THE BERNARDI FORMULA FOR NON-TRANSITIVE DEFORMATIONS OF THE BRAID ARRANGEMENT

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Abstract. Bernardi has given a general formula to compute the number of regions of a deformation of the braid arrangement as a signed sum over *boxed trees*. We prove that the contribution to this sum of the set of boxed trees sharing an underlying rooted labeled tree is 0 or ± 1 and give an algorithm for computing this value. We then restrict to arrangements which we call *almost transitive* and construct a sign-reversing involution which reduces Bernardi's signed sum to enumeration of a set of rooted labeled trees in this case. We conclude by explicitly enumerating the trees corresponding to the regions of certain nested Ish arrangements which we call *non-negative*, recovering their known counting formula.

1. Introduction

This paper furthers the study of the connection between a family of hyperplane arrangements, known as deformations of the braid arrangement, and a set of partitions of the nodes of rooted labeled trees from [4]. Deformations of the braid arrangement contain many well-studied hyperplane arrangements, for example the braid arrangement, the Shi arrangement, the Linial arrangement, and the Ish arrangement.

The main result of [4] gives a general counting formula for the number of regions of a deformation of the braid arrangement as a signed sum over so called *boxed trees*, a set of partitions of the nodes of certain rooted labeled trees (see Definition 2.3 below for the precise definition). It is further shown that this formula simplifies to an unsigned sum when the arrangement satisfies a condition known as *transitivity* (see Definition 2.5), recapturing known counting formulas for several arrangements. The aim of this paper is to better understand this formula for arrangements which are not transitive. Our main contribution is an algorthm for computing the contribution of the set of boxed trees with the same underlying rooted labeled tree to the signed sum of Bernardi (see Definition 3.4 and Theorem 3.14). We then reduce Bernardi's signed sum to an enumeration formula for arrangements which are *almost* transitive (we make this notion precise in Definition 4.1). This class notably contains the *Ish arrangement*, which is known to admit a nice counting formula (namely the Cayley numbers, see [3] and the discussion in Section 2).

1.1. **Organization and main results.**

This paper is organized as follows: In Section 2, we review the necessary background and definitions. In Section 3, we study the contribution of all **S**-*boxings* of a given (rooted labeled) tree to the signed sum of Bernardi in general and prove our first main theorem:

Theorem 1.1 (Theorem 3.14, simplified version). *For an arbitrary deformation of the braid arrangement, the contribution of the set of boxed trees sharing an underlying rooted labeled tree to the formula of Bernardi is 0 or ± 1. Moreover, there exists an algorithm for computing this contribution.*

Trees which contribute -1 are not present in the transitive case, but we conjecture that they are ubiquitous in the non-transitive case (see Remark 3.16).

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In Section 4, we define *almost transitive* arrangements (Definition 4.1) and specialize the result of Theorem 3.14 to these arrangements. In Section 5, we again consider almost transitive arrangements. In this case, we define for every tree contributing to the signed sum two parameters: the *boxed 1-length* and *inefficiency* (Definitions 5.1, 5.3). We then construct a sign-reversing involution on the set of trees with boxed 1-length different from 1 or nonzero inefficiency, allowing us to prove our second main theorem:

Theorem 1.2 (Theorem 5.6)**.** *The regions of an almost transitive deformation of the braid arrangement are equinumerous with the corresponding rooted labeled trees with boxed 1-length 1 and inefficiency 0.*

In Section 6, we prove our final main theorem, showing explicitly for any non-negative nested Ish arrangement (Definition 2.1) that our sign-reversing involution simplifies the Bernardi formula to the known counting formula.

Theorem 1.3 (Theorem 6.2). *Let $\mathcal{A}_{\mathbf{S}}$ be a non-negative nested Ish arrangement in $\mathbb{R}^n$. For $2 \leq j \leq n$, let $S_{1,k}$ be the set of hyperplanes in $\mathcal{A}_{\mathbf{S}}$ of the form $x_1 - x_k = s$ for some $s \in \mathbb{R}$. Then the number of regions of $\mathcal{A}_{\mathbf{S}}$ is equal to*

$$\prod_{k=2}^{n}(n+1+|S_{1,k}|-k). \tag{1}$$

2. Background

In this section, we recall notations, constructions, and results that will be used in this paper. For detailed definitions and background pertaining to hyperplane arrangements, we refer readers to [9]. For the purposes of introduction, we follow much of the exposition in [4].

We consider *hyperplane arrangements* consisting of hyperplanes in $\mathbb{R}^n$ of the form

$$H_{i,j,s}: \ x_i - x_j = s$$

for some $1 \leq i < j \leq n$ and $s \in \mathbb{Z}$. These are known as *deformations of the braid arrangement*, where the *braid arrangement* consists of $\{H_{i,j,0}\}$ for all $1 \leq i < j \leq n$. A common question about a set of hyperplanes is the number of *regions* that they divide $\mathbb{R}^n$ into, where a region is a connected component of the complement of the hyperplanes in $\mathbb{R}^n$.

