The Prime Number Theorem
A PRIMES Exposition

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• Euclid (300 BC): There are infinitely many primes
• Legendre (1808): for primes less than 1,000,000:
\[ \pi(x) \sim \frac{x}{\log x} \]
Progress on the Distribution of Prime Numbers

- Euler: The product formula

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}
\]

so (heuristically)

\[
\prod_p \frac{1}{1 - p^{-1}} = \log \infty
\]

- Chebyshev (1848-1850): if the ratio of \( \pi(x) \) and \( x/\log x \) has a limit, it must be 1

- Riemann (1859): *On the Number of Primes Less Than a Given Magnitude*, related \( \pi(x) \) to the zeros of \( \zeta(s) \) using complex analysis

- Hadamard, de la Vallée Poussin (1896): Proved independently the prime number theorem by showing \( \zeta(s) \) has no zeros of the form \( 1 + it \), hence the celebrated prime number theorem
Theorem (Maximum Principle)

Let $\Omega$ be a domain, and let $f$ be holomorphic on $\Omega$.

(A) $|f(z)|$ cannot attain its maximum inside $\Omega$ unless $f$ is constant.

(B) The real part of $f$ cannot attain its maximum inside $\Omega$ unless $f$ is a constant.

Theorem (Jensen’s Inequality)

Suppose $f$ is holomorphic on the whole complex plane and $f(0) = 1$. Let $M_f(R) = \max_{|z|=R} |f(z)|$. Let $N_f(t)$ be the number of zeros of $f$ with norm $\leq t$ where a zero of multiplicity $n$ is counted $n$ times. Then

$$\int_0^R \frac{N_f(t)}{t} dt \leq \log M_f(R).$$

- Relates growth of a holomorphic function to distribution of its zeroes
- Used to bound the number of zeroes of an entire function
Theorem (Borel-Carathéodory Lemma)

Suppose $f = u + iv$ is holomorphic on the whole complex plane. Suppose $u \leq A$ on $\partial B(0, R)$. Then

$$|f^{(n)}(0)| \leq \frac{2n!}{R^n} (A - u(0))$$

- Bounds all derivatives of $f$ at 0 using only the real part of $f$
- Used in proof of Hadamard Factorization Theorem to prove that function is a polynomial by taking limit and showing that nth derivative approaches 0
Definition (Order)

The order of an entire function, $f$, is the infimum of all possible $\lambda > 0$ such that there exists constants $A$ and $B$ that satisfy

$$|f(z)| \leq Ae^{B|z|^\lambda}$$

- $\sin z$, $\cos z$ have order 1
- $\cos \sqrt{z}$ has order $1/2$
Theorem

Let \( f \) be an entire function of order \( \lambda \) with \( f(0) = 1 \). Then, for any \( \varepsilon > 0 \) there exists a constant, \( C_\varepsilon \), that satisfies

\[
N_f(R) \leq C_\varepsilon R^{\lambda + \varepsilon}
\]

Theorem

Let \( f \) be an entire function of order \( \lambda \) with \( f(0) = 1 \) and \( a_1, a_2, \ldots \) be the zeroes of \( f \) in non-decreasing order of norms. Then, for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda + \varepsilon} < \infty
\]

In other words, the convergence index of the zeros is at most \( \lambda \).

For example, \( \sin z \) and \( \cos z \) have order 1, and their zeroes grow linearly while \( \cos \sqrt{z} \) has order \( 1/2 \), and its zeroes grow quadratically.
Theorem (Hadamard Factorization Theorem)

A complex entire function $f(z)$ of finite order $\lambda$ and roots $a_i$ can be written as

$$f(z) = e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp \left(\sum_{k=1}^{p} \frac{z^k}{k a_n^k}\right)$$

with $p = \lfloor \lambda \rfloor$, and $Q(z)$ being some polynomial of degree at most $p$.

