FEW DISTANCE SETS IN ℓ_p SPACES AND ℓ_p PRODUCT SPACES

RICHARD CHEN, FENG GUI, JASON TANG, AND NATHAN XIONG

ABSTRACT. Kusner asked if $n + 1$ points is the maximum number of points in $\mathbb{R}^n$ such that the ℓ_p distance between any two points is 1. We present an improvement to the best known upper bound when p is large in terms of n, as well as a generalization of the bound to s-distance sets. We also study equilateral sets in the ℓ_p sums of Euclidean spaces, deriving upper bounds on the size of an equilateral set for when $p = \infty$, p is even, and for any $1 \le p < \infty$.

1. INTRODUCTION

1.1. Background

A classic exercise in linear algebra asks for the maximum number of points in $\mathbb{R}^n$ such that the pairwise distances take only two values. One can associate a polynomial to each point in the set such that the polynomials are linearly independent. Then, one can show that the polynomials all lie in a subspace of dimension $(n + 1)(n + 4)/2$. Since the number of linearly independent vectors cannot exceed the dimension of the subspace, the cardinality of the set is at most $(n+1)(n+4)/2$. This beautiful argument was found by Larman, Rogers, and Seidel [LRS77] in 1977, illustrating the power of algebraic techniques in combinatorics.

We can ask the much more general question: "In a metric space X, what is the maximum number of points such that the pairwise distances take only s values?" We use $e_s(X)$, or just $e(X)$ if $s = 1$, to denote the answer to this question (by convention, we do not count 0 as a distance). A set of points $S \subseteq X$ satisfying this question's conditions, i.e., the cardinality of the set $\{d(x, y) : x, y \in S, x \ne y\}$ is s, is called an s-distance set. A 1-distance set is also known as an equilateral set. Also, we typically restrict the metric space to a normed space, so that the specific distances used do not matter. Thus, we will always assume that the largest of the s distances is 1.

The posed question has been studied on many different spaces. The most famous result would be the upper bound $\binom{n+s}{s}$ on an s-distance set in n-dimensional Euclidean space, found by Bannai, Bannai, and Stanton [BBS83]. This result was also discovered independently by Blokhuis [Blo83]. Another important case is when all points lie on the n-dimensional sphere. There is strong motivation for this problem as it has many applications in coding theory and design theory [DGS77]. In particular, having tight upper bounds can help us find extremal configurations which satisfy unique properties.

Similar results have been obtained in the hyperbolic space [Blo83], the Hamming space [MN11], and the Johnson space [MN11]. Not as much is known in an arbitrary finite-dimensional normed space, known as a *Minkowski space*, other than Petty's [Pet71] general bound of $e(X) \le 2^n$, where $n = \dim X$. Swanepoel [Swa99] then conjectured that $e_s(X) \le (s + 1)^n$ for a Minkowski space X with dimension n and proved it for the $n = 2$ case. We should mention that equilateral sets in

Minkowski spaces have been applied in differential geometry, where they are used to find minimal surfaces [Mor92].

1.2. Definitions

In our paper, we investigate this problem on $\mathbb{R}^n$ with the ℓ_p norm, as well as on the ℓ_p sum of Euclidean spaces. For a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and a $p \geq 1$, the ℓ_p norm is defined to be

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}},$$

and the ℓ_∞ norm is defined to be

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

The ℓ_1 norm is the well-known "taxicab" norm, and the ℓ_2 norm is the standard Euclidean norm. Throughout the paper, we write $\|\cdot\|$ instead of $\|\cdot\|_2$ to emphasize that we are using the Euclidean norm. We also use $\mathbb{E}^n$ to emphasize that we are in n-dimensional Euclidean space.

For Euclidean spaces $\mathbb{E}^{a_1}, \ldots, \mathbb{E}^{a_n}$, we define their ℓ_p sum as the product space $\mathbb{E}^{a_1} \times \cdots \times \mathbb{E}^{a_n}$ equipped with the norm

$$\|(x_1, \ldots, x_n)\|_p = \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}},$$

where $x_i \in \mathbb{E}^{a_i}$ for $i = 1, \ldots, n$. We use $\mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}$ to describe this space.

When $p = \infty$, the norm is just

$$\|(x_1, \ldots, x_n)\|_\infty = \max_{1 \leq i \leq n} \|x_i\|.$$

Although our notation for the norm in ℓ_p spaces and in ℓ_p sums is the same, the norm being used should be clear from context.

1.3. Previous Work and Our Results

We first study s-distance sets in $\mathbb{R}^n$ equipped with the ℓ_p norm. This space is denoted by $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$. The two most famous questions pertaining to this problem are Kusner's [Guy83] conjectures on equilateral sets.

Conjecture 1 (Kusner). $e(\ell_1^n) = 2n$.

Conjecture 2 (Kusner). $e(\ell_p^n) = n + 1$ for $1 < p < \infty$.

For Conjecture 1, note that the set $\{\pm e_i : i = 1, \ldots, n\}$, where e_i is the i-th standard basis vector, is equilateral in ℓ_1^n, so $e(\ell_1^n) \geq 2n$. Currently, the best known upper bound is $e(\ell_1^n) \leq cn \log n$ due to Alon and Pudlák [AP03]. It is also known that Conjecture 1 holds for $n = 3$ (Bandelt, Chepoi, and Laurent [BCL98]) and $n = 4$ (Koolen, Laurent, and Schrijver [KLS00]).

