Evolution of Curves via Curve Shortening Flow

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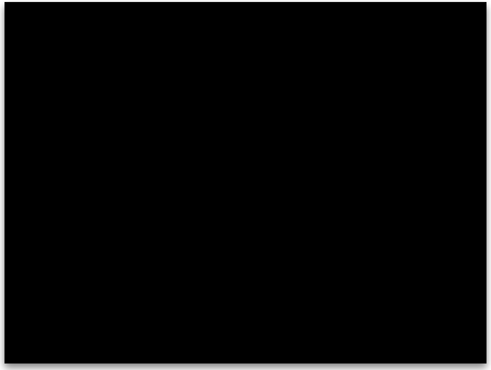
Introduction

Curve shortening flow (CSF) is an intriguing phenomenon that affects the geometric properties of curves in a highly interesting manner. It has numerous applications, from other geometric flows to how heat diffuses over an area and shape analysis.

In this slideshow, we discuss interesting areas and results in prominent ideas of curve shortening flow.



Warm greetings irrevocably dissolving away as they undergo CSF

CSF demonstration created and captured using https://a.carapetis.com/csf_record/#

Curve Shortening Flow (CSF): Definition

The curve-shortening flow equation is defined with as

$$\partial_t \gamma = kN$$

We will establish the basis to comprehend this equation as well as our following discussion.

For our purposes, we will be working in two dimensions.

An immersed curve with a case of CSF

CSF demonstration created and captured using https://a.carapetis.com/csf_record/#

What Comprises a Curve?

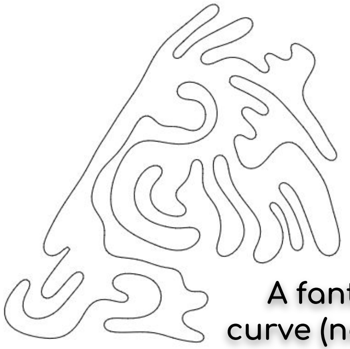
Definition:

A *curve* is a differentiable function

$$\alpha : I \in \mathbb{R} \to \mathbb{R}^2$$

The *arclength* of a curve is defined as the curve's total length, denoted L

A curve is *convex* if any two points on the curve can be connected with a straight line that does not intersect with the curve.



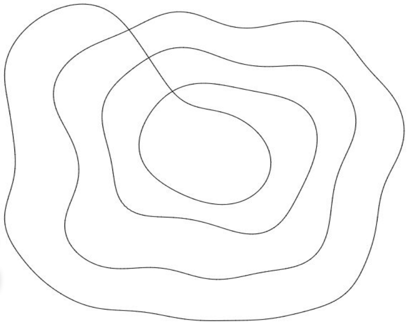
A fantastical curve (not convex)

Curve illustration created and captured using https://a.carapetis.com/csf_record/#

Curve Types: Closed Curves

Definition:

A *closed curve* is a curve that has identical endpoints and is smooth at those endpoints



A cantankerously correctly closed curve

Curve illustration created and captured using https://a.carapetis.com/csf_record/#

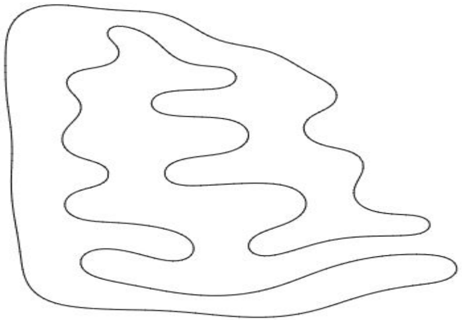
Curve Types: Embedded Curves

Definition:

An *embedded curve* is a curve that is a bijection of the interval

$$I \in \mathbb{R} \to \mathbb{R}^2$$

onto its image



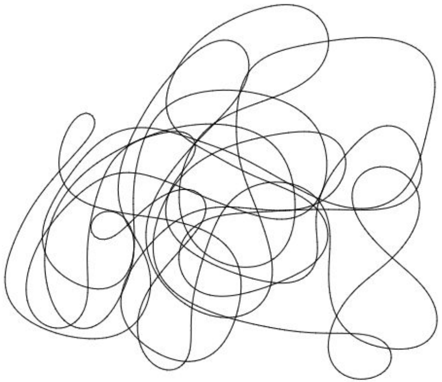
An empty and emphatic embedded curve

Curve illustration created and captured using https://a.carapetis.com/csf_record/#

Curve Types: Immersed Curves

Definition:

An *immersed curve* is one where the derivative is nonzero.
They admit self-intersections.
They may have infinite arclength.