Deformations of the braid arrangement include several families of hyperplane arrangements with historically known counting formulas for their number of regions. Examples include the *braid arrangement*, *Shi arrangement* [8], and *Linial arrangement* [7]. We refer to [4, Sections 1-2] for additional examples and references.

Consider a deformation of the braid arrangement $\mathcal{A} = \{x_i - x_j = s\}$ in $\mathbb{R}^n$. We identify $\mathcal{A}$ with the tuple of sets $\mathbf{S} = (S_{i,j})_{1 \leq i < j \leq n}$, where for $1 \leq i < j \leq n$,

$$S_{i,j} := \{s : (x_i - x_j = s) \in \mathcal{A}\}.$$

We likewise denote

$$S_{j,i} := S_{i,j}, \qquad S^-_{i,j} := \{s \geq 0 | -s \in S_{i,j}\}, \qquad S^-_{j,i} := \{0\} \cup \{s > 0 | s \in S_{i,j}\} \tag{2}$$

Such a hyperplane arrangement is called an $\mathbf{S}$-*braid arrangement* and we write $\mathcal{A} = \mathcal{A}_{\mathbf{S}}$. The number of regions of $\mathcal{A}_{\mathbf{S}}$ is denoted $r_{\mathbf{S}}$. For use later, we fix the notation

$$m := \max\left\{|s| :\in \bigcup_{1 \leq i < j \leq n} S_{i,j}\right\}. \tag{3}$$

In this paper, we consider *nested Ish arrangements* as our prototypical example.

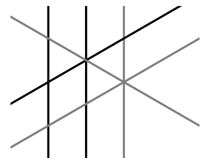

FIGURE 1. The projection of the Ish arrangement for $n = 3$ onto the plane $x_0 + x_1 + x_2 = 0$, viewed from the direction $(1, 1, 1)$. The three hyperplanes $x_i - x_j = 0$ are drawn in gray. The complement of the arrangement consists of $16 = 4^2$ connected components.

Definition 2.1. Let $\mathbf{S}$ be a tuple of sets as above such that $0 \in S_{i,j}$ for all $1 \leq i < j \leq n$, $S_{i,j} = \{0\}$ whenever $i \neq 1$, and $S_{1,j} \subseteq S_{1,k}$ for all $1 < j < k \leq n$. Then $\mathcal{A}_\mathbf{S}$ is called a *nested Ish arrangement.* If in addition $S_{1,j}$ consists of only non-negative integers for all j, we say that $\mathcal{A}_\mathbf{S}$ is *non-negative.* If $S_{1,j} = \{0, 1, \dots, j-1\}$ for all j, then $\mathcal{A}_\mathcal{S}$ is simply called the (n-dimensional) *Ish arrangement.*

The Ish arrangement with $n = 3$ is shown in Figure 1. This arrangement was defined in [2]. Its regions were studied in connection with those of the Shi arrangement in [3, 6]. In particular, it is shown that the number of regions formed by the Ish arrangement is the same as for the Shi arrangement, namely it is given by the Cayley formula $(n+1)^{n-1}$. The more general nested Ish arrangements are introduced in [1], where it is shown that their regions are enumerated by the formula

$$r_\mathbf{S} = \prod_{k=2}^{n} (n + 1 + |S_{1,k}| - k).$$

We now give an overview of the constructions and definitions in [4] that lay the foundation of this paper.

A *rooted plane tree* is a tree (a graph with no cycles) with some vertex designated as the root, and with an ordering imposed on the children of each vertex. If u, v are vertices joined by an edge such that the path from v to the root goes through u, u is the *parent* of v and v is one of the *children* of u. A vertex is a called a *node* if it has a child, and a *leaf* otherwise. We consider *rooted labeled (plane) trees*, where the nodes are labeled with distinct positive integers from 1 to the number of nodes.

We draw rooted labeled trees with the root of the tree at the bottom, each child of a vertex above the original vertex, and the children of any vertex ordered from left to right. Given a node u, its *cadet* is its rightmost child that is also a node, and is denoted $\mathsf{cadet}(u)$. If v has a parent u, we define the *left siblings* of v to be the vertices that are to the left of it in the ordering of the children of u. We denote let $\mathsf{lsib}(v)$ be the number of left siblings of v, and define *right siblings* similarly. An example of these concepts is shown in Figure 2.

We denote by $\mathcal{T}^{(m)}(n)$ the set of rooted labeled (plane) trees with n nodes such that each node has $m+1$ children.

Definition 2.2. [4, Definition 4.1] A sequence $(v_1, v_2, \dots, v_k)$ of nodes in a tree $T \in \mathcal{T}^{(m)}(n)$ is a *cadet sequence* if $v_j = \mathsf{cadet}(v_{j-1})$ for all $1 < j \leq k$. If in addition $\sum_{p=i+1}^{j} \mathsf{lsib}(v_p) \notin S^-_{i,j}$ for all $1 \leq i < j \leq k$, then the sequence $(v_1, v_2, \dots, v_k)$ is called an $\mathbf{S}$-*cadet sequence*

Examples of cadet sequences are shown as boxed sets of nodes in Figure 3.

