Graphs and Homomorphisms
A graph $X$ is a collection of vertices (dots) and edges (line segments or arrows).

Notation:

- $V(X)$: the set of vertices.
- $E(X)$: the set of edges.
- $u \sim v$: edge $\{u, v\} \in E(X)$. 

![Graph example image]

Minghan S., Andrew W., Christopher Z. (MIT PRIMES Reading Group Mentor: Younhun Kim)

Homomorphisms of Graphs

June 6, 2020 3 / 25
Definition

Let $X$ and $Y$ be graphs. A map $\varphi : V(X) \to V(Y)$ is a homomorphism if $\varphi(x) \sim \varphi(y)$ whenever $x \sim y$. Less formally, a homomorphism maps edges to edges.

Example

\[
\varphi : \begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ
\end{array}
\rightarrow
\begin{array}{c}
\circ \\
\circ
\end{array}
\]
Colorings

**Definition**

Let $I$ be a subset of the vertex set $V(G)$ of a graph. We say that $I$ is an **independent set** if there exists no edge that joins two vertices in $I$.

**Definition**

For a positive integer $c$, a **$c$-coloring** of a graph $G$ is a partition of $V(G)$ into $c$ independent sets. The **chromatic number** of a graph, $\chi(G)$, is the smallest integer $n$ such that $G$ has a $n$-coloring.

We can think of a $c$-coloring of $G$ as a homomorphism $G \rightarrow K_c$ that identifies each independent set with a distinct vertex of $K_c$. 

![Diagram](image.png)
Hedetniemi’s Conjecture

- \( \psi : X \rightarrow Y \) exists, \( \implies \chi(X) \leq \chi(Y) \), because there is 
  \( \pi : Y \rightarrow K_{\chi(Y)} \), and \( \pi \circ \psi \) is a homomorphism \( X \rightarrow K_{\chi(Y)} \).
- Since the map that sends \((x, y)\) to \(x\) is a homomorphism 
  \( X \times Y \rightarrow X \implies \chi(X \times Y) \leq \min\{\chi(X), \chi(Y)\} \).

Conjecture (Hedetniemi, 1966)

For all graphs \( X, Y \), we have \( \chi(X \times Y) = \min\{\chi(X), \chi(Y)\} \).

\[ \begin{array}{c}
  A \\
  B
\end{array} \times \begin{array}{c}
  1 \\
  2 \\
  3
\end{array} \implies \begin{array}{c}
  A1 \\
  A2 \\
  A3 \\
  B1 \\
  B2 \\
  B3
\end{array} \]

**Figure:** \( K_2 \times K_3 \cong C_6 \)
Shitov’s counterexample (2019)

Main Idea:

1. Shitov proves that if $G$ contains a large cycle but no short ones,

$$\chi(\varepsilon_c(G \boxtimes K_q)) > c$$

where $c = \lceil 3.1 \cdot q \rceil$.

2. Can also show:

$$\chi(G \boxtimes K_q) > c$$

(2)

3. and yet...

$$\chi((G \boxtimes K_q) \times \varepsilon_c(G \boxtimes K_q)) = c.$$  

(3)
Future Directions

There have been attempts to modify Hedetniemi’s Conjecture, in terms of the *Poljak–Rödl function*.

**Definition**

The **Poljak–Rödl function** $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfies

$$f(n) = \min_{\chi(G), \chi(H) \geq n} \chi(G \times H).$$

(4)

Hedetniemi is false $\implies f(n) < n$ for some $n \in \mathbb{N}$.

**Weak Hedetniemi Conjecture**

$$\lim_{n \to \infty} f(n) = \infty.$$ 

(5)
Colorings and Cliques
Generalizing Colorings

Standard $k$-coloring of a graph = a $\{0, 1\}$-valued function on independent sets

Generalization: a nonnegative function on all independent sets of a graph.

1 1 1
Fractional Colorings: Examples

(0, 1, 0, 0, 0, 0, 1, 1)

(1, 1, 1, 1, 1, 1, 1, 1)

(0, 0.5, 0.5, 0, 0, 0.5, 0.5, 1)

(0, 0.5, 0.5, 0, 0, 0.5, 0.5, 1)
Fractional Colorings: Definitions

Let $\mathcal{I}(X)$ denote the set of all independent sets of a graph $X$, and let $\mathcal{I}(X, u)$ denote all the independent sets that also contain the vertex $u$.