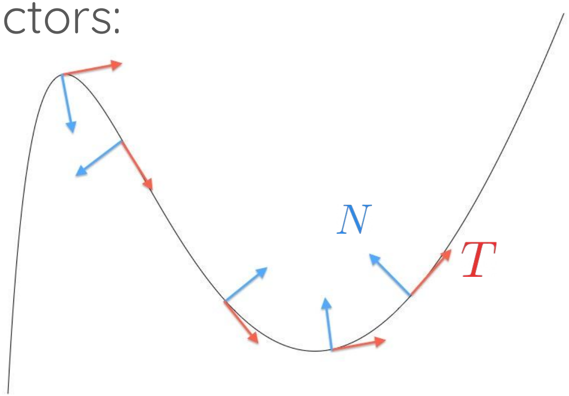


A hairball of an immersed curve

Curve illustration created and captured using https://a.carapetis.com/csf_record/#

Important Vectors: the Tangent and Normal

We will be using a frame field comprised of the tangent and normal vectors:



Tangent Vector: T
Normal Vector: N

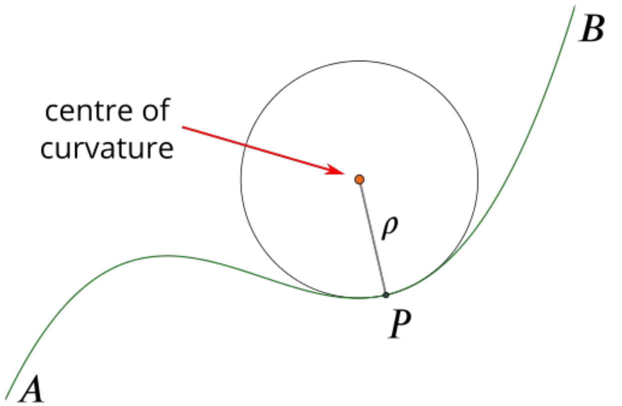
Here, the red vector is the tangent and the blue vector is the normal

Vectors illustration adapted from “Supplemental Modules (Calculus).” Author unknown. <math.libretexts.org>. Date of publication unknown. Accessed on 5/29/20.

Defining Curvature (continued)

The unique circle touching the curve at only point P on this side of the curve

$$k = \frac{1}{\rho}$$



Curvature illustration adapted from "Curvature|Glossary." Author unknown. *Underground Mathematics, University of Cambridge.* <undergroundmathematics.org/glossary/curvature>. Date of publication unknown. Accessed on 6/01/20.

Defining the Commonly Crucial Curvature

Curvature is

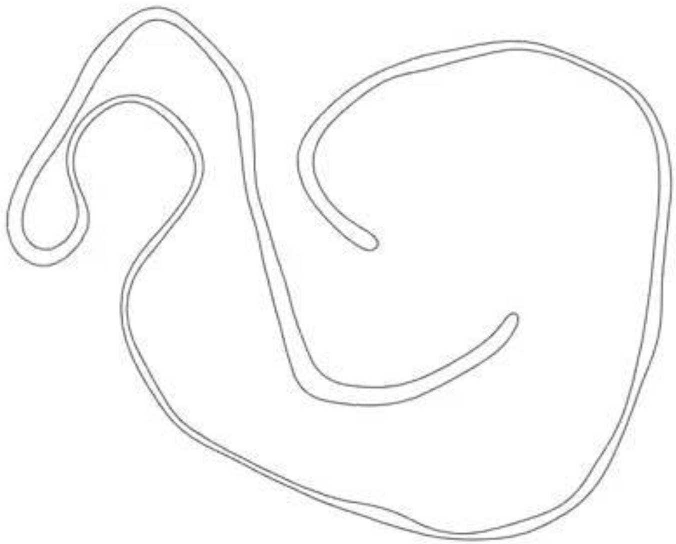
$$k = T'/N$$

Described as how much a curve “twists” at a given point

CSF Examples

CSF demonstration created and captured using
https://a.carapetis.com/csf_record/#