Definition 2.3. [4, Definition 4.1] A *boxed tree* is a pair (T, B), where $T \in \mathcal{T}^{(m)}(n)$ and B is a partitioning of the nodes of T into cadet sequences. We say that (T, B) is $\mathbf{S}$-*boxed* if each cadet sequence is also an $\mathbf{S}$-cadet sequence. The set of $\mathbf{S}$-boxed trees is denoted $\mathcal{U}_\mathbf{S}(n)$.

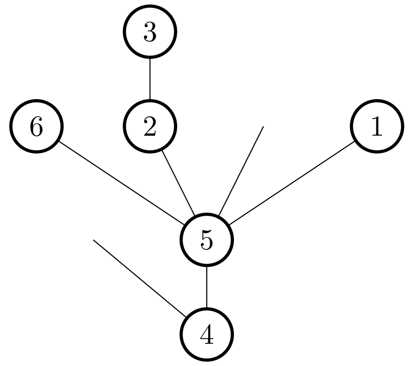

FIGURE 2. A rooted labeled (plane) tree with 6 nodes. Note that all leaves with no nodes to the right of them are omitted, which will be the case in the remaining examples as well. In this tree, 4 is the root, $\mathsf{cadet}(4) = 5$, $\mathsf{cadet}(5) = 1$, $\mathsf{lsib}(5) = 1$, $\mathsf{lsib}(1) = 3$, $\mathsf{lsib}(3) = 0$, $\text{parent}(6) = 5$, and 1 is a right sibling of 6.

Given a boxed tree (resp. **S**-boxed tree) (T, B), we will sometimes refer to B as a *boxing* (resp. **S**-*boxing*) of T and refer to the partition elements of B as *boxes* (resp. **S**-*boxes*). Examples of **S**-boxings can be found in Example 3.7 below.

Theorem 2.4. [4, Theorem 4.2] *The number of regions $r_{\mathbf{S}}$ in the arrangement $\mathcal{A}_{\mathbf{S}}$ is given by*

$$r_{\mathbf{S}} = \sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}(n)} (-1)^{n-|B|}, \tag{4}$$

where $|B|$ is the number of partition elements in B.

We refer to Equation 4 as the *Bernardi formula.* While the formula holds in general, there are many hyperplane arrangements with more explicit counting formulas (for example, the Ish arrangement). One of the main results in [4] is to recover such formulas when the set **S** satisfies a condition called *transitivity.*

Definition 2.5. [4, Definition 4.3] We call the tuple **S** *transitive* if for all distinct $i, j, k \in \{1, 2, \ldots, n\}$ and for all non-negative integers s and t, if $s \notin S^-_{i,j}$ and $t \notin S^-_{j,k}$, then $s + t \notin S^-_{i,k}$.

Theorem 2.6. [4, Theorem 4.6] *If **S** is transitive, then there exists a set of trees $\mathcal{T}_{\mathbf{S}} \subseteq \mathcal{T}^{(m)}(n)$ such that $\sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}(n)} (-1)^{n-|B|} = |\mathcal{T}_{\mathbf{S}}|$. Moreover, there is a sign-reversing involution on $\mathcal{U}_{\mathbf{S}}(n) \setminus \mathcal{T}_{\mathbf{S}}$.*

This result beautifully unifies and expands upon many known results for the number regions of certain hyperplane arrangements, in particular answering a question of Gessel (see [4, Section 2.3], [5, Section 1]).

As stated in the theorem, Bernardi's proof involves the construction of a sign-reversing involution; that is, a bijection $\mathcal{U}_{\mathbf{S}}(n) \setminus \mathcal{T}_{\mathbf{S}} \to \mathcal{U}_{\mathbf{S}}(n) \setminus \mathcal{T}_{\mathbf{S}}$ that swaps each **S**-boxed tree (T_1, B_1) with some other **S**-boxed tree (T_2, B_2) such that $(-1)^{n-|B_1|} + (-1)^{n-|B_2|} = 0$. This makes the corresponding terms in the Bernardi formula "cancel out". We adopt a similar approach in Section 5. More detailed explanation and examples of sign-reversing involutions can be found in [10, Section 1.8].

Remark 2.7. In the case where $\mathcal{A}_{\mathbf{S}}$ is the Ish arrangement and $n \geq 4$, we have $S^-_{4,2} = \{0\}$, $S^-_{2,1} = \{0, 1\}$, and $S^-_{4,1} = \{0, 1, 2, 3\}$. Observe that $1 \notin S^-_{4,2}$, $2 \notin S^-_{2,1}$, but $1 + 2 \in S^-_{4,2}$. Therefore, for $n \geq 4$, the Ish arrangement is **not** transitive.

