Induced Representations of Finite Groups

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Definition

A **linear representation** of a group $G$ over $\mathbb{C}$ is a complex vector space $V$ together with a group homomorphism $\rho : G \to \text{GL}(V)$. 
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$V$ is called a **representation space** and has the structure of a left $\mathbb{C}G$-module.
Linear Representations

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$V$ is called a **representation space** and has the structure of a left $\mathbb{C}G$-module.

Example

- Let $C_n = \{g^m \mid 0 \leq m < n\}$ be the cyclic group.
  
  $\rho : C_n \to \mathbb{C}^\times$, $\rho(g^k) = e^{2\pi i \frac{k}{n}}$, $0 \leq k < n$, for every $g \in G$. 
G-invariant Subspaces

Let $\rho : G \rightarrow \text{GL}(V)$ be a linear representation over $\mathbb{C}$.

**Definition (G-invariant subspace)**

A linear subspace $W$ of $V$ is called $G$-invariant if $\rho(g)(W) \subseteq W$ for all $g \in G$. 

Example $\rho : \mathbb{C}^2 \rightarrow \text{GL}_2(\mathbb{C})$, $\gamma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Eigenvectors to $+1$ and $-1$: $v_1 = (1,1)$, $v_2 = (-1,1)$.
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Definition (Subrepresentation)

A **subrepresentation** of $\rho$ is a $G$-invariant linear subspace $W$ of $V$ together with the restricted group homomorphism $\rho^W : G \to \text{GL}(W)$. 
Definitions and Maschke’s Theorem

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**Theorem (Maschke)**

Every complex linear representation is the direct sum of irreducible representations.
Definition (Character of a representation)

The **character** of a linear representation $\rho : G \to GL(V)$ is the complex valued function $\chi : G \to \mathbb{C}$, given by

$$\chi_\rho(s) := \text{Tr}(\rho(s))$$

for every $s \in G$. 

The character is a class function on $G$. The space $H$ of class functions on $G$ has a scalar product given by

$$\langle f, f' \rangle := \frac{1}{|G|} \sum_{g \in G} f(g) f'(g),$$

for $f, f' \in H$. 

Cai, Xiao

Induced Representations
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- The space $\mathcal{H}$ of class functions on $G$ has a scalar product given by $\langle f , f' \rangle := \frac{1}{|G|} \sum_{g \in G} f(g) \overline{f'(g)}$, for $f, f' \in \mathcal{H}$. 
Character Theory

Theorem (Orthogonality of Characters)

Let \( \chi_\rho \) and \( \chi_{\rho'} \) be the characters of the irreducible representations \( \rho \) and \( \rho' \), respectively. Then, \( \langle \chi_\rho, \chi_{\rho'} \rangle = 1 \) if \( \rho \) and \( \rho' \) are equivalent and \( \langle \chi_\rho, \chi_{\rho'} \rangle = 0 \) if they are not.
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$\langle \chi_W, \chi_V \rangle = \dim \text{Hom}_G(W, V)$ for a $\mathbb{C}G$-module $V$ and a simple $\mathbb{C}G$-module $W$. 
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- $\langle \chi_W, \chi_V \rangle = \dim \text{Hom}_G(W, V)$ for a $\mathbb{C}G$-module $V$ and a simple $\mathbb{C}G$-module $W$.
- Two representations $\rho$ and $\rho'$ are equivalent iff $\chi_\rho = \chi_{\rho'}$.
- The characters of all irreducible representations of $G$ form an orthonormal basis of $H$.
- The number of irreducible representations of $G$ is equal to the number of conjugacy classes of $G$. 
Let \( \theta : H \to \text{GL}(W) \) be a representation. Select a system of representatives \( R := \{ \sigma \in gH : gH \in G/H \} \) of \( G/H \) and set \( W_{\sigma} := C \sigma \otimes C W \). Construct a new representation

\[
\tau : G \to \text{GL}\left( \bigoplus_{\sigma \in R} W_{\sigma} \right)
\]

by

\[
\tau(t)(\sigma \otimes w) = t\sigma \otimes w = \sigma' \otimes \theta(h)w
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where \( t\sigma = \sigma'h \) with \( \sigma' \in R \) and \( h \in H \).
Induced Representations - Definition