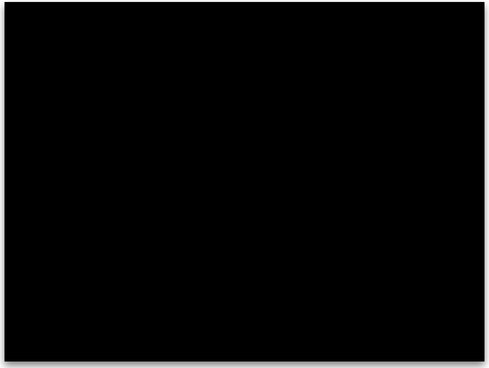
CSF Examples (continued)



CSF demonstration created and captured using
https://a.carapetis.com/csf_record/#

Evolution of Geometric Properties in CSF: Area

Area decreases at a uniform rate of 2π. Given A(t) is function of time that outputs the area of a closed curve, undergoing CSF, it becomes extinct at time A(0)/2π.

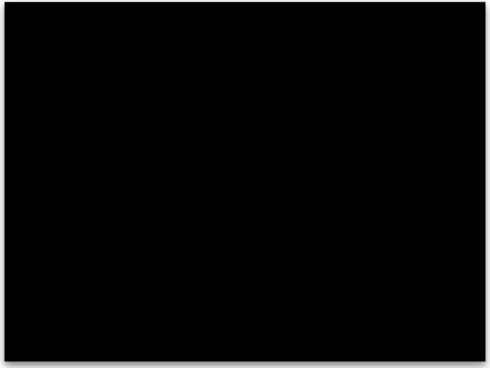


CSF demonstration created and captured using https://a.carapetis.com/csf_record/#

Evolution of Geometric Properties in CSF: Arclength

Arclength evolution over time:

$$\frac{dL}{dt} = -\int k^2 \, ds$$

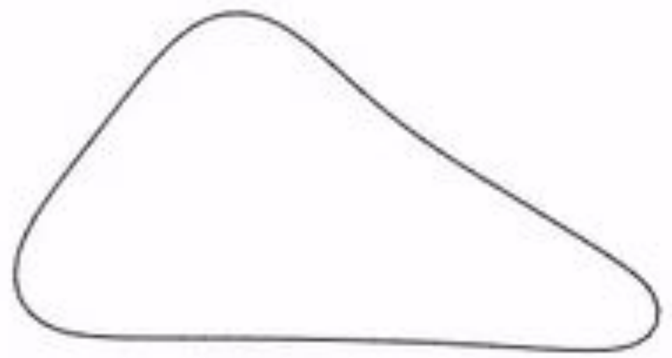


CSF demonstration created and captured using https://a.carapetis.com/csf_record/#

Important Theorems - Gage-Hamilton

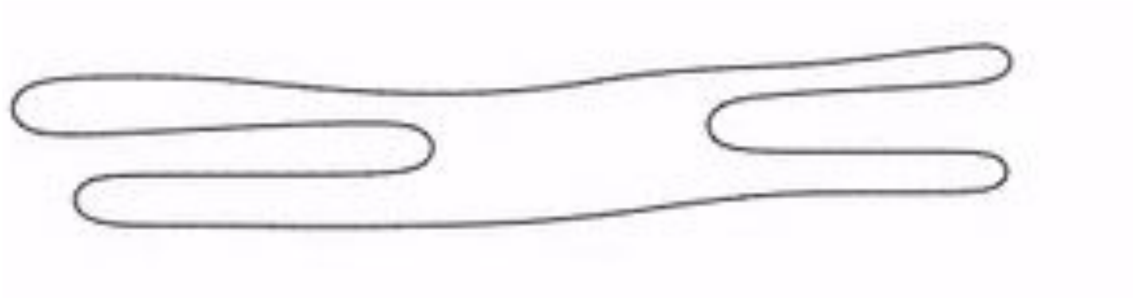
A **convex** curve that is embedded, closed, and smooth, shrinks to a single point

as t $\longrightarrow A_0/2\pi$



Important Theorems - Grayson’s Theorem

A closed and embedded curve **becomes convex &** converges to a round shrinking circle



Special Solutions to the CSF Problem

Special Solutions → unique geometric properties under CSF

Self-Similar → solutions which preserve their shape and are translated under CSF

Translating Solutions:

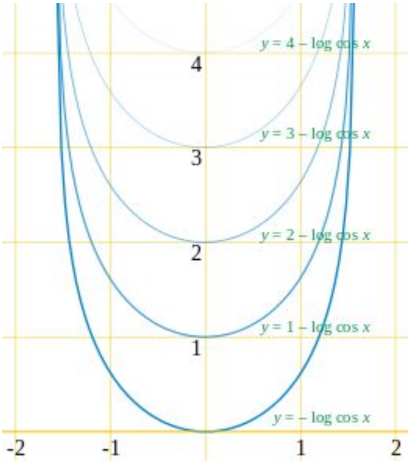
$$v(x,t) = v(x) + ct$$

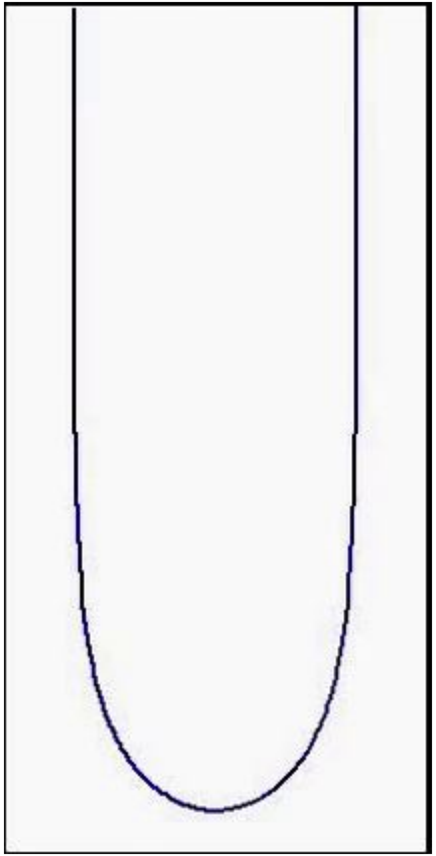
Special Solutions: Grim Reaper

$$y = -\log\cos(x), x \in (-\pi/2, \pi/2)$$

The Grim Reaper's shape is preserved

Only travelling curve



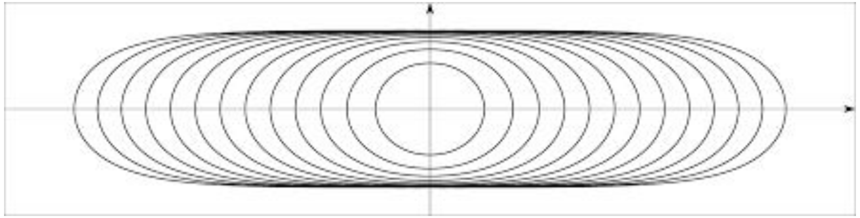


Special Solutions: Paperclip

$$\cosh(y(t)) = e^{t}\cos(x(t))$$

$t \to 0$ Single Point

$t \to -\infty$ Oval made up of 2 grim reapers

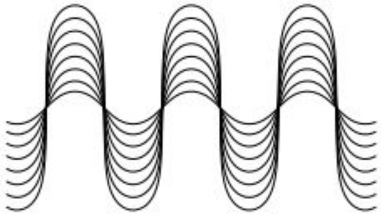


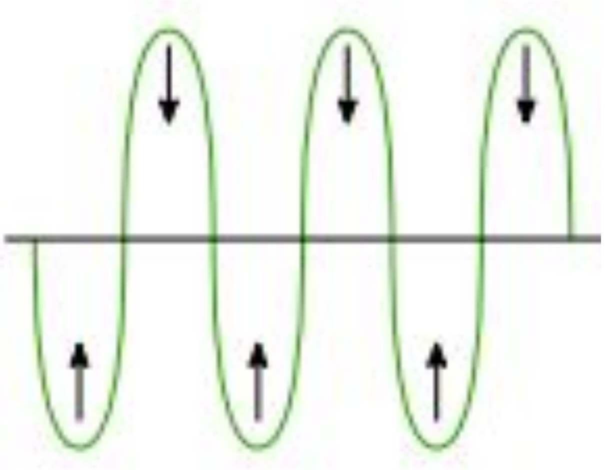
Special Solutions: Hairclip

$$\sinh(y(t)) = e^t \cos x(t)$$

$t \rightarrow 0$ Horizontal line

$t \rightarrow -\infty$ Connected grim reapers alternating between translating up and down