As for Conjecture 2, note that the set $\{e_1, \ldots, e_n, \lambda \sum_{i=1}^n e_i\}$ is equilateral for a suitable choice of λ, so $e(\ell_p^n) \geq n + 1$. For $1 < p < 2$ and n large enough, Swanepoel [Swa04b] actually showed that $e(\ell_p^n) > n + 1$, disproving Conjecture 2 in this case. The first nontrivial upper bound of $e(\ell_p^n) \leq c_p n^{(p+1)/(p-1)}$ was found by Smyth [Smy01] using an approximation argument. This result

was later improved by Alon and Pudlák [AP03] and is currently the best known upper bound on $e(\ell_p^n)$ for arbitrary n and p.

Theorem 1 (Alon and Pudlák). *For every $p \geq 1$, we have $e(\ell_p^n) \leq c_p n^{(2p+2)/(2p-1)}$, where one may take $c_p = cp$ for an absolute $c > 0$.*

Our first result is an improvement of Theorem 1 when p is large in terms of n. One can check that $c > 2$, so Theorem 2 is indeed an improvement.

Theorem 2. *Let $c > 0$ be the constant from Theorem 1. If $n > 1$ and $p \geq c(n \log n)^2$, then $e(\ell_p^n) \leq 2(p+1)n$.*

We should mention that when p satisfies other special properties, Theorem 1 can be strengthened. If p is an even integer, Swanepoel [Swa04b] used a linear independence argument to show that

$$e(\ell_p^n) \leq \begin{cases} (\frac{p}{2} - 1)n + 1 & \text{if } p \equiv 0 \pmod 4, \\ \frac{p}{2}n + 1 & \text{if } p \equiv 2 \pmod 4. \end{cases}$$

If p is an odd integer, Alon and Pudlák's argument for $e(\ell_1^n)$ extends to $e(\ell_p^n)$, giving the bound $e(\ell_p^n) < c_p n \log n$ in this case.

Our next result is a generalization of Theorem 1 to s-distance sets. As far as we know, the only literature on s-distance sets in ℓ_p^n is by Smyth [Smy13]. Our theorem below is strictly stronger than Conjecture 5 in [Smy13].

Theorem 3. *If s is a positive integer and p is a real number satisfying $2p > s$, then $e_s(\ell_p^n) \leq c_{p,s} n^{(2ps+2s)/(2p-s)}$ for a constant $c_{p,s}$ depending on p and s.*

Our next three results are on equilateral sets in the ℓ_p sum of Euclidean spaces. As far as we know, this problem has not been well studied in this space. Swanepoel [Swa18] showed that $e(X \oplus_\infty Y) \leq e(X)b_f(Y)$ for normed spaces X and Y, where b_f is the finite Borsuk number. However, explicit bounds are still unknown when X and Y are Euclidean spaces and when we take an ℓ_p sum instead of an ℓ_∞ sum. Our first result in this area almost completely resolves the problem for $\mathbb{E}^a \oplus_\infty \mathbb{E}^b$. Note that we have the obvious lower bound $e(\mathbb{E}^a \oplus_\infty \mathbb{E}^b) \geq e(\mathbb{E}^a)e(\mathbb{E}^b) = (a+1)(b+1)$ by taking the Cartesian product of the two equilateral sets. This lower bound actually meets the upper bound when $a = 2, 3$ [Swa18].

Theorem 4. *Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces. Then, $e(\mathbb{E}^a \oplus_\infty \mathbb{E}^b) \leq (a+1)(b+1) + 1$.*

Our second result in this area provides an upper bound when p is even.

Theorem 5. *Let $\mathbb{E}^a$ and $\mathbb{E}^b$ be Euclidean spaces, and let p be an even integer. Then, $e(\mathbb{E}^a \oplus_p \mathbb{E}^b) \leq \binom{a+p/2}{a} + \binom{b+p/2}{b}$.*

Finally, we extend Alon and Pudlák's [AP03] result on equilateral sets in ℓ_p^n to ℓ_p product spaces. We present an upper bound on the ℓ_p sum of n Euclidean spaces for any $1 \leq p < \infty$. Observe that Theorem 1 is a special case of our theorem below when $a_1 = \cdots = a_n = 1$.

Theorem 6. *Let $\mathbb{E}^{a_1}, \ldots, \mathbb{E}^{a_n}$ be Euclidean spaces and set $a = \max\limits_{1 \leq i \leq n} a_i$. If $2p > a$, then $e(\mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}) \leq c_{p,a} n^{\frac{2p+2a}{2p-a}}$ for a constant $c_{p,a}$ depending on p and a.*

Our paper is structured as follows. In Section 2, we introduce two important tools used in our proofs. Then, we prove Theorems 2, 3, 4, 5, and 6 in Sections 3, 4, 5, 6, and 7 respectively.

2. Preliminaries

In this section, we present two famous results that we use later. The first is the Rank Lemma, which allows us to estimate the rank of a symmetric matrix. The second is Jackson's Theorem, a celebrated result from approximation theory. This theorem allows us to approximate $|x|^p$ by a polynomial with sufficiently small error.

2.1. The Rank Lemma

Lemma 1 (Rank Lemma). *For a real symmetric $n \times n$ nonzero matrix A,*

$$\operatorname{rank} A \geq \frac{\left(\sum_{i=1}^{n} a_{ii}\right)^2}{\sum_{i,j=1}^{n} a_{ij}^2}.$$

We will frequently make use of the following corollary.

Corollary. *Let A be a real symmetric $n \times n$ matrix with $a_{ii} = 1$ and $|a_{ij}| \leq \varepsilon$ for all $i \neq j$. Then*

$$\operatorname{rank} A \geq \frac{n}{1 + (n-1)\varepsilon^2}.$$

Choosing $\varepsilon = n^{-1/2}$ gives $\operatorname{rank} A \geq n/2$.