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**Definition**

A representation $\rho : G \rightarrow \text{GL}(V)$ is **induced** by $\theta : H \rightarrow \text{GL}(W)$ if $V \cong \bigoplus_{\sigma \in R} W_\sigma$ as representations of $G$. 
Induced Representations - Alternative Definition

**Definition**

Let \( \theta : H \to \text{GL}(W) \) be a linear representation which equips \( W \) with the structure of a left \( \mathbb{C}H \)-module. Set \( V = \mathbb{C}G \otimes_{\mathbb{C}H} W \). The representation \( \text{Ind}_{H}^{G}(\theta) : G \to \text{GL}(V) \) given by

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\text{Ind}_{H}^{G}(\theta)(g)(\sigma \otimes w) = g \sigma \otimes w
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is called **induced** by \( \theta \).
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is called \textit{induced} by $\theta$.

- As $\bigoplus_{\sigma \in R} W_\sigma \cong \mathbb{C}G \otimes_{\mathbb{C}H} W$, both definitions are equivalent.
- If $f$ is a class function on $H$, the function defined by

$$\text{Ind}_G^H(f)(u) := \frac{1}{|H|} \sum_{t \in G} f(t^{-1}ut)$$

for every $u \in G$ is the \textit{induced class function} on $G$. 
Examples of Induced Representations

Example

The regular representation $r_G$ of $G$ is induced by the regular representation $r_H$ of every $H \subset G$: $\mathbb{C}G \cong \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}H$. 

Example

Let $G = S_3$, $H = \mathbb{Z}_2$. Let $\tau$ be the signum rep. of $H$, let $\epsilon$ be the signum rep. of $G$ and let $\rho$ be the standard rep. of $G$. Then $\text{Ind}_{G}^{H}(\tau) = \epsilon \oplus \rho$. 

Example

For representations $\theta_i: H \rightarrow \text{GL}(W_i)$, $i = 1, 2$, of $H$, \[ \text{Ind}_{G}^{H}(\theta_1 \oplus \theta_2) = \text{Ind}_{G}^{H}(\theta_1) \oplus \text{Ind}_{G}^{H}(\theta_2). \]
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Characters of Induced Representations

**Theorem**

Let \( \theta : H \rightarrow \text{GL}(W) \) be a representation of \( H \subset G \) and \( R \) a system of representatives of \( G/H \). Then, for each \( u \in G \), we have

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\text{Ind}^G_H(\chi \theta)(u) = \sum_{r \in R} \chi \theta(r^{-1}ur) = \chi_{\text{Ind}^G_H(\theta)}(u).
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**Example**

- \( \text{Ind}_H^G(\chi_{r_H}) = \chi_{r_G} \).
- \( \text{Ind}_H^G(\chi_{\theta_1 \oplus \theta_2}) = \text{Ind}_H^G(\chi_{\theta_1}) \oplus \text{Ind}_H^G(\chi_{\theta_2}) = \chi_{\text{Ind}_H^G(\theta_1)} \oplus \chi_{\text{Ind}_H^G(\theta_2)}. \)
Frobenius Reciprocity

Theorem (Frobenius Reciprocity)

Let $E$ and $W$ be a $\mathbb{C}G$-module and a $\mathbb{C}H$-module, respectively. Then, we have

$$\text{Hom}_G(\text{Ind}_H^G W, E) \cong \text{Hom}_H(W, \text{Res}_H^G E).$$
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**Corollary (Frobenius Reciprocity for Characters)**

$$\langle \text{Ind}_H^G \chi_\rho, \chi_\rho' \rangle_G = \langle \chi_\rho, \text{Res}_H^G \chi_\rho' \rangle_H.$$
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**Corollary (Frobenius Reciprocity for Characters)**

$$\langle \text{Ind}_H^G \chi_{\rho}, \chi_{\rho'} \rangle_G = \langle \chi_{\rho}, \text{Res}_H^G \chi_{\rho'} \rangle_H.$$