**Definition**

A fractional coloring of a graph $X$ is a function $f : \mathcal{I}(X) \rightarrow \mathbb{R}_{\geq 0}$ such that for all vertices $x \in X$, $\sum_{S \in \mathcal{I}(X,x)} f(S) \geq 1$.

**Definition**

The weight of a fractional coloring is defined as $\sum_{S \in \mathcal{I}(X)} f(S)$. The fractional chromatic number $\chi^*(X)$ of the graph $X$ is the minimum possible weight of a fractional coloring.
Generalizing Cliques

Cliques (complete subgraphs) = \{0, 1\}-valued functions on vertices.

Generalization: sum up nonnegative functions over vertices.
Fractional Cliques: Examples

- For the graph with vertices labeled 0, 0, 0, 0, the fractional clique is given by the vector (0, 0, 0, 0, 0, 0, 0, 0).

- For the graph with vertices labeled 0, 1, 1, 1, the fractional clique is given by the vector (0, 1, 1, 1, 0, 1, 1, 1).

- For the graph with vertices labeled 0.3, 0.7, 0.7, 0.5, 0.5, the fractional clique is given by the vector (0.3, 0.7, 0.7, 0.5, 0.5, 1, 1, 1).
Fractional Cliques: Definitions

Definition

A fractional clique of a graph $X$ is a function $f : V(X) \rightarrow \mathbb{R}_{\geq 0}$ such that $\sum_{v \in V(S)} f(v) \leq 1$ for all independent sets $S \in \mathcal{I}(X)$.

Definition

The weight of a fractional clique is defined as $\sum_{v \in V(X)} f(v)$. The fractional clique number of $\omega^*(X)$ of the graph $X$ is the maximum possible weight of a fractional clique.
Duality

Proposition

For any graph $X$, we have $\omega^*(X) \leq \chi^*(X)$.
Symmetry of graphs: Transitivity
**Definition**

A **graph automorphism** is a permutation of the vertices that takes edges to edges and nonedges to nonedges. They form a group, $\text{Aut}(X)$.

**Example**

A graph with vertices $1, 2, 3, 4, 5$ has automorphism group $\text{Aut}(X) = \{ (1), (14)(23) \}$.

$\hspace{2cm}$

$\text{Aut}(X) = S_4$

**Proposition**

A graph automorphism preserves the degree of a vertex.
Transitivity

$Aut(X)$ acts on the set of vertices, the set of edges, and the set of arcs (ordered pairs of two adjacent vertices).

**Definition**

Given a set $A$ on which $Aut(X)$ acts, we say that a graph is $A$-transitive if for every $a, b \in A$, there is a graph automorphism taking $a$ to $b$.

**Example**

Any cycle $C_n$ is vertex, edge, and arc transitive.

The star graph $K_{1,4}$ is edge but not arc transitive since $(1, 2) \not\sim (2, 1)$.

The graph $C_2 \cup C_1$ is arc and edge transitive but not vertex transitive.
s-arc Transitivity

**Definition**

An s-arc is a sequence \((v_0, v_1, \ldots, v_s)\) of adjacent vertices such that \(v_{i-1} \neq v_{i+1}\) for all \(i\).

Note that 0-arc transitivity is the same as vertex transitivity, and 1-arc transitivity is the same as arc transitivity.

**Example**

A cycle \(C_n, n \geq 3\) is s-arc transitive for all \(s\).

The star graph \(K_{1,4}\) is 2-arc transitive.
**Example**

The cube is 0-, 1-, and 2-arc transitive, but not 3-arc transitive.

![Graph Diagram]

**Proposition**

If every connected component of $X$ contains a cycle, then

$$s$$-arc transitive $\implies (s - 1)$-arc transitive.

If $X$ satisfies this condition and is $s$-transitive for some $s$, then $X$ is vertex transitive, so every vertex has the same degree.

We will consider graphs of degree at least 3.
Restrictions on $s$

**Theorem (Tutte, 1947)**

Let $X$ be an $s$-arc transitive graph of degree equal to 3. Then $s \leq 5$.

**Example**

The Tutte-Coxeter graph achieves $s = 5$.  

![Graph Diagram](image_url)
Restrictions on $s$

**Theorem (Weiss, 1981)**

Let $X$ be an $s$-arc transitive graph of degree at least 3. Then $s \leq 7$. Furthermore, if $s = 6$ then $X$ is 7-arc transitive.

**Example**

The smallest known example of a nontrivial 7-arc transitive graph has degree four and is on 728 vertices.
Thank you

We would like to thank:

- Younhun Kim
- Dr. Slava Gerovitch
- Prof. Pavel Etingof
- The MIT PRIMES program