2.2. Jackson's Theorem

Theorem 7 (Jackson, [Jac30]). *For any $f \in C^k[-1, 1]$ and positive integer d, there exists a polynomial P with degree at most d such that*

$$|f(x) - P(x)| \leq \frac{c^k}{(n+1)_k} \omega(f^{(k)}, 1/d) \qquad \text{for all } x \in [-1, 1],$$

where $c > 0$ is an absolute constant, $\omega(f, \delta)$ denotes the modulus of continuity of f, and $(n+1)_k = (n+1)n \cdots (n-k+2)$ uses falling factorial notation.

From Theorem 7, we can recover the following lemma. For its proof, see [Swa04a].

Lemma 2. *For any $p \geq 1$ and $d \geq \lceil p \rceil$, there exists a polynomial P with degree at most d such that*

$$(1) \qquad |P(x) - |x|^p| \leq \frac{B(p)}{d^p} \qquad \text{for all } x \in [-1, 1]$$

where $B(p) = (\lceil p \rceil^p (1 + \pi^2/2)^{\lceil p \rceil} (p)_{\lceil p \rceil - 1})/\lceil p \rceil!$.

We will always assume that the polynomial P in Lemma 2 is even and that $P(0) = 0$. If P is not even, we can take the even part of P. If $P(0) \neq 0$, we can take the polynomial $Q(x) = P(x) - P(0)$ as this only increases the error term by a factor of 2.

3. Bound on equilateral sets in ℓ_p^n for large p

We start with an important lemma about the ℓ_p norm.

Lemma 3. *Suppose $1 \le p \le q$. Then for any $x \in \mathbb{R}^n$,*

$$\|x\|_q \le \|x\|_p \le n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q.$$

Proof. The left hand inequality is a well-known property of the ℓ_p norm. The right hand inequality is the Power Mean Inequality, which can be proven using Jensen's Inequality. $\square$

Suppose our equilateral set is $\{p_1, \ldots, p_m\}$. Let k be the closest even integer to p, rounding up if p is odd. The main idea is that by Lemma 3, we can approximate $\|\cdot\|_p$ by $\|\cdot\|_k$. If p is large enough in terms of n, the error term is sufficiently small. From there, it suffices to bound the size of an approximately-equilateral set in ℓ_k^n, which can be done with a linear algebra argument.

Proof of Theorem 2. We use the notation explained above. There are two cases to consider depending on whether k is greater than or less than p.

Case 1. We have $\lfloor p \rfloor$ is odd, so $p < k$.

From our bound on p and the fact that $p \ge 2$,

$$\begin{aligned} m &\le cpn^{(2p+2)/(2p-1)} \\ &\le cpn^2 \\ &\le \frac{p^2}{(\log n)^2} \\ &\le (1 - n^{-1/p})^{-2}, \end{aligned}$$

where the first inequality holds by Theorem 1. Assuming without loss of generality that $m > 1$, we may rearrange the result above into

$$-p \log_n \left(1 - \frac{1}{\sqrt{m}}\right) \ge 1.$$

Because k is the smallest even integer greater than p, we have

$$(2) \qquad p < k \le p + 1 \le p - p\log_n\left(1 - \frac{1}{\sqrt{m}}\right).$$

Now, embed the m points into ℓ_k^n. Since $p < k$, Lemma 3 implies

$$(3) \qquad n^{\frac{1}{k}-\frac{1}{p}} \le \|p_i - p_j\|_k \le 1$$

for all $i \neq j$. Consider the m functions $f_i : \mathbb{R}^n \to \mathbb{R}$ given by $f_i(x) = 1 - \|p_i - x\|_k^k$, and let A be the matrix with $a_{ij} = f_i(p_j)$. Clearly, $a_{ii} = 1$, and from (2) and (3),

$$|a_{ij}| \leq 1 - n^{1-\frac{k}{p}}$$
$$\leq 1 - n^{1-(1-\log_n(1-1/\sqrt{m}))}$$
$$= \frac{1}{\sqrt{m}},$$

for all $i \neq j$. Since A is symmetric, Lemma 1 tells us rank $A \geq m/2$. For the upper bound on the rank, note that every f_i lies in the span of the set of polynomials

$$\mathcal{S} = \{1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^2, \ldots, x_1^{k-1}, \ldots, x_n^{k-1}, \sum_{t=1}^{n} x_t^k\}.$$

The i-th row vector of A is $(f_i(p_1), \ldots, f_i(p_m))$. Hence, every row vector belongs to the subspace spanned by $\{(f(p_1), \ldots, f(p_m)) : f \in \mathcal{S}\}$, which has dimension at most $|\mathcal{S}| = (k-1)n + 2 \leq kn$. It follows that rank $A \leq kn \leq (p+1)n$. Combining the upper and lower bound, we have

$$m \leq 2(p+1)n,$$

as desired.

Case 2. We have $\lfloor p \rfloor$ is even, so $p \geq k$.

Since $c, n \geq 2$, $p > 4$. So, similar to in Case 1, we have

$$m \leq cpn^{(2p+2)/(2p-1)}$$
$$\leq cpn^{2-2/p}$$
$$\leq n^{-2/p} \cdot \frac{p^2}{(\log n)^2}$$
$$\leq n^{-2/p}(1 - n^{-1/p})^{-2}$$
$$= (n^{1/p} - 1)^{-2}.$$

Rearranging the result gives us

$$p \log_n \left(1 + \frac{1}{\sqrt{m}}\right) \geq 1.$$

Because k is the largest even integer less than or equal to p, we have

$$(4) \qquad\qquad p \geq k \geq p - 1 \geq p - p\log_n\left(1 + \frac{1}{\sqrt{m}}\right).$$