The theorem extends the property of polynomials to be factored based on their roots as

$$k \prod_{i=1}^{n} \left(1 - \frac{z}{a_i}\right).$$
The proof is based off of truncating the first $p$ terms of the series

$$\log \left( 1 - \frac{z}{a_n} \right) = -\sum_{k=1}^{\infty} \frac{z^k}{k a_n^k}$$

which bounds the magnitude to $O(R^{\lambda + \epsilon})$ and gives rise to the exponential factor. Now,

$$g(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) \exp \left( \sum_{k=1}^{p} \frac{z^k}{k a_n^k} \right)$$

has the same roots as $f(z)$ and the polynomial $Q(z)$ is found by taking the logarithm of $f(z)/g(z)$. The degree of $Q$ is determined by bounding the derivatives using the Borel-Carathéodory lemma.
Riemann Zeta Function

Definition (Reimann \( \zeta \) Function)

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \quad \sigma > 1.
\]

Theorem (Euler Product Formula)

The zeta function can also be written as

\[
\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}
\]

The Euler product formula is the analytic equivalent of the unique factorization theorem for integers.
Chebyshev Functions

Definition (Van-Mangoldt Function)

\[ \Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1 \\ 
0 & \text{otherwise} 
\end{cases} \]

Definition (Chebyshev Functions)

\[ \psi(x) = \sum_{n \leq x} \Lambda(n), \quad \vartheta(x) = \sum_{p \leq x} \log p. \]

Theorem

\[ \frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \]
The Chebyshev functions can be related to $\pi(x)$ by the following integral expressions.

**Theorem**

\[
\vartheta(x) = \pi(x) \log x + \int_2^x \frac{\pi(t)}{t} dt
\]

\[
\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt
\]

Studying the asymptotic behavior of the formulas, we see that all of the following expressions are logically equivalent:

\[
\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1
\]

\[
\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1
\]

\[
\lim_{x \to \infty} \frac{\psi(x)}{x} = 1
\]
The Perron Formula acts as a filter to isolate the first finitely many terms from a Dirichlet series.

**Theorem (Perron Formula)**

Let \( F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \) be absolutely convergent for \( \sigma > \sigma_a \). Then for arbitrary \( c, x > 0, \) if \( \sigma > \sigma_a - c \),

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s+z) \frac{x^z}{z} \, dz = \sum_{n \leq x} \* \frac{f(n)}{n^s}
\]

where \( \sum^\* \) means that the last term is halved when \( x \) is an integer.

**Corollary**

For \( x > 0, \)

\[
\int_0^x \frac{\psi(y)}{y} \, dy = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s^2} \, ds
\]
We define a function $\xi$ in terms of $\Gamma$, $\zeta$, and $\pi$ to obtain a functional equation that gives information about the symmetry of zero distribution.

**Definition**

$$\xi(s) = \frac{\Gamma\left(\frac{s}{2}\right) \zeta(s)}{\pi^{s/2}}.$$

**Theorem (Functional Equation, Riemann 1859)**

$$\xi(s) = \xi(1 - s)$$

The proof relies on the Poisson summation formula from Fourier analysis.
Proof of the Functional Equation

\[ \frac{\Gamma \left( \frac{s}{2} \right) \zeta(s)}{\pi^{s/2}} = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s/2-1} e^{-n^2 \pi x} \, dx \]

\[ = \int_{0}^{\infty} x^{s/2-1} \Psi(x) \, dx = \int_{0}^{1} + \int_{1}^{\infty} \]

where

\[ \Psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}. \]

Using the Poisson summing formula,

\[ 2\Psi(x) + 1 = \frac{1}{\sqrt{x}} \left( 2\Psi \left( \frac{1}{x} \right) + 1 \right), \]

substituting into the integral from 0 to 1,

\[ \xi(s) = \frac{1}{s(s-1)} + \int_{0}^{1} x^{s/2-3/2} \Psi \left( \frac{1}{x} \right) \, dx + \int_{1}^{\infty} x^{s/2-1} \Psi(x) \, dx. \]
Changing the variable $x \rightarrow \frac{1}{x}$

$$\xi(s) = \frac{\Gamma\left(\frac{s}{2}\right) \zeta(s)}{\pi^{s/2}} = \frac{1}{s(s-1)} + \int_{1}^{\infty} \left( x^{-s/2-1/2} + x^{s/2-1} \right) \Psi(x) \, dx$$

Substituting $s = 1 - s$ gives $\xi(s) = \xi(1 - s)$.
The functional relations of $\zeta(s)$ can remarkably be obtained by studying the theory of $p$-adic numbers.

