Walks on Young's Lattice

USA-PRIMES Reading Group

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About Algebraic Combinatorics

Text: *Algebraic Combinatorics* by Richard Stanley

Apply tools from linear algebra to combinatorial problems

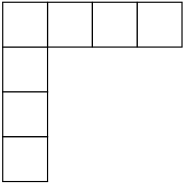
- Walks on Graphs
- Group Actions
- Spanning Trees
- Electrical Networks
- *Young Diagrams and Tableaux*

Young Diagrams

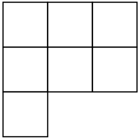
Definition

A *Young Diagram* is a collection of cells on a grid that are NW justified.

We refer to a Young Diagram by a nonincreasing sequence of numbers, representing the size of each row.



$\sim (4,1,1,1) \vdash 7$

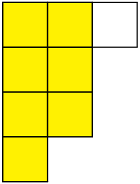


$\sim (3,3,1) \vdash 7$

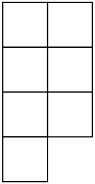
Covering Relations

We say a Young Diagram λ *covers* μ if μ fits into λ and λ has exactly one more square than μ.

Example:



covers:



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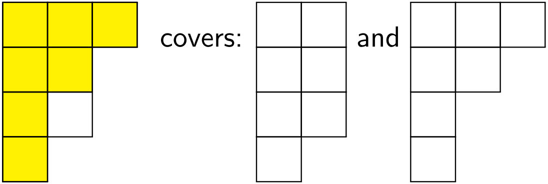

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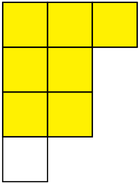
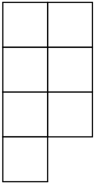
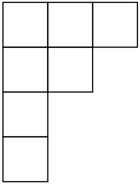


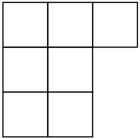
covers:



and



and



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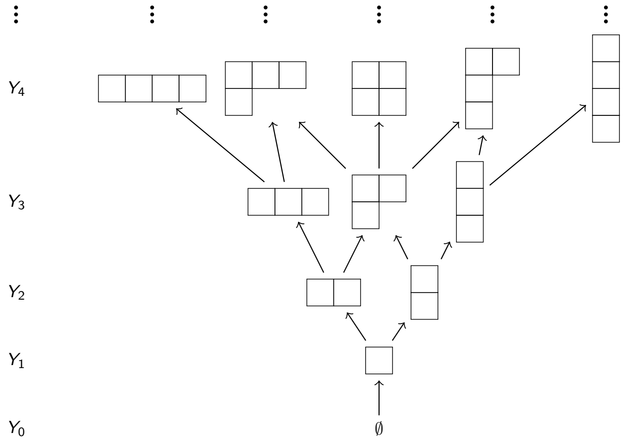
Young's Lattice

Young's Lattice is a visual representation of the covering relations.



Formal Sums

$\mathbb{R}Y$ is the real vector space generated by the elements of Y. These are *formal sums* of Young Diagrams.

A typical element looks like

$$\alpha = 2 \cdot \ydiagram{3,3,2} - 1.5 \cdot \ydiagram{4,1} + 3 \cdot \emptyset$$

.

We let α_λ refer to the coefficient of λ in α. For example,

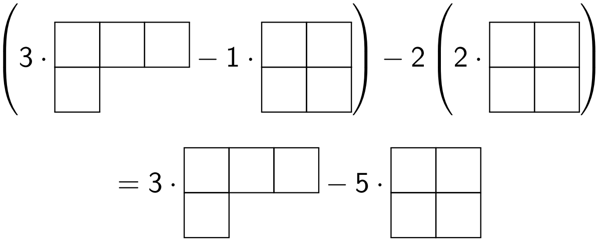
$$\alpha_{\ydiagram{4,1}} = -1.5.$$

Formal Sums

Note the distinction between the basis vector $\emptyset$ and the vector $\vec{0}$.

$$\emptyset + \emptyset = 2 \cdot \emptyset$$
$$\vec{0} + \vec{0} = \vec{0}$$

The usual properties of scalar multiplication and vector addition hold.