Embed the m points into ℓ_k^n. This time, Lemma 3 implies

$$(5) \qquad\qquad 1 \leq \|p_i - p_j\|_k \leq n^{\frac{1}{k} - \frac{1}{p}}.$$

We define f_i and A like in Case 1. Again, $a_{ii} = 1$, and from (4) and (5),

$$|a_{ij}| \leq n^{1-\frac{k}{p}} - 1$$
$$\leq n^{1-(1-\log_n(1+1/\sqrt{m}))} - 1$$
$$= \frac{1}{\sqrt{m}},$$

for all $i \neq j$. Applying Lemma 1, rank $A \geq m/2$. Similar to in Case 1, we have the bound rank $A \leq kn \leq pn$. The final result is

$$m \leq 2pn < 2(p+1)n.$$

Having considered all cases, the proof is complete. $\qquad\square$

4. Bound on s-distance sets in ℓ_p^n

We would like to extend Alon and Pudlák's [AP03] idea of combining Jackson's Theorem with the rank lemma to the s-distance case. However, as pointed out by Smyth [Smy13], if one of the distances is arbitrarily small, we need arbitrarily high degrees of approximation. So, we want to impose a lower bound on our distances.

Consider a two-distance set with distances 1 and a, where a is very small. Intuitively, this set should look like several "clusters" of points, such that the distance between each cluster is 1. So, if we look globally, this set looks like an equilateral set with distance 1, but if we look locally at each cluster, we have an equilateral set with distance a. This means that we have essentially reduced the problem from one about two-distance sets to one about equilateral sets. We carry this intuition to s-distance sets and formalize this argument with an induction.

Proof of Theorem 3. We use strong induction on s. The base case is $s = 1$, which was proven by Alon and Pudlák [AP03]. For the inductive step, assume that the statement holds true for all $1 \leq s < k$. We prove that it is true for $s = k$.

Let S be a k-distance set in ℓ_p^n. Let our k distances be $1 = a_1 > a_2 > \cdots > a_k$. There are two cases to consider depending on whether the k distances are lower bounded or not.

Case 1. The smallest distance a_k is less than 2^{1-k}.

There exists an index $1 \leq i < k$ such that $a_i > 2a_{i+1}$. Let $S' \subseteq S$ be a maximal i-distance set using the distances $a_1, \ldots, a_i$. From the maximality of S', every point $p \in S \setminus S'$ is within a_{i+1} of some point in S'. Draw a closed ball of radius a_{i+1} around every point in S'. These balls are disjoint from our condition. The condition also implies that within each ball, the distance between any two points is at most a_{i+1}. Thus, within every ball, we have a $(k-i)$-distance set using the distances $a_{i+1}, \ldots, a_k$. This implies the bound

$$e_k(\ell_p^n) \leq e_i(\ell_p^n) \cdot e_{k-i}(\ell_p^n).$$

Applying the inductive hypothesis gives us

$$e_k(\ell_p^n) \leq c_{p,i} n^{\frac{2pi+2i}{2p-i}} \cdot c_{p,k-i} n^{\frac{2p(k-i)+2(k-i)}{2p-(k-i)}}$$
$$\leq c_{p,k} n^{\frac{2pk+2k}{2p-k}},$$

as desired. The second inequality follows by analyzing the function

$$f(x) = \frac{2px + 2x}{2p - x} + \frac{2p(k-x) + 2(k-x)}{2p - (k-x)}$$

on the interval $[0, k]$. It is symmetric about $x = \frac{k}{2}$, and its first derivative is

$$f'(x) = \frac{4p(p+1)}{(2p-x)^2} - \frac{4p(p+1)}{(2p-(k-x))^2}.$$

Since $2p > k$, f is decreasing on $[0, \frac{k}{2}]$ and increasing on $[\frac{k}{2}, k]$. Thus, it is maximised at $x = 0$ and $x = k$.

Case 2. The smallest distance a_k is at least 2^{1-k}.

Suppose S consists of the points $\{p_1, \ldots, p_m\}$. For convenience, let $\pi = a_1^p a_2^p \cdots a_k^p$. Fix $B(p)$ as the constant from Lemma 2. Define c as

$$c = \max(B(p)k \cdot 2^{pk^2 - pk + 2k}, (2^{1/p} - 1)^{-p}).$$

Then, let d be a positive integer satisfying

$$cn\sqrt{m} < d^p < 2cn\sqrt{m},$$

which is possible since $c \geq (2^{1/p} - 1)^{-p}$.

Lemma 2 allows us to pick an even polynomial P with $P(0) = 0$ and degree at most d such that

$$|P(x) - |x|^p| \leq \frac{B(p)}{d^p} \qquad \text{for all } x \in [-1, 1].$$

Consider the m functions $f_i : \mathbb{R}^n \to \mathbb{R}$ given by

$$f_i(x) = \frac{1}{\pi} \prod_{u=1}^{k} \left(a_u^p - \sum_{t=1}^{n} |x_t - p_{it}|^p \right),$$

and their polynomial approximations

$$g_i(x) = \frac{1}{\pi} \prod_{u=1}^{k} \left(a_u^p - \sum_{t=1}^{n} P(x_t - p_{it}) \right).$$

Let A be the $m \times m$ matrix given by $a_{ij} = g_i(p_j)$. First, since $P(0) = 0$, $a_{ii} = 1$ for all i. We now estimate a_{ij} for $i \neq j$. Expand

$$f_i(x) = \frac{1}{\pi} \sum_{\ell=0}^{k} (-1)^{k+\ell} \sigma(\ell) \left(\sum_{t=1}^{n} |x_t - p_{it}|^p \right)^{k-\ell},$$

and

$$g_i(x) = \frac{1}{\pi} \sum_{\ell=0}^{k} (-1)^{k+\ell} \sigma(\ell) \left(\sum_{t=1}^{n} P(x_t - p_{it}) \right)^{k-\ell},$$

where $\sigma(\ell)$ denotes the ℓ-th elementary symmetric polynomial in $a_1^p, \ldots, a_k^p$. For convenience, we define $X_{ij} = \sum_{t=1}^n |p_{jt} - p_{it}|^p$ and $Y_{ij} = \sum_{t=1}^n P(p_{jt} - p_{it})$. Applying the triangle inequality with (1), we have

$$|X_{ij} - Y_{ij}| \le \frac{nB(p)}{d^p}.$$

Since $i \ne j$, recall that $X_{ij} \ge 2^{1-k}$. Combining this with the fact that $c > B(p) \cdot 2^{k-1}$ implies that Y_{ij} is positive. Now, we are ready to attack a_{ij}.