Motivated by Theorem 2.6 and this remark, we will show directly in Theorems 5.6 and 6.2 that for any non-negative nested Ish arrangement, the Bernardi formula simplifies to the known counting

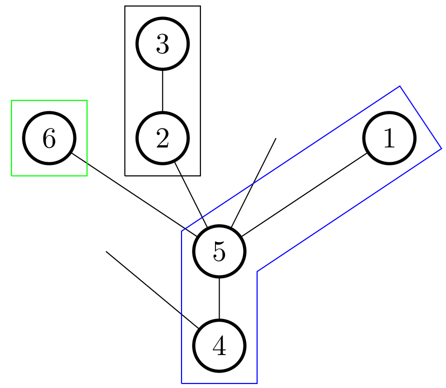

FIGURE 3. A rooted labeled tree partitioned into maximal cadet sequences. A box is drawn around the vertices of each sequence (but this is not necessarily an **S**-boxing). The last node of the maximal cadet sequence in the blue box is 1, and the first node is 4.

formula by the use of a sign-reversing involution. This is perhaps somewhat expected because the relation between the Ish arrangement and the transitive Shi arrangement.

3. The Contribution of a Tree

In this section, we define and characterize the contribution of a rooted labeled tree to the Bernardi formula. We fix for the remainder of this section an arbitrary deformation of the braid arrangement $\mathcal{A}_{\mathbf{S}}$.

Definition 3.1. Let $T \in \mathcal{T}^{(m)}(n)$. For any sequence $(v_1, v_2, \ldots, v_k)$ of the nodes of T, define the *last node* to be v_k, and the *first node* to be v_1. We refer to k as the *length* of the sequence.

Definition 3.2. Let $T \in \mathcal{T}^{(m)}(n)$. Define a *maximal cadet sequence* of T to be a sequence $(v_1, v_2, \ldots, v_k)$ of the nodes of T such that $\mathsf{cadet}(v_j) = v_{j+1}$, all the children of v_k are leaves, and there is no node u such that $\mathsf{cadet}(u) = v_1$.

An example of these concepts is shown in Figure 3.

Remark 3.3. As in Figure 3, the nodes of any rooted labeled tree can be partitioned into maximal cadet sequences. Indeed, since no two nodes can have the same cadet and each node has at most one cadet, we can complete any cadet sequence (in the forward and reverse directions) to a unique maximal cadet sequence. Possibly, such a sequence will contain only one node.

Given a (possibly maximal) cadet sequence in a rooted plane tree, we can consider an **S**-boxing *of the sequence* as in Definition 2.3. This leads to the following definition.

Definition 3.4. Let $T \in \mathcal{T}^{(m)}(n)$. Define the *contribution* of a (possibly maximal) cadet sequence of T of length k to be the sum of $(-1)^{k-|B|}$ over all **S**-boxings B of the sequence. Similarly, define the *contribution* of the tree T to be the sum of $(-1)^{n-|B|}$ over all **S**-boxings of T.

Observe that, by definition, the Bernardi formula (Equation 4) says that the number of regions of $\mathcal{A}_{\mathbf{S}}$ is the sum over $T \in \mathcal{T}^{(m)}(n)$ of the contribution of T.

Lemma 3.5. *Let $T \in \mathcal{T}^{(m)}(n)$. Then the contribution of T is equal to the product of the contributions of its maximal cadet sequences.*

Proof. Choosing an **S**-boxing of the tree is equivalent to choosing an **S**-boxing of each maximal cadet sequence. $\square$

Definition 3.6. Let $T \in \mathcal{T}^{(m)}(n)$. Let $(v_i, v_{i+1}, \ldots, v_j)$ be an $\mathbf{S}$-cadet sequence of T and let $(v_1, v_2, \ldots, v_k)$ be the maximal cadet sequence containing it. We say $(v_i, v_{i+1}, \ldots, v_j)$ is a *maximal box* if (a) either $i = 1$ or $(v_{i-1}, v_i, \ldots, v_{i+j})$ is not an $\mathbf{S}$-cadet sequence and (b) either $j = k$ or $(v_i, \ldots, v_j, v_{j+1})$ is not an $\mathbf{S}$-cadet sequence.

Example 3.7. In the tree from Figure 3, suppose that $S^-_{2,3} = S^-_{4,5} = S^-_{5,1} = \{0\}$, and $S^-_{4,1} = \{0, 1, 2, 3, 4, 5\}$. Then, writing ($\mathbf{S}$-)boxings as the list of sets of nodes in each $\mathbf{S}$-box,

(1) The only valid $\mathbf{S}$-boxing of the maximal cadet sequence in the green box is the partition $\{6\}$, for a contribution of $(-1)^{1-1} = 1$.
(2) The only valid $\mathbf{S}$-boxing of the maximal cadet sequence in the black box is the partition $\{2\}\{3\}$, for a contribution of $(-1)^{2-2} = 1$.
(3) The valid $\mathbf{S}$-boxings of the maximal cadet sequence in the blue box are the partitions $\{4\}\{5\}\{1\}$, $\{4, 5\}\{1\}$, and $\{4\}\{5, 1\}$ for a contribution of $(-1)^{3-3}+(-1)^{3-2}+(-1)^{3-2} = -1$.
(4) The valid $\mathbf{S}$-boxings of the tree are $\{6\}\{2\}\{3\}\{4\}\{5\}\{1\}$, $\{6\}\{2\}\{3\}\{4, 5\}\{1\}$, and $\{6\}\{2\}\{3\}\{4\}\{5, 1\}$ for a sum of

$$\begin{aligned} -1 &= (-1)^{6-6} + (-1)^{6-5} + (-1)^{6-5} \\ &= \left((-1)^{1-1}\right)\left((-1)^{2-2}\right)\left((-1)^{3-3} + (-1)^{3-2} + (-1)^{3-2}\right), \end{aligned}$$

which can be seen to be the product of the contributions of the maximal cadet sequences.