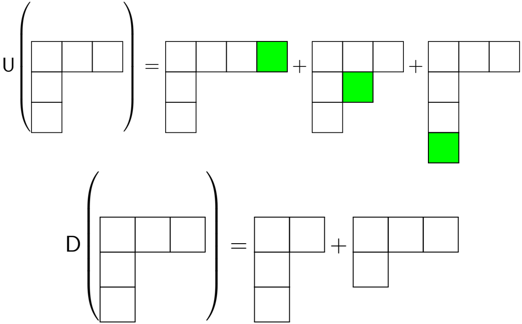


Order-Raising and Order-Lowering Operators

Linear transformations on $\mathbb{R}Y$ defined using the covering relations:

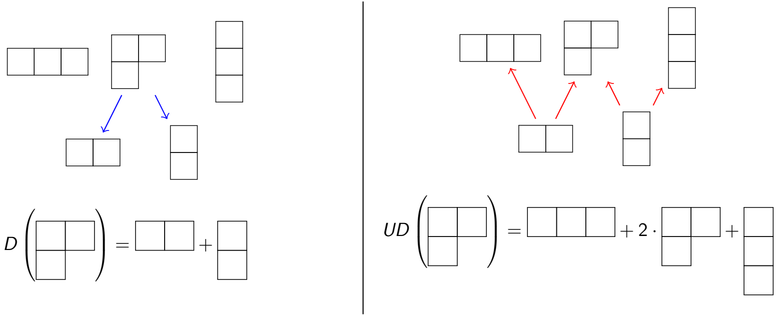
$$U : \mathbb{R}Y \to \mathbb{R}Y \qquad D : \mathbb{R}Y \to \mathbb{R}Y$$

$$U(\lambda) = \sum_{\mu \text{ covers } \lambda} \mu \qquad D(\lambda) = \sum_{\lambda \text{ covers } \mu} \mu$$



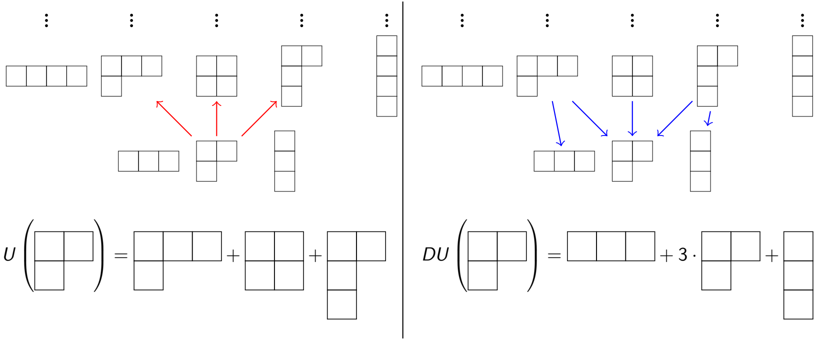
Counting Paths

The *order-raising* and *order-lowering operators* U and D essentially model walking upwards and downwards on the lattice.



$UD(\lambda)_\mu$ is the number of paths from λ to μ that go down then up.

Counting Paths

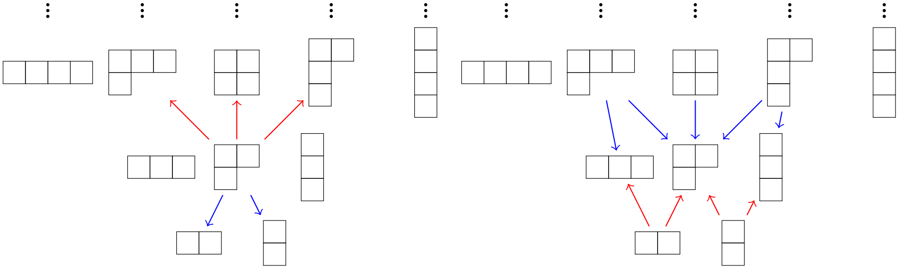


$DU(\lambda)_\mu$ is the number of paths from λ to μ that go up then down.

Operator Identity

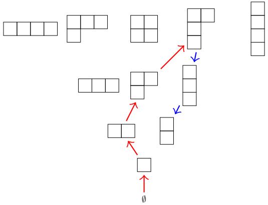
Theorem

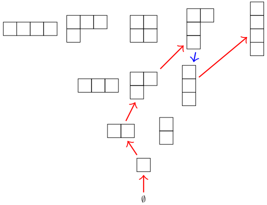
$DU - UD = I_{\mathbb{R}Y}$



Examples of Walks

More generally, let P be a sequence of Us and Ds that correspond to a type of path on Young's Lattice starting from $\emptyset$.