$$|a_{ij}| = \frac{1}{\pi} \left| \sum_{\ell=0}^k (-1)^{k+\ell} \sigma(\ell)(Y_{ij}^{k-\ell} - X_{ij}^{k-\ell}) \right|$$

$$\le \frac{1}{\pi} \cdot \frac{nB(p)}{d^p} \sum_{\ell=0}^{k-1} \sigma(\ell) \sum_{r=1}^{k-\ell} |X_{ij}^{k-\ell-r} Y_{ij}^{r-1}|$$

$$\le \frac{1}{\pi} \cdot \frac{nB(p)}{d^p} \sum_{\ell=0}^{k-1} \sigma(\ell) \sum_{r=1}^{k-\ell} \left| \frac{(k-\ell-r)X_{ij} + (r-1)Y_{ij}}{k-\ell-1} \right|^{k-\ell-1}$$

$$= \frac{1}{\pi} \cdot \frac{nB(p)}{d^p} \sum_{\ell=0}^{k-1} \sigma(\ell) \sum_{r=1}^{k-\ell} \left| \frac{(r-1)(Y_{ij} - X_{ij})}{k-\ell-1} + X_{ij} \right|^{k-\ell-1}$$

$$\le \frac{1}{\pi} \cdot \frac{nB(p)}{d^p} \sum_{\ell=0}^{k-1} \sigma(\ell)(k-\ell) \left(\frac{nB(p)}{d^p} + 1 \right)^{k-\ell-1}$$

$$\le \frac{1}{\pi} \cdot \frac{nB(p)k}{d^p} \cdot 2^k \sum_{\ell=0}^{k-1} \sigma(\ell).$$

Now, since $a_u \ge 2^{1-k}$, we can lower bound π with

$$\pi = a_1^p a_2^p \cdots a_k^p \ge (2^{(1-k)p})^k = 2^{pk - pk^2}.$$

On the other hand, since $a_t \le 1$, we can upper bound $\sigma(\ell)$ with

$$\sigma(\ell) = \sum_{1 \le j_1 < j_2 < \cdots < j_\ell \le k} a_{j_1}^p a_{j_2}^p \cdots a_{j_\ell}^p \le \binom{k}{\ell}.$$

This gives us $\sum_{\ell=0}^{k-1} \sigma(\ell) < 2^k$. Putting everything together, we have

$$|a_{ij}| < 2^{pk^2 - pk + 2k} \cdot \frac{nB(p)k}{d^p}$$

$$< \frac{1}{\sqrt{m}}$$

from our choice of c and d.

Recall that P is even, so the matrix A is symmetric. Lemma 1 then tells us that rank $A \ge m/2$. We now find an upper bound for the rank.

Note that the polynomials $a_u^p - \sum_{t=1}^n P(x_t - p_{it})$ belong to the span of the set

$$\{1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^2, \ldots, x_1^{d-1}, \ldots, x_n^{d-1}, \sum_{t=1}^n x_t^d\}.$$

This set consists of $(d-1)n + 2 \le dn$ polynomials. Thus, all the g_i belong to the span of a set $\mathcal{S}$ of at most $(dn)^k$ polynomials. The i-th row vector of A is $(g_i(p_1), \ldots, g_i(p_m))$. Hence, every row vector belongs to the subspace spanned by

$$\{(f(p_1), \ldots, f(p_m)) : f \in \mathcal{S}\},$$

which has dimension at most $|\mathcal{S}| \le (dn)^k$. This implies rank $A \le (dn)^k$.

The upper and lower bound combine to give $(dn)^k \ge m/2$. Using the upper bound on d^p and rearranging the inequality, we obtain

$$m \le c_{p,k} n^{(2pk+2k)/(2p-k)}.$$

This completes the induction. □

5. Bound on equilateral sets in ℓ_∞ product spaces

Proof of Theorem 4. Write $x = (\widetilde{x}_1, \widetilde{x}_2)$ for each point $x \in \mathbb{E}^a \oplus_\infty \mathbb{E}^b$, where $\widetilde{x}_1 \in \mathbb{E}^a$ and $\widetilde{x}_2 \in \mathbb{E}^b$. Let S be our equilateral set with cardinality m. Consider the m functions $f_u : \mathbb{R}^{a+b} \to \mathbb{R}$ defined by

$$f_u(x) = \left(1 - \|\widetilde{x}_1 - \widetilde{u}_1\|^2\right)\left(1 - \|\widetilde{x}_2 - \widetilde{u}_2\|^2\right),$$

for all $u \in S$. Note that $f_u(v) = \delta_{uv}$, so the f_u are linearly independent.