Moreover, the maximal boxes of the blue maximal cadet sequence are $\{4, 5\}$ and $\{5, 1\}$ since $\{4, 5, 1\}$ is not $\mathbf{S}$-boxed.

Definition 3.8. Let $T \in \mathcal{T}^{(m)}(n)$. Call a (nonempty) cadet sequence X of T *connected* if

(1) Given a maximal box Y of T, either $X \cap Y = \emptyset$ or $Y \subseteq X$.
(2) X contains no proper, nonempty subsequence which satisfies (1).

Lemma 3.9. *Let $T \in \mathcal{T}^{(m)}(n)$ and let $X = (v_1, \ldots, v_k)$ be a maximal cadet sequence of T. Then for all $1 \leq i \leq k$, there exist unique indices $1 \leq i' \leq i \leq i'' \leq k$ so that $(v_{i'}, \ldots, v_i, \ldots, v_{i''})$ is a connected cadet sequence. That is, any maximal cadet sequence can be uniquely partitioned into connected cadet sequences.*

Proof. Let $v_i \in X = (v_1, \ldots, v_k)$ and let $\mathcal{Y}_i$ be the set of cadet sequences containing v_i satisfying (1) in the definition of a connected cadet sequence. Since $X \in \mathcal{Y}_i$ and $\mathcal{Y}_i$ is closed under intersections, we see that $Y_i = \bigcap_{Y \in \mathcal{Y}_i} Y$ is the unique connected cadet sequence containing v_i. □

Lemma 3.10. *The contribution of a maximal cadet sequence is equal to the product of the contributions of its connected cadet sequences.*

Proof. Recall that any $\mathbf{S}$-box must be contained within a maximal box and any maximal box must be contained within some connected cadet sequence. Thus choosing an $\mathbf{S}$-boxing of a maximal cadet sequence is equivalent to choosing an $\mathbf{S}$-boxing of each of its connected cadet sequences. □

Example 3.11. Figure 4 shows a maximal cadet sequence and its maximal $\mathbf{S}$-boxes (we have suppressed the numbers of left siblings and the definition of $\mathbf{S}$). This can be partitioned into the connected cadet sequences with nodes labeled $\{1, 2, 3\}$, $\{4\}$, and $\{5, 6\}$.

(1) The valid boxings of $\{1, 2, 3\}$ are $\{1\}\{2\}\{3\}$, $\{1, 2\}\{3\}$, and $\{1\}\{2, 3\}$, so the contribution is $(-1)^{3-3} + (-1)^{3-2} + (-1)^{3-2} = -1$.
(2) The only valid boxing of $\{4\}$ is $\{4\}$, for a contribution of $(-1)^{1-1} = 1$.
(3) The valid boxings of $\{5, 6\}$ are $\{5\}\{6\}$ and $\{5, 6\}$, for a contribution of $(-1)^{2-2}+(-1)^{2-1} = 0$.
(4) The valid $\mathbf{S}$-boxings of the entire maximal cadet sequence are $\{1, 2, 3\}\{4\}\{5\}\{6\}$, $\{1, 2, 3\}\{4\}\{5, 6\}$ $\{1, 2\}\{3\}\{4\}\{5\}\{6\}$, $\{1, 2\}\{3\}\{4\}\{5, 6\}$, $\{1\}\{2, 3\}\{4\}\{5\}\{6\}$, and

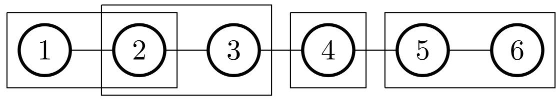

FIGURE 4. A maximal cadet sequence and its maximal **S**-boxes. Cadet relationships move left to right; that is, if there is an edge from u to v and v is right of u, then $\mathsf{cadet}(u) = v$.

$\{1\}\{2,3\}\{4\}\{5,6\}$, for a contribution of

$$\begin{aligned}&(-1)^{6-4} + (-1)^{6-3} + (-1)^{6-5} + (-1)^{6-4} + (-1)^{6-5} + (-1)^{6-4}\\ &= ((-1)^{3-3} + (-1)^{3-2} + (-1)^{3-2})((-1)^{1-1})((-1)^{2-2} + (-1)^{2-1}) = 0\end{aligned}$$

Definition 3.12. Let $T \in \mathcal{T}^{(m)}(n)$ and let $(v_1, v_2, \ldots, v_k)$ be a connected cadet sequence in T with maximal boxes $(X_1, X_2, \ldots, X_{k'})$, in increasing order of index of last node. For convenience, define X_0 and $X_{k'+1}$ to be empty sets of nodes and define $\mathsf{parent}(v_1) = \emptyset$ and $\mathsf{cadet}(v_k) = \emptyset$. For $0 \le i < j \le k'$, we say X_i *reaches* X_j if the parent of the largest indexed node that is in $X_j \setminus X_{j+1}$ is contained in X_i. We also say a (not necessarily maximal) box X *precedes* X_j if the cadet of the last node in X is the last node in $X_j \setminus X_{j+1}$.