A walk of type D^2U^4

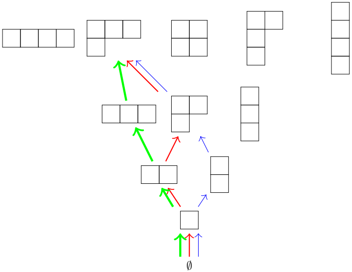
A walk of type UDU^4

$UDU^4(\lambda)_\mu$ is the number of walks of type UDU^4 from λ to μ.

Note that the operators are applied right to left

U^n and D^n

Let's compute $(U^4\emptyset)_{(3,1)}$



1	2	3
4		

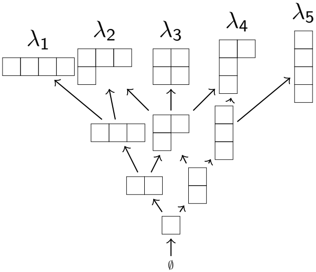
1	2	4
3		

1	3	4
2		

Let $f^\lambda = (U^n\emptyset)_\lambda$ where $\lambda \vdash n$. Similarly, $f^\lambda = (D^n\lambda)_\emptyset$ by reversing the arrows.

Note that f^λ is the number of Standard Young Tableaux.

$D^n U^n$ Walks



- $f^{\lambda_1} = 1.$
- $f^{\lambda_2} = 3$
- $f^{\lambda_3} = 2$
- $f^{\lambda_4} = f^{\lambda_2}$ and $f^{\lambda_5} = f^{\lambda_1}$

$$D^4 U^4 \emptyset = D^4 (\lambda_1 + 3\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5)$$
$$= \left(1^2 + 3^2 + 2^2 + 3^2 + 1^2\right) \emptyset$$

In general,

$$(D^n U^n \emptyset)_\emptyset = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2 .$$

Operator Lemma

Lemma

$DU^k = U^k D + kU^{k-1}$.

Proof. Each time we swap a DU with a UD, we introduce a new U^{k-1} term, since we start with k U's to the right of D, we end up with $U^k D$ and kU^{k-1} terms. For example,

$$\begin{aligned} DU^3 &= (UD + I)U^2 && = UDU^2 + U^2 \\ &= U(UD + I)U + U^2 && = U^2 DU + 2U^2 \\ &= U^2(UD + I) + 2U^2 && = U^3 D + 3U^2 \end{aligned}$$

Counting Formula

Lemma

$DU^k = U^k D + kU^{k-1}$

We can compute $(D^n U^n \emptyset)_\emptyset$ another way using our lemma.

$$\begin{aligned} D^n U^n \emptyset &= \cancel{D^{n-1} U^n D\emptyset} + nD^{n-1} U^{n-1} \emptyset \\ &= \cancel{D^{n-2} U^{n-1} D\emptyset} + n(n-1) D^{n-2} U^{n-2} \emptyset \\ &= \dots \\ &= \cancel{DU^2 D\emptyset} + n(n-1)\cdots(2) DU\emptyset \\ &= \cancel{UD\emptyset} + n(n-1)\cdots(2)(1) I\emptyset \\ &= n!\emptyset \end{aligned}$$

Hence, $\sum_{\lambda \vdash n} \left(f^\lambda\right)^2 = n!$.

Paths on Young's Lattice

We can use this argument to count paths of arbitrary type. For example,

$$\begin{aligned} UD^2U^3DU^4\emptyset &= \cancel{UD^2U^7D\emptyset} + 4UD^2U^6\emptyset \\ &= \cancel{4UDU^6D\emptyset} + 24UDU^5\emptyset \\ &= \cancel{24U^6D\emptyset} + 120U^5\emptyset \end{aligned}$$

And so, for $\lambda \vdash 5$,

$$\left(UD^2U^3DU^4\emptyset\right)_\lambda = 120f^\lambda.$$

Theorem

Let y_i be the level of Young's Lattice we occupy before taking the ith downward step. Then the number of paths of type P starting from $\emptyset$ and ending at λ is given by

$$f^\lambda \prod y_i$$

Connections to Representation Theory

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n!$$

This is a specific case of the more general fact that if $\rho_1, \ldots \rho_r$ are the *irreducible representations* of a finite group G, then

$$\sum_{1 \leq i \leq r} (\dim(\rho_i))^2 = |G|.$$

A common theme throughout this book is studying group-like structures through linear algebra, so some of the results are heavily linked to representation theory.

Acknowledgements

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- Our mentor, Yan Sheng, who provided continual support and guidance throughout the year.
- MIT PRIMES for providing such a wonderful opportunity.
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