We can expand f_u as

$$f_u(x) = \left(1 - \sum_{t=1}^a (\widetilde{x}_{1t} - \widetilde{u}_{1t})^2\right)\left(1 - \sum_{t=1}^b (\widetilde{x}_{2t} - \widetilde{u}_{2t})^2\right)$$

$$= \left(1 - \|\widetilde{u}_1\|^2 - \|\widetilde{x}_1\|^2 + 2\sum_{t=1}^a \widetilde{x}_{1t}\widetilde{u}_{1t}\right)\left(1 - \|\widetilde{u}_2\|^2 - \|\widetilde{x}_2\|^2 + 2\sum_{t=1}^b \widetilde{x}_{2t}\widetilde{u}_{2t}\right)$$

So, the f_u are all spanned by the following set of $(a+2)(b+2)$ polynomials

$$\{1, \widetilde{x}_{1i}, \widetilde{x}_{2j}, \widetilde{x}_{1i}\widetilde{x}_{2j}, \|\widetilde{x}_1\|^2, \|\widetilde{x}_2\|^2, \|\widetilde{x}_2\|^2\,\widetilde{x}_{1i}, \|\widetilde{x}_1\|^2\,\widetilde{x}_{2j}, \|\widetilde{x}_1\|^2\,\|\widetilde{x}_2\|^2 : 1 \le i \le a, 1 \le j \le b\}.$$

This implies the bound $m \le (a+2)(b+2)$.

We will now prove that the set of polynomials $\{f_u, 1, x_k, \|\widetilde{x}_1\|^2 : u \in S, 1 \le k \le a+b\}$ is linearly independent. Assume for the sake of contradiction that we have a dependence

(6) $$\sum_{u \in S} \alpha_u f_u + \sum_{k=1}^{a+b} \beta_k x_k + \gamma \|\widetilde{x}_1\|^2 + \delta = 0.$$

The left hand side of (6) is a polynomial in the x_k that is identically zero. Thus, extracting the coefficient of $\|\widetilde{x}_1\|^2 \|\widetilde{x}_2\|^2$, we have

$$(7) \qquad \sum_{u \in S} \alpha_u = 0.$$

Extracting the coefficient of $\|\widetilde{x}_r\|^2 x_k$, for the appropriate $r \in \{1, 2\}$, we have

$$(8) \qquad \sum_{u \in S} \alpha_u u_k = 0.$$

Extracting the coefficient of $\|\widetilde{x}_2\|^2$ and applying (7), we have

$$(9) \qquad \sum_{u \in S} \alpha_u \|\widetilde{u}_1\|^2 = 0.$$

Now, plug u into (6), multiply both sides by α_u, and sum over all $u \in S$.

$$\sum_{u \in S} \alpha_u^2 + \sum_{k=1}^{a+b} \beta_k \sum_{u \in S} \alpha_u u_k + \gamma \sum_{u \in S} \alpha_u \|\widetilde{u}_1\|^2 + \delta \sum_{u \in S} \alpha_u = 0.$$

Applying (7), (8), and (9) implies $\alpha_u = 0$ for all $u \in S$. It easily follows that all other coefficients are zero, as desired.

Now, we know that $m + a + b + 2 \le (a+2)(b+2)$. This rearranges into $m \le (a+1)(b+1) + 1$. $\square$

6. Bound on equilateral sets in ℓ_p product spaces for even p

Proof of Theorem 5. Let S be an equilateral set in $\mathbb{E}^a \oplus_p \mathbb{E}^b$ with m points. For every $u \in S$, define the function $f_u : \mathbb{R}^{a+b} \to \mathbb{R}$ by

$$f_u(x) = 1 - \|\widetilde{x}_1 - \widetilde{u}_1\|^p - \|\widetilde{x}_2 - \widetilde{u}_2\|^p$$

for all $x = (\widetilde{x_1}, \widetilde{x_2}) \in \mathbb{E}^a \oplus_p \mathbb{E}^b$ so that $f_u(v) = \delta_{uv}$. It follows that the m polynomials are linearly independent. Now, we can expand f_u as

$$f_u(x) = 1 - \left(\sum_{t=1}^a (\widetilde{x}_{1t} - \widetilde{u}_{1t})^2 \right)^{p/2} - \left(\sum_{t=1}^b (\widetilde{x}_{2t} - \widetilde{u}_{2t})^2 \right)^{p/2}$$

$$= 1 - \left(\|\widetilde{x}_1\|^2 + \|\widetilde{u}_1\|^2 - 2 \sum_{t=1}^a \widetilde{x}_{1t} \widetilde{u}_{1t} \right)^{p/2} - \left(\|\widetilde{x}_2\|^2 + \|\widetilde{u}_2\|^2 - 2 \sum_{t=1}^b \widetilde{x}_{2t} \widetilde{u}_{2t} \right)^{p/2}$$

$$= 1 - \widetilde{f}_{u1}(x) - \widetilde{f}_{u2}(x)$$

Utilizing multi-index notation, we can expand

$$\widetilde{f}_{u1}(x) = \sum_{\substack{\varepsilon, g \\ \varepsilon + \gamma \le p/2}} (-2)^\gamma \binom{\gamma}{g} \binom{p/2}{\varepsilon, \gamma, p/2 - \varepsilon - \gamma} \widetilde{u}_1^g \|\widetilde{u}_1\|^{p-2\varepsilon-2\gamma} \widetilde{x}_1^g \|\widetilde{x}_1\|^{2\varepsilon}.$$

The sum is taken over all non-negative integers ε and non-negative integer vectors g with a entries such that $\varepsilon + \gamma \leq p/2$, where γ is the sum of the components of g. Similarly, we can expand

$$\widetilde{f}_{u2}(x) = \sum_{\substack{\varepsilon, g \\ \varepsilon + \gamma \leq p/2}} (-2)^\gamma \binom{\gamma}{g} \binom{p/2}{\varepsilon, \gamma, p/2 - \varepsilon - \gamma} \widetilde{u}_2^g \, \|\widetilde{u}_2\|^{p-2\varepsilon - 2\gamma} \, \widetilde{x}_2^g \, \|\widetilde{x}_2\|^{2\varepsilon} \, .$$