Remark 3.13. We note that the condition $\mathsf{parent}(v_0) = \emptyset$ implies that X_0 reaches X_1 if $X_2 \setminus X_1$ contains a single node and X_0 does not reach any maximal box otherwise. Likewise, the condition $\mathsf{cadet}(v_k) = \emptyset$ implies that $X_{k'-1}$ reaches $X_{k'}$ if $X_{k'} \setminus X_{k'-1}$ contains a single node and $X_{k'-1}$ does not reach any maximal box otherwise.

Theorem 3.14. *Let $T \in \mathcal{T}^{(m)}(n)$ and let $(v_1, v_2, \ldots, v_k)$ be a connected cadet sequence in T with maximal boxes $(X_1, X_2, \ldots, X_{k'})$, in increasing order of index of last node. For convenience, define X_0 and $X_{k'+1}$ to be empty sets of nodes. Generate a subsequence $(X_{i_0} X_{i_1}, \ldots, X_{i_t})$ of $(X_0, X_1, X_2, \ldots, X_{k'})$ by the following algorithm:*

(1) *Define $X_{i_0} = X_0$*
(2) *For $j \ge 0$ such that $i_j \ne k'$, let $X_{i_{j+1}}$ be the first maximal box reached by X_{i_j} but not reached by X_{i_k} for $k < j$, if it exists. If no such maximal box exists, the algorithm fails.*
(3) *If $i_j = k'$, take $t = j$.*

Now, if any step of this fails, the contribution is 0. Otherwise, the contribution is $(-1)^{k-t}$.

Our proof uses ideas similar to that of [4, Theorem 4.6].

Proof. We first claim that if $X_2 \setminus X_1$ contains more than a single node, then the algorithm fails and the contribution of X is 0. Indeed, if there exist two nodes $v_i, v_{i+1} \in X_1 \setminus X_2$, then there is an involution on the **S**-boxings of X given as follows:

- If v_i is in the **S**-box Y and v_{i+1} is in the **S**-box $Y' \ne Y$, replace Y and Y' with $Y \cup Y'$.
- If there exists an **S**-box Y containing v_i and v_{i+1}, replace it with $Y' = \{v_j \in Y | j \le i\}$ and $Y'' = \{v_j \in Y | j \ge i+1\}$.

This involution changes the parity of $|B|$ (the number of **S**-boxes) and thus implies the contribution of X is 0. Moreover, we recall that if $X_2 \setminus X_1$ contains more than a single node, then X_0 does not reach any maximal box, so the algorithm fails in this case.

Now let $\mathcal{B}$ be the set of **S**-boxings of X for which every **S**-box except the last one (the one containing v_k) precedes some maximal box X_j for $j \le k'$.

We claim there is a sign-reversing involution on the **S**-boxings which are not in $\mathcal{B}$. Indeed, let B be such an **S**-boxing and let $Y \in B$ be the first box that does not precede any maximal box. Let v the last node of Y. Now since Y does not precede any maximal box, we have that $\mathsf{cadet}(v)$

and $\mathsf{cadet}(\mathsf{cadet}(v))$ are contained in $X_j \setminus X_{j+1}$ for some j. As before, we can then construct a sign-reversing involution by combing or splitting the boxes containing $\mathsf{cadet}(v)$ and $\mathsf{cadet}(\mathsf{cadet}(v))$.

We have thus far shown that the contribution of X is $\sum_{B\in\mathcal{B}}(-1)^{k-|B|}$. We wish to reduce this to the sum over a single $\mathbf{S}$-boxing (of size $t+1$).

We claim there is a bijection between $\mathcal{B}$ and subsequences $(X_{i_0},\ldots,X_{i_r})$ of $(X_0,X_1,\ldots,X_{k'+1})$ such that $X_{i_0}=X_0, X_{i_r}=X_{k'}$, and X_{i_j} reaches $X_{i_{j+1}}$ for all j. Moreover, the number of boxes in B is the same as the length (i.e., the value of $r+1$) in the corresponding sequence.

For readability, we use X_{1+i_j} for the maximal box with index $1+i_j$ and $X_{i_{j+1}}$ for the maximal box with index i_{j+1} (the maximal box for which i has index $j+1$).

First consider an $\mathbf{S}$-boxing $B=(Y_1,\ldots,Y_r)\in\mathcal{B}$. For $1\le j\le r-1$, let $X_{i_{j+1}}$ be the maximal box that Y_j precedes. Let $X_{i_0}=X_0$ and $X_{i_1}=X_1$. Note that $X_{i_r}=X_{k'}$ because $v_k\in X_{k'}\setminus X_{k'-1}$, so Y_r must be contained within $X_{k'}$, so the cadet of the last node in Y_{r-1} must be in $X_{k'}$.