Generally, the distance between two numbers is considered using the usual metric $|x - y|$, but for every prime $p$, a separate notion of distance can be made for $\mathbb{Q}$.

For a rational number $x = p^n a/b$, $p \nmid a, b$, we define the $p$-adic absolute value as $|x|_p = p^{-n}$. Then, the $p$-adic distance between two numbers is defined as $|x - y|_p$.

The $p$-adic absolute value is multiplicative, positive definite, and satisfies the strong triangle inequality: $|x - y|_p \leq \max(|x|_p, |y|_p) \leq |x|_p + |y|_p$

Examples:

$$|7/24|_2 = 8$$

$$|2 - 27|_5 = 1/25$$
• \( \mathbb{Q} \) can be completed under this metric to form the field \( \mathbb{Q}_p \)

• \( \mathbb{Q}_p \) contains a subring, \( \mathbb{Z}_p \), which is the completion of \( \mathbb{Z} \) under \( | \cdot |_p \)

• \( \mathbb{Z}_p \) can be written using an "infinite \( p \)-adic expansion" with no negative powers and numbers in \( \mathbb{Q}_p \) have finite negative powers

• Example: in \( \mathbb{Q}_5 \),

\[
\ldots + 5^2 + 5 + 1 = \sum_{n=0}^{\infty} 5^n = \frac{1}{1 - 5} = -\frac{1}{4} \in \mathbb{Z}_5
\]

and \(-1/5 = \ldots 111 \times \frac{4}{5} = \ldots 444.4\)

• Letting \( \mathbb{Z}_p^\times \) denote the elements of \( \mathbb{Z}_p \) for which \( |x|_p = 1 \), \( \mathbb{Q}_p^\times = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p^\times \)
Adèles and Idèles

- The fields $\mathbb{R}$ and $\mathbb{Q}_p$ can be put together to form what is called the ring of Adèles.
- The Adèle ring $\mathbb{A}_\mathbb{Q}$ is the collection of sequences $x = \{x_p\}_{p \in \mathbb{P} \cup \{\infty\}}$ for the primes $\mathbb{P}$ and where for each $p \in \mathbb{P}$, $x_p \in \mathbb{Q}_p$ and $x_\infty \in \mathbb{R}$ with almost all $x_p \in \mathbb{Z}_p$.
- The Idèle group $\mathbb{I}_\mathbb{Q}$ is the collection of Adèles for which almost all $x_p \in \mathbb{Z}_p^\times$ and forms a group under componentwise multiplication.
- We can also introduce a topology on $\mathbb{I}_\mathbb{Q}$ with open sets being the product of open sets in $\mathbb{R}_\times^\times$ and $\mathbb{Q}_p^\times$, making $\mathbb{I}_\mathbb{Q}$ is a locally compact abelian group, a Haar measure, $\mu$, can be formed for it.
- Introduce the volume function $\|x\| := |x_\infty| \times \prod_{p \in \mathbb{P}} |x_p|_p$ and the function $\varphi(x) := \exp(-\pi |x_\infty|^2) \prod 1_{\mathbb{Z}_p}(x_p)$ and consider the integral

$$\int_{\mathbb{I}_\mathbb{Q}} \varphi(x) \|x\|^s \, d\mu(x)$$
The integral can be split into the real and $p$-adic components to obtain

$$\xi(s) = \left( \int_{\mathbb{R} \times \mathbb{R}} e^{-\pi |t|^2} |t|^s \frac{dt}{t} \right) \times \left( \prod_p \int_{\mathbb{Z}_p^\times} |x_p|^s d\mu_p \left( x_p \right) \right).$$

The first integral turns out to be $\Gamma \left( \frac{s}{2} \right) \pi^{-s/2}$ and the latter ones are $\frac{1}{1-p^{-s}}$ which combine to form $\zeta(s)$. So the integral coincides with the $\xi(s)$ introduced before:

$$\xi(s) = \Gamma \left( \frac{s}{2} \right) \zeta(s) \pi^{-s/2}.$$
The \( \xi \) Function Revisited

- The integral can also be evaluated in a different way using the fact that \( \mathbb{Q}^\times \) naturally embeds into \( \mathbb{I}_\mathbb{Q} \) with constant sequences to form the group of principal idèles.