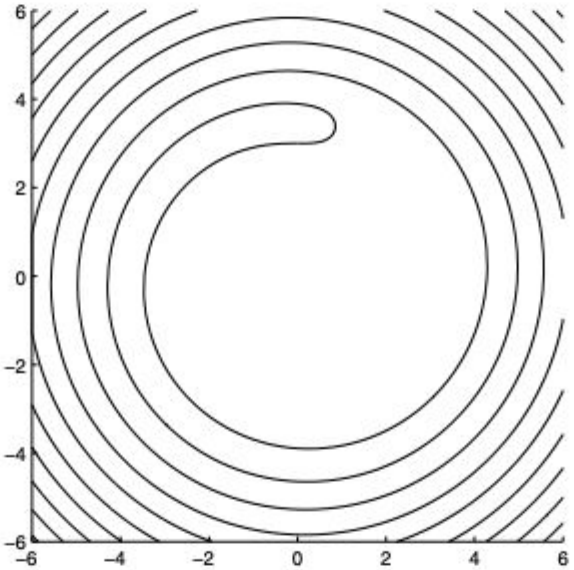


Special Solutions: Spirals

Rotate about the origin in the polar angle

$$r\cos(\alpha(r) + ct), r\sin(\alpha(r) + ct)$$

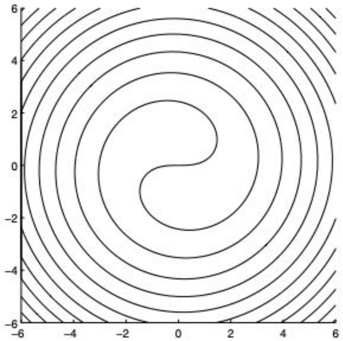
Rotate with constant speed c



Special Solutions: Spirals - Yin Yang

Yin Yang

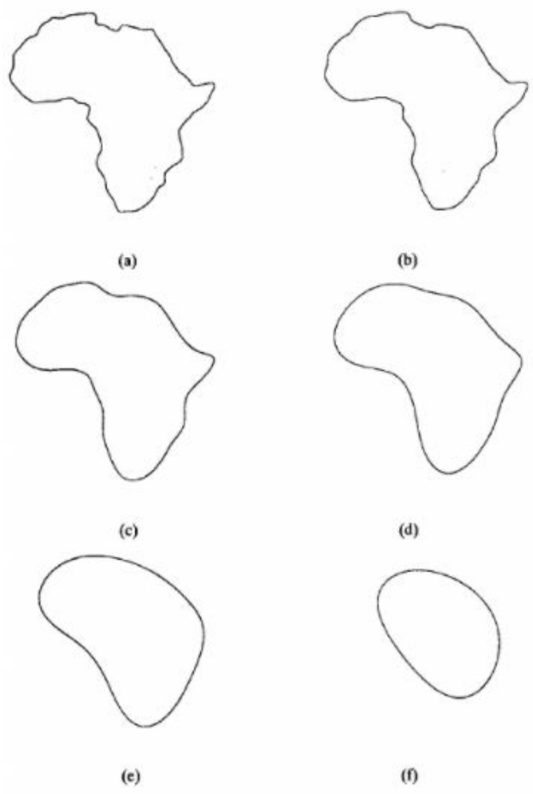
1 Inflection point that remains at the same location



Applications: Shape Analysis

Image Processing & computer vision

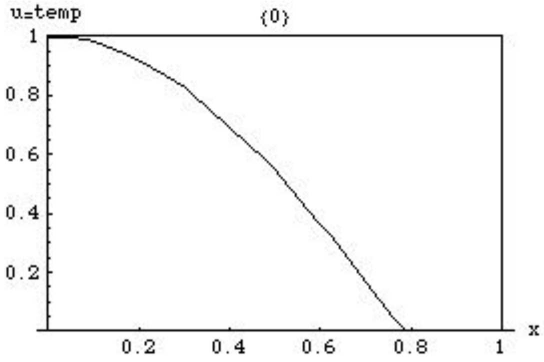
Remove "noise" from a shape derived from a digital image



Applications: Heat Diffusion

n-dimension heat equation:

$$u_t - \Delta u = 0$$



1-dimensional heat equation behavioral demonstration adapted from "Heat Equation." Authors unknown. *Wikipedia.org*. <https://en.wikipedia.org/wiki/Heat_equation>. Date of publication unknown. Accessed on 6/06/20.

Thank You for Your Attention! Questions?

We are grateful for this opportunity to **PRIMES Circle**.

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We are thankful to **Peter Haine** for coordinating this program and providing valuable feedback on our progress.