To see that X_{i_j} reaches $X_{i_{j+1}}$, let v be the last node in Y_{j-1}. Then since Y_{j-1} precedes X_{i_j}, we have $\mathsf{cadet}(v)\in X_{i_j}\setminus X_{1+i_j}$. Moreover, we have $\mathsf{cadet}(v)\in Y_j$ which means $Y_j\subseteq X_{i_j}$. Now let u be the last node in Y_j. Then since Y_j precedes $X_{i_{j+1}}$, we have $\mathsf{cadet}(u)\in X_{i_{j+1}}\setminus X_{1+i_{j+1}}$, and $\mathsf{cadet}(u)$ is also the largest indexed node with this property. Now, since the node u with this property is contained in Y_j, it is contained in X_{i_j}, so X_{i_j} reaches $X_{i_{j+1}}$ by definition.

Now consider a sequence $(X_{i_0},\ldots,X_{i_r})$ so that $X_{i_0}=X_0, X_{i_r}=X_{k'}$, and X_{i_j} reaches $X_{i_{j+1}}$ for all j. Now for $1\le j\le r-1$, define Y_j to have last node $u\in X_{i_j}$ so that $\mathsf{cadet}(u)\in X_{i_{j+1}}\setminus X_{1+i_{j+1}}$ and $\mathsf{cadet}(\mathsf{cadet}(u))\in X_{1+i_{j+1}}$, which exists since X_{i_j} reaches $X_{i_{j+1}}$. Define the last node of Y_r to be v_k.

Now that we have identified our remaining $\mathbf{S}$-boxings with certain sequences of maximal boxes, we wish to find a sign-reversing involution on all such sequences except the one generated by the algorithm. For such a sequence let (1_j) denote the condition that $X_{i_{j+2}}$ is not reached by X_{i_j}, and let (2_j) denote the condition that $X_{i_{j+1}}$ is the first box in the full sequence reached by X_{i_j} but not $X_{i_{j-1}}$.

Suppose there is some smallest index q for which either (1_q) is false or (2_q) is false. If (2_q) is false, we can insert X_p after X_{i_j} such that X_p is the smallest indexed box reached by X_{i_j} but not $X_{i_{j-1}}$. This increases the length by 1 and makes (1_q) false and (2_q) true.

Otherwise, (2_q) is true and (1_q) is false. We can then delete X_{q+1} from the subsequence. This decreases the length by 1 and makes (2_q) false. As these two operations are inverse to each other, we can consider only $\mathbf{S}$-boxings $B\in\mathcal{B}$ corresponding to sequences satisfying (1_j) and (2_j) for all j. This leaves only the sequence generated by the algorithm. □

We observe that Theorem 3.14, together with Lemmas 3.5 and 3.10, provide an algorithm for computing the contribution of any tree $T\in\mathcal{T}^{(m)}(n)$. We add the following claim, which makes it easier to compute the contribution of many trees.

Corollary 3.15. *In a tree with nonzero contribution, any maximal box of size more than one must intersect some other maximal box.*

Proof. Let X be a maximal box which does not intersect some other maximal box. We observe that X is a connected cadet sequence. Moreover, if there exist two nodes $v,\mathsf{cadet}(v)\in X$, then as in the proof of the theorem, we can construct a sign-reversing involution by combining or splitting the boxes containing v and $\mathsf{cadet}(v)$. Thus any such tree has contribution 0. □

Remark 3.16. We observe that for transitive arrangements, maximal boxes cannot intersect and hence every connected cadet sequence consists of exactly one maximal box. This implies that the contribution of each connected cadet sequence (and by extension each tree) is either 0 or 1. This fact was leveraged by Bernardi to prove Theorem 2.6. We suspect that this is never the case for a non-transitive arrangement. For example, if $s\notin S^-_{i,j}$, $t\notin S^-_{j,k}$, and $s+t\in S^-_{i,k}$, then a connected cadet sequence (v_i,v_j,v_k) with $\mathsf{lsib}(v_j)=s$ and $\mathsf{lsib}(v_k)=t$ would contribute -1.

4. Contributions for Almost Transitive Arrangements

In this section, we define *almost transitive* arrangements, show that the Ish arrangement is almost transitive, and characterize the contributions of trees in the case of the almost transitive arrangements.

Definition 4.1. We define a hyperplane arrangement $\mathcal{A}_{\mathbf{S}}$ to be *almost transitive* if for all distinct $i, j, k \in \{1, \dots, n\}$ with $k \neq 1$ and for all non-negative integers $s \notin S^-_{i,j}$ and $t \notin S^-_{j,k}$, we have $s + t \notin S^-_{i,k}$.

Remark 4.2. Since permuting the axes does not affect the number of regions in the hyperplane arrangement or the signed sum of trees, the condition $c \neq 1$ could be replaced with $c \neq k$ for any fixed k.

Lemma 4.3. *Any non-negative nested Ish arrangement is almost transitive.*

Proof. Let $\mathcal{A}_{\mathbf{S}}$ be a non-negative nested Ish arrangement. From Equation 2 and the non-negativity assumption, we have $S^-_{i,j} = \{0\}$ whenever $j \neq 1$ and $S^-_{i,1} = S_{1,i}$ for all $1 < i \leq n$.

