbb{F}_2$. Then $\beta_N(x) = -\infty$ if $x \neq 1$, and $\beta_N(1) = n$.

Example A.4.3. Let $N = H^*(\mathbb{RP}^4)$. Then $\beta_N(1) = 7, \beta_N(Sq^1) = 3, \beta_N(Sq^2) = 3, \beta_N(Sq^2Sq^1) = 1$, and $\beta_N(x) = -\infty$ for all other x.

We are interested in $\beta_N(1)$. In general, $\beta_N(1)$ falls in a predictable range.

Lemma A.5. *Suppose that* $0 < \langle 1, N\rangle = c_N < \infty$. *Then* $c_N \leq \beta_N(1) \leq 2c_N - 1$.

Proposition A.6. *The composite* $\mathcal{U} \xrightarrow{(-):N} \mathcal{U} \to \mathcal{U}_n$ *factors through* $\mathcal{U}_{n+\beta_N(1)}$. $\mathcal{U}_{-1}$ *can be taken to be graded* $\mathbb{F}_2$*-modules.*

Proof. The complete proof is not given here, rather just a sketch. The point is that we can construct the factorization through $\mathcal{U}_{n+\beta_N(1)-1}$ by using the same generators as in Proposition A.4, with relations (1), (2), and recursively expand out the relation (4) : $Sq_i d_f(x) = 0$ for $i < 0$ using relation (3).

Relation (3) says that in order to compute $Sq_i d_f(x)$, you need to know $d_f Sq_{i+|f|}(x)$ as well as $Sq_{i+2k} d_{fSq^k}(x)$. This recursion relation is exactly complementary to the one defining $\beta_n(x)$; Thus we learn that if you know the operations Sq_i for $i \leq \beta_N(x) + k$, then you can compute $Sq_k d_{fx}(a)$ for any f and a. Thus to compute Sq_i for $i \leq n$ (i.e factor the functor through $\mathcal{U}_n$) we need to know Sq_i for $i \leq n + \beta_N(1)$. □

We believe that the value $n + \beta_N(1)$ in the above proposition is the optimal one making it true, but do not attempt to prove this.

Next we run the the same analysis for the category $\mathcal{K}$. $(-) :_{\mathcal{K}} N$ is the left adjoint of the functor $A \to A \otimes N$ from $\mathcal{K}$ to itself.

Let UF denote the free unstable algebra generated by a graded vector space, analogous to the functor F, except for $\mathcal{K}$. For example $UF(\Sigma^n \mathbb{F}_2)$ is a polynomial algebra on $\mathrm{Sq}^I x$, where x is the generator in degree n and I runs over all admissable sequences of excess $< n$ (i.e the cohomology of $K(\mathbb{Z}/2, n)$). We can replace F with UF in the proof of Lemma A.2 to obtain the next lemma.

Lemma A.7. $UF(V_*) : N$ *is canonically isomorphic to* $UF(V_* \otimes N^*)$. *The adjunction isomorphism* $\nu : \mathrm{Hom}(UF(V_* \otimes N^*), M) \to \mathrm{Hom}(UF(V_*), N \otimes M)$ *is given by the formula* $f(\nu(h)(v)) = h(v \otimes f)$ *where* $f \in N^*, v \in M$.

Once again, we have classes $d_f(v)$ that satisfy all the relations satisfied in case of modules. However now there are more relations coming from the multiplicative structure. Since N has a multiplication, N^* has a comultiplication, which is denoted Δ.

Lemma A.8. *Suppose that* $\Delta(f) = \sum_i (\alpha_i \otimes \beta_i)$. *Then* $d_f(ab) = \sum_i d_{\alpha_i}(a) d_{\beta_i}(b)$.

Proof. The proof is similar to Lemma A.3. Let ν be as in Lemma A.2, $f \in N^*$ and $h : (M' : N) \to M$ a map. Then we have $h(d_f(ab)) = f(\eta(h)(ab)) = f(\eta(h)(a)\eta(h)(b)) = \sum_i \alpha_i(\eta(h)(a))\beta_i(\eta(h)(b)) = h(\sum_i d_{\alpha_i}(a)d_{\beta_i}(b))$. We can take h to be the identity, to obtain the result. ☐

Note that since d_f is linear in f, the above lemma does not depend on how $\Delta(f)$ is presented as a sum of simple tensors. Thus we denote $\sum_i d_{\alpha_i}(a)d_{\beta_i}(b)$ by $(\Delta d_f)(a \otimes b)$, so that the above Lemma can concisely be written as $d_f(ab) = (\Delta d_f)(a \otimes b)$.

The next Proposition is the analog of Proposition A.4 for $\mathcal{K}$.