We want to count the number of monomials in each polynomial. Since $\widetilde{f}_{u1}$ is a polynomial in $\widetilde{x}_{11}, \ldots, \widetilde{x}_{1a}$ and $\widetilde{f}_{u2}$ is a polynomial in $\widetilde{x}_{21}, \ldots, \widetilde{x}_{2b}$, the two sets of monomials are disjoint, except for the constant monomial. Let us consider $\widetilde{f}_{u1}$ first. By choosing $\varepsilon = 0$, we must count all the monomials with degree at most $p/2$. There are $\binom{a+p/2}{a}$ of these. If the degree is greater than $p/2$, say $p/2 + c$, we only need to count the monomials formed when $\varepsilon = c$ and $\gamma = p/2 - c$. Hence, the total number of monomials in $\widetilde{f}_{u1}$ is

$$\binom{a+p/2}{a} + \binom{a+p/2-2}{a-1} + \binom{a+p/2-3}{a-1} + \cdots + \binom{a-1}{a-1} = \binom{a+p/2}{a} + \binom{a+p/2-1}{a}.$$

Similarly, there are $\binom{b+p/2}{b} + \binom{b+p/2-1}{b}$ monomials in $\widetilde{f}_{u2}$. In total, f_u has $\binom{a+p/2}{a} + \binom{a+p/2-1}{a} + \binom{b+p/2}{b} + \binom{b+p/2-1}{b} - 1$ monomials (we subtract 1 as the constant monomial is counted twice). This gives an upper bound on m.

By finding a larger linearly independent set of polynomials, a trick first used by Blokhuis [Blo81], we can lower this bound to $\binom{a+p/2}{a} + \binom{b+p/2}{b}$. We prove that the set of polynomials

$$\{f_u, \widetilde{x}_1^m, \widetilde{x}_2^n, 1 : u \in S, 0 < \mu < p/2, 0 < \nu < p/2\}$$

is linearly independent. The details are very similar to those in Blokhuis's [Blo83] bound for the s-distance set in $\mathbb{R}^n$.

Suppose we have a dependence

$$(10) \qquad \sum_{u \in S} a_u f_u(x) + \sum_{0 < \mu < p/2} a_m \widetilde{x}_1^m + \sum_{0 < \nu < p/2} a_n \widetilde{x}_2^n + \delta = 0.$$

The main claim is the following lemma.

Lemma 4. *For all m with $\mu < p/2$,*

$$\sum_{u \in S} a_u \widetilde{u}_1^m = 0.$$

Similarly, for all n with $\nu < p/2$,

$$\sum_{u \in S} a_u \widetilde{u}_2^n = 0.$$

Proof. We only prove the first statement. The argument for the second statement is identical.

Suppose $\mu = t < p/2$. Consider the part of the left hand side of (10) that is homogeneous in $\widetilde{x}_{11}, \ldots, \widetilde{x}_{1a}$ with degree $p - t$. Note that the monomials $\widetilde{x}_1^m$ do not contribute to this, so we only have to look at the part from the f_u. Using our expansion of f_u above, the part of f_u homogeneous

in the a variables with degree $p - t$ is

$$-\sum_{\substack{\varepsilon, g \\ 2\varepsilon+\gamma=p-t \\ \varepsilon+\gamma\leq p/2}} (-2)^\gamma \binom{\gamma}{g} \binom{p/2}{\varepsilon, \gamma, p/2 - \varepsilon - \gamma} \widetilde{u}_1^g \|\widetilde{u}_1\|^{p-2\varepsilon-2\gamma} \widetilde{x}_1^g \|\widetilde{x}_1\|^{2\varepsilon}.$$

The left hand side of (10) is a polynomial in the x_i which is identically zero. Thus, we have

$$\sum_{\substack{\varepsilon, g \\ 2\varepsilon+\gamma=p-t \\ \varepsilon+\gamma\leq p/2}} (-2)^\gamma \binom{\gamma}{g} \binom{p/2}{\varepsilon, \gamma, p/2 - \varepsilon - \gamma} \sum_{u\in S} a_u \widetilde{u}_1^g \|\widetilde{u}_1\|^{p-2\varepsilon-2\gamma} \widetilde{x}_1^g \|\widetilde{x}_1\|^{2\varepsilon} = 0.$$

Now, substitute $x = v$, multiply by $a_v \|\widetilde{v}_1\|^{2t-p}$, and sum over all $v \in S$.

$$\sum_{\substack{\varepsilon, g \\ 2\varepsilon+\gamma=p-t \\ \varepsilon+\gamma\leq p/2}} (-2)^\gamma \binom{\gamma}{g} \binom{p/2}{\varepsilon, \gamma, p/2 - \varepsilon - \gamma} \left(\sum_{u\in S} a_u \|\widetilde{u}_1\|^{p-2\varepsilon-2\gamma} \widetilde{u}_1^g \right)^2 = 0.$$

This is a sum of squares where all coefficients have the same sign. So,

$$\sum_{u\in S} a_u \|\widetilde{u}_1\|^{p-2\varepsilon-2\gamma} \widetilde{u}_1^g = 0.$$

Plugging in $\gamma = t$ proves the lemma. $\qquad\square$

Now, plug in u into (10), multiply by a_u and sum over all $u \in S$,

$$\sum_{u\in S} a_u^2 + \sum_{0<\mu<p/2} a_m \sum_{u\in S} a_u \widetilde{u}_1^m + \sum_{0<\nu<p/2} a_n \sum_{u\in S} a_u \widetilde{u}_2^n + \delta \sum_{u\in S} a_u = 0.$$

Applying Lemma 4, we obtain $a_u = 0$ for all u. We already know that the other polynomials in our set are linearly independent, so all coefficients vanish, as desired. This implies $m \leq \binom{a+p/2}{a} + \binom{b+p/2}{b}$ $\qquad\square$

Remark. This proof can be easily extended to the ℓ_p sum of n Euclidean spaces. If we consider the product space $\mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}$, our bound is just $\binom{a_1+p/2}{a_1} + \cdots + \binom{a_n+p/2}{a_n}$.