- Taking the integral by considering equivalences classes, \( \bar{x} \), of \( \mathbb{I}_\mathbb{Q} \) over the principal idèles, it can be shown that

\[
\xi(s) = \int_{\|\bar{x}\| > 1} (\|\bar{x}\|^s + \|\bar{x}\|^{1-s})(\Theta(\bar{x}) - 1)d\bar{\mu}(\bar{x})
\]

\[
+ \int_{\|\bar{x}\| > 1} (\|\bar{x}\|^{1-s} - \|\bar{x}\|^{-s})d\bar{\mu}(\bar{x})
\]

for the Jacobi theta function

\[
\Theta(\bar{x}) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \|\bar{x}\|^2}
\]

which gives \( \xi(s) = \xi(1 - s) \) by using the Poisson summation formula.

- The Poisson identity translates into Fourier analysis on \( \mathbb{I}_\mathbb{Q} \).
Factorization of $\zeta(s)$

- We find that this function is asymptotically related to $\Gamma$ by
  $$\xi(s) \sim \Gamma \left( \frac{s}{2} \right)$$
as $s \to +\infty$.

- Using the Hadamard Factorization Theorem, we obtain
  $$\xi(s) = e^{as+b} \frac{s(1-s)}{s(1-s)} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho}.$$ 

- Hence
  $$\zeta(s) = \frac{e^{A+Ds}}{s-1} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{2n} \right) e^{-s/(2n)}.$$
Theorem

For \( x > 0 \), not equal to an integer,

\[
\int_0^x \frac{\psi(y)}{y} \, dy = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{\zeta'(s)}{s} \frac{x^s}{s^2} \, ds
\]

Substituting in

\[
\frac{\zeta'(s)}{\zeta(s)} = D - \frac{1}{s - 1} + \sum_p \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right),
\]

calculating the integrals using the residue theorem, We finally get the explicit formula for primes:

\[
\int_0^x \frac{\psi(y)}{y} \, dy = x - (D + 2) \log x - \sum_\rho \frac{x^\rho}{\rho^2} + \left( \frac{\pi^2}{24} + \sum_\rho \frac{1}{\rho^2} \right) - \sum_{n=1}^{\infty} \frac{x^{-2n}}{4n^2}.
\]
Theorem (de la Vallée Poussin, Hadamard, 1896)

No zero of $\zeta(s)$ has real part 1.

Proof.

Taking the logarithm of the Euler product representation of $\zeta(s)$, we get

$$\log |\zeta(\sigma + it)| = -\Re \sum_{p} \log(1 - p^{-(\sigma+it)}) = \sum_{p} \sum_{n=1}^{\infty} \frac{1}{np^{n\sigma}} \cos(n\sigma \log p)$$

So

$$3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| = \sum_{p} \sum_{n=1}^{\infty} \frac{1}{np^{n\sigma}} 2(\cos(n\sigma \log p) + 1)^2 \geq 0$$

Thus,

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1, \sigma > 1.$$  

So if $1 + it$ is a zero of $\zeta(s)$, then letting $\sigma \downarrow 1$, we arrive at a contradiction.
Thus, all $\rho$ lie in the strip $0 < Re(\rho) < 1$. So there is a continuous non-increasing function $h : [0, \infty) \rightarrow (0, 1)$, such that $\zeta(s)$ is zero-free in the region $\sigma < h(t)$.

We use this fact to bound

$$\left| \sum_{\rho} \frac{x^\rho}{\rho^2} \right| = o(x)$$

Thus,

$$\int_0^x \frac{\psi(y)}{y} dy = x + o(x),$$

which is equivalent to the prime number theorem.
The more we increase the bounds on the zero-free region, the better our precision of our estimate will be.

**Theorem (de la Vallée Poussin, 1898)**

There is a constant $A > 0$ such that $\zeta(s)$ has no zero in the region

$$\sigma < 1 - \frac{A}{\log(2 + t)}, \ t \geq 0.$$  

We use this to bound $\psi(y)$ and use the relations between the Chebychev functions to $\pi(x)$ to get

$$\pi(x) = \text{Li}(x) + O(xe^{-c\sqrt{\log x}})$$

which is the prime number theorem with error term.