Proposition A.9. *$M :_{\mathcal{K}} N$ is presented by $d_f(v)$ for $f \in N^*, v \in M$ along with the following relations:*

(1) $d_f(v + v') = d_f(v) + d_f(v')$
(2) $d_{f+f'}(v) = d_f(v) + d_{f'}(v)$
(3) $Sq^i(d_f(v)) = d_f Sq^i(v) - \sum_1^i Sq^{i-j} d_{fSq^j} v$
(4) $Sq_i d_f(v) = 0$ *for* $i < 0$
(5) $Sq_0 d_f(v) = (d_f(v))^2$
(6) $d_f(vv') = (\Delta d_f)(v \otimes v')$.

Proof. M can be presented as the free algebra generated by M as a vector space, along with relations that enforce the algebra structure as well as the Steenrod action. We can view this as the following pushout diagram in $\mathcal{K}$:

$$\begin{tikzcd}
UF(M \otimes M) \otimes UF(\bigoplus_0^\infty \Sigma^i M) \arrow[r] \arrow[d] & UF(M) \arrow[d] \\
0 \arrow[r] & M
\end{tikzcd}$$

The vertical nonzero map is given by $[m] \mapsto m$ and the horizontal map sends $[m \otimes m']$ to $[m][m'] - [mm']$, and $m \in \Sigma^i M$ to $Sq^i[m] - [Sq^i m]$.

Applying $(-) : N$, since $(-) : N$ preserves pushouts, we get the pushout square

$$\begin{tikzcd}
UF(M \otimes M \otimes N^*) \otimes UF(\bigoplus_0^\infty \Sigma^i M \otimes N^*) \arrow[r] \arrow[d] & UF(M \otimes N^*) \arrow[d] \\
0 \arrow[r] & M : N
\end{tikzcd}$$

As in Proposition A.4, relations $(1) - (4)$ follow as in the case for $\mathcal{U}$. The relation (5) is enforced by being an object in $\mathcal{K}$, and (6) is the relation enforced by the $UF(M \otimes M \otimes N^*)$ according to Lemma A.9. ☐

We can repeat our analysis of the structure needed to compute division for algebras.

Proposition A.10. *The composite* $\mathcal{K} \xrightarrow{(-):N} \mathcal{K} \to \mathcal{K}_n$ *factors through* $\mathcal{K}_{n+\beta_N(1)}$.

Appendix B. Relations

In our definition of ℓ_n we only give the associated graded relations, which are the real relations mod the ideal generated by the w_i. However in some cases, we are able to find and prove the real relations among the classes.

Theorem B.1. *The following relations hold among the real classes ($\delta_{i,j}$ is defined to be 1 for $i = j$ and 0 otherwise).*

- $\phi_i(a+b) = \phi_i(a) + \phi_i(b) + w_{n-i}\delta(ab)$ *(*$w_1 = 0, w_0 = 1$*)*
- $\delta(a+b) = \delta(a) + \delta(b)$
- $\phi_k(ab) = \sum_{i+j=k} \phi_i(a)\phi_j(b) + \sum_{\ell=n+1}^{2n} \sum_{i+j=\ell} \phi_i(a)\phi_j(b) \left(\sum_{\substack{2\le\alpha_1\ldots\alpha_m\le n+1 \\ \alpha_1+\ldots\alpha_m=\ell-k \\ \alpha_m>n-k}} \prod_{f=1}^m w_{\alpha_f} \right)$

Proof. Observe that in $H^*(Z^2_{hO(n)})$, $\tau^{O(n)}_{SO(n)}(a\otimes b) = P(a+b) - P(a) - P(b)$. Thus multiplying by t^i and looking at the image in $H^*(Z^{S^n}_{hSO(n+1)})$, we can use the definition of the ϕ_i to get the first relation. Since ev_0^* and $\tau^{SO(n+1)}_{SO(n)}$ are additive, the second relation is true. To see the third, note that the image of $P(a)$ in $Z^{S^n}_{hO(n)}$ is $\sum_i^n t^i\phi_i(a)$. Since $P(ab) = P(a)P(b)$, and $t^{n+1} = t^{n-1}q^*w_2 + t^{n-2}q^*w_3 + \cdots + q^*w_{n+1}$, isolating the t^k coefficient gives

$$\phi_k(ab) = \sum_{i+j=k} \phi_i(a)\phi_j(b) + \sum_{\ell=n+1}^{2n} \sum_{i+j=\ell} \phi_i(a)\phi_j(b) \left(\sum_{\substack{2\le\alpha_1\ldots\alpha_m\le n+1 \\ \alpha_1+\ldots\alpha_m=\ell-k \\ \alpha_m>n-k}} \prod_{f=1}^m w_{\alpha_f} \right).$$

$\square$

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