Frobenius Reciprocity states that if $\rho$ and $\rho'$ are irreducible representations of $H$ and $G$, respectively, then the multiplicity of $\rho'$ in $\text{Ind}_H^G(\rho)$ equals the multiplicity of $\rho$ in $\text{Res}_H^G(\rho')$. 
Example (2-dimensional irreducible representation of $D_4$)

$$\rho : r^k \mapsto \begin{pmatrix} e^{2\pi ik/4} & 0 \\ 0 & e^{-2\pi ik/4} \end{pmatrix}$$

$$sr^k \mapsto \begin{pmatrix} 0 & e^{-2\pi ik/4} \\ e^{2\pi ik/4} & 0 \end{pmatrix}$$ for all $k = 0, 1, 2, 3$.

The cyclic subgroup $C_4 \leq D_4$ has an irreducible representation $\rho_1 : C_4 \to \mathbb{C}^\times$ with character $\chi_{\rho_1}(r^k) = e^{2\pi ik/4}$ for $k = 0, 1, 2, 3$. By Frobenius reciprocity,

$$\langle \text{Ind}_{C_4}^{D_4}(\chi_{\rho_1}), \chi_\rho \rangle = \langle \chi_{\rho_1}, \text{Res}_{C_4}^{D_4}(\chi_\rho) \rangle = \frac{1}{4}(2 + 1 + e^{\pi i} + 1 + e^{2\pi i} + 1 + e^{3\pi i}) = 1.$$ 

Hence, the irreducible $\rho$ of $D_4$ is induced by the irreducible $\rho_1$ of $C_4$. 
**Counterexample**

**Question:** Is the induced representation of an irreducible representation always irreducible?
Counterexample

**Question:** Is the induced representation of an irreducible representation always irreducible?

**Answer:** No!

**Example**

Let $G = S_3$, $H = \mathbb{Z}_2$. Let $\tau$ be the signum representation of $H$, let $\epsilon$ be the signum representation of $G$ and let $\rho$ be the standard representation of $G$. We can compute

\[
\begin{align*}
\chi_{\text{Ind}_{H}^{G}(\tau)}(\text{Id}) &= 3, & \chi_{\text{Ind}_{H}^{G}(\tau)}((12)) &= -1, & \chi_{\text{Ind}_{H}^{G}(\tau)}((123)) &= 0. \\
\chi_{\epsilon}(\text{Id}) &= 1, & \chi_{\epsilon}((12)) &= -1, & \chi_{\epsilon}((123)) &= 1. \\
\chi_{\rho}(\text{Id}) &= 2, & \chi_{\rho}((12)) &= 0, & \chi_{\rho}((123)) &= -1.
\end{align*}
\]

\[
\chi_{\text{Ind}_{H}^{G}(\tau)} = \chi_{\epsilon} + \chi_{\rho}.
\]
Mackey’s Irreducibility Criterion

Let \( \rho : H \rightarrow GL(W) \), \( H \leq G \), be a representation. Let \( H_s := sHs^{-1} \cap H \) for \( s \in G \).

Let \( \rho^s : H_s \rightarrow GL(W) \) be a representation given by \( \rho^s(x) := \rho(s^{-1}xs) \) for \( x \in H_s \).

Let \( \text{Res}_s(\rho) \) denote the restriction of \( \rho \) to \( H_s \).
Mackey’s Irreducibility Criterion

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Theorem (Mackey’s Irreducibility Criterion)

*In order that $\text{Ind}_H^G(\rho)$ is an irreducible representation, it is necessary and sufficient that the following two conditions be satisfied:*

(i) $W$ is a simple left $\mathbb{C}H$-module.
(ii) For every $s \in G - H$, we have $\langle \rho^s, \text{Res}_s(\rho) \rangle = 0$. 
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