7. Bound on equilateral sets in ℓ_p product spaces for $1 \leq p < \infty$

Proof of Theorem 6. Let $\{p_1, \ldots, p_m\}$ be an equilateral set in $\mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}$, and write $p_i = (\widetilde{p}_{i1}, \ldots, \widetilde{p}_{in})$ where $\widetilde{p}_{ik} \in \mathbb{E}^{a_k}$ for $k = 1, \ldots, n$.

Let $B(p)$ be the constant from Lemma 2. Take $c = \max(B(p), (2^{1/p} - 1)^{-p})$ and set d to be a positive integer satisfying

$$cn\sqrt{m} < d^p < 2cn\sqrt{m}.$$

By Lemma 2, there exists an even polynomial P with $P(0) = 0$ and degree at most d that approximates $|x|^p$. Thus, for $i \neq j$ and $k = 1, \ldots, n$,

$$\left| P(\|\widetilde{p}_{ik} - \widetilde{p}_{jk}\|) - \|\widetilde{p}_{ik} - \widetilde{p}_{jk}\|^p \right| \leq \frac{B(p)}{d^p}.$$

Next, for $i = 1, \ldots, m$ define the functions $f_i : \mathbb{R}^{a_1 + \cdots + a_n} \to \mathbb{R}$ by

$$f_i(x) = 1 - \sum_{k=1}^{n} P(\|\widetilde{p}_{ik} - \widetilde{x}_k\|).$$

Let M be the $m \times m$ matrix given by $m_{ij} = f_i(p_j)$. Since $P(0) = 0$, we have $m_{ii} = 1$. For $i \neq j$,

$$
\begin{aligned}
|m_{ij}| &= \left| \sum_{k=1}^{n} \|\widetilde{p}_{ik} - \widetilde{p}_{jk}\|^p - \sum_{k=1}^{n} P(\|\widetilde{p}_{ik} - \widetilde{p}_{jk}\|) \right| \\
&\leq \sum_{k=1}^{n} \left| \|\widetilde{p}_{ik} - \widetilde{p}_{jk}\|^p - P(\|\widetilde{p}_{ik} - \widetilde{p}_{jk}\|) \right| \\
&\leq \frac{nB(p)}{d^p} \\
&< \frac{1}{\sqrt{m}},
\end{aligned}
$$

with the last inequality following from our conditions on c and d. Because M is symmetric, we can apply Lemma 1 to get $\operatorname{rank} M \geq m/2$.

Now, we look for an upper bound on the rank. We can write any point $x \in \mathbb{E}^{a_1} \oplus_p \cdots \oplus_p \mathbb{E}^{a_n}$ as

$$x = (\widetilde{x}_1^{(1)}, \widetilde{x}_1^{(2)}, \ldots, \widetilde{x}_1^{(a_1)}, \widetilde{x}_2^{(1)}, \widetilde{x}_2^{(2)}, \ldots, \widetilde{x}_2^{(a_2)}, \ldots, \widetilde{x}_n^{(1)}, \widetilde{x}_n^{(2)}, \ldots, \widetilde{x}_n^{(a_n)}).$$

Because P is even, all terms in the expansion of $P(\|\widetilde{p}_{ik} - \widetilde{x}_k\|)$ have integer exponent, i.e., f_i is actually a polynomial. Additionally, since P has degree at most d, each f_i is in the span of the set

$$\mathcal{S} = \{1, \sum_{k=1}^{n} \|\widetilde{x}_k\|^d\} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_n,$$

where each set $\mathcal{S}_i$ consists of all the monomials with degree less than d formed by $\widetilde{x}_i^{(1)}, \ldots, \widetilde{x}_i^{(a_i)}$. Thus, the f_i are spanned by at most $2 + \sum_{k=1}^{n} \binom{a_k + d - 1}{a_k} - n$ polynomials. Because every row vector of M, each of the form $(f_i(p_1), \ldots, f_i(p_m))$, belongs to the subspace spanned by the set $\{(f(p_1), \ldots, f(p_m)) : f \in \mathcal{S}\}$, we have

$$
\begin{aligned}
\operatorname{rank} M &\leq 2 + \sum_{k=1}^{n} \binom{a_k + d - 1}{a_k} - n \\
&\leq \sum_{k=1}^{n} \binom{a_k + d - 1}{a_k} \\
&\leq n \binom{a + d - 1}{a} \\
&< n e^a \left(1 + \frac{d}{a}\right)^a,
\end{aligned}
$$

where we let $a = \max\limits_{1 \le i \le n} a_i$. Now, recall that $d \ge B(p)^{1/p} \ge p$ and $2p > a$, so $d > a/2$. Thus,

$$\frac{m}{2} \le \operatorname{rank} M < n \left(\frac{3ed}{a}\right)^a.$$

Combining this inequality with the condition $d^p < 2cn\sqrt{m}$ yields

$$m < c_{p,a} n^{\frac{2p+2a}{2p-a}},$$

as desired. □

8. Acknowledgements

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Lexington High School, Lexington, MA 02421, USA
Email address: 21chen26@lexingtonma.org

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
Email address: fenggui@mit.edu

Belmont High School, Belmont, MA 02478, USA
Email address: jtang21@belmontschools.net

Phillips Academy Andover, Andover, MA 01810, USA
Email address: nxiong22@andover.edu