Now let $i, j, k \in \{1, \dots, n\}$ be distinct with $k \neq 1$ and let $s \notin S^-_{i,j}$ and $t \notin S^-_{j,k}$ be non-negative integers. Note that $0 \in S^-_{i,j} \cap S^-_{j,k}$, so we must have $s > 0$ and $t > 0$. Since $k \neq 1$, this means $s + t \notin S^-_{i,k} = \{0\}$, as desired. □

Remark 4.4. In general, there exist nested Ish arrangements which are not almost transitive. For example, in $\mathbb{R}^3$, suppose $\mathcal{A}_{\mathbf{S}}$ is a nested Ish arrangement with $S_{1,2} = \{0, 1\}$ and $S_{1,3} = \{-2, 0, 1\}$. Then we have $S^-_{1,2} = \{0\}$ and $S^-_{1,3} = \{0, 2\}$. In particular, $1 \notin S^-_{1,2} \cup S^-_{2,3}$, but $2 \in S^-_{1,2}$, meaning $\mathbf{S}$ is not transitive.

Lemma 4.5. *Let $\mathcal{A}_{\mathbf{S}}$ be almost transitive and let $T \in \mathcal{T}^{(m)}(n)$. Suppose that $(v_1, v_2, \dots, v_k)$ and (v_k, v_{k+1}) are $\mathbf{S}$-cadet sequences in T, where $v_{k+1} \neq 1$. Then, $(v_1, v_2, \dots, v_{k+1})$ is an $\mathbf{S}$-cadet sequence in T.*

Proof. We need to show that for each $1 \leq j < k$, given that $\sum_{i=j+1}^{k} \operatorname{lsib}(v_i) \notin S^-_{v_j,v_k}$ and $\operatorname{lsib}(v_{k+1}) \notin S^-_{v_k,v_{k+1}}$, then $\sum_{i=j+1}^{k+1} \operatorname{lsib}(v_i) \notin S^-_{v_j,v_{k+1}}$, which follows directly from the definition. □

Corollary 4.6. *Let $\mathcal{A}_{\mathbf{S}}$ be almost transitive and let $T \in \mathcal{T}^{(m)}(n)$. Suppose there exist positive integers a, b, c such that $(u_1, u_2, \dots, u_{a+b})$ and $(u_a, u_{a+1}, \dots, u_{a+b+c})$ are maximal boxes of T. Then $u_{a+b+1} = 1$.*

Proof. Suppose, for the sake of contradiction, that $u_{a+b+1} \neq 1$. Then, since $(u_1, u_2, \dots, u_{a+b})$ and (u_{a+b}, u_{a+b+1}) are $\mathbf{S}$-cadet sequences (because they are contained in maximal boxes), by Lemma 4.5, $(u_1, u_2, \dots, u_{a+b}, u_{a+b+1})$ is a $\mathbf{S}$-cadet sequence, contradicting the fact that $(u_1, u_2, \dots, u_{a+b})$ is a maximal box. □

Now, we can fully characterize the contribution of a tree.

Proposition 4.7. *Let $\mathcal{A}_{\mathbf{S}}$ be almost transitive and let $T \in \mathcal{T}^{(m)}(n)$. Then the contribution of T is nonzero if and only the following conditions hold:*

(1) Any maximal cadet sequence in T not containing 1 does not contain any maximal boxes of size greater than 1.
(2) The maximal cadet sequence $(v_1, v_2, \dots, v_k)$ containing the node $1 = v_j$ either has no maximal boxes of size greater than 1, or has maximal boxes

$$\{v_1\}, \{v_2\}, \dots, \{v_{i-1}\}\{v_i, v_{i+1}, \dots, v_{j-1}\}, \{v_{i+1}, v_{i+2}, \dots, v_j\},$$
$$\{v_{j+1}\}, \{v_{j+2}\}, \dots, \{v_k\}$$

for some j.

Furthermore, a tree satisfying these conditions contributes $(-1)^{j-i+1}$.

Proof. Let $T \in \mathcal{T}^{(m)}(n)$ have nonzero contribution. Then each of its maximal cadet sequences must contribute ± 1 by Lemma 3.5. By Corollary 3.15, this means any maximal box of size greater than 1 intersects another maximal box, and so condition (1) is a direct consequence of Corollary 4.6.

Now, consider the maximal cadet sequence $X = (v_1, v_2, \ldots, v_k)$ containing $v_j = 1$. Suppose X has maximal boxes $(X_1, X_2, \ldots, X_{k'})$ in increasing order of index of last node. Amongst those maximal boxes with size greater than 1, all of them except the last one must have cadet 1 by Corollary 4.6. Since no two maximal boxes can have the same last node (or one is contained in the other, contradiction), there are at most two maximal boxes of size greater than one. Since any maximal box of size greater than one intersects another (see Corollary 3.15), there are thus either 0 or 2 such maximal boxes.

In the case that there are no maximal boxes of size greater than 1, we are done. If there are two intersecting maximal boxes $X_r = (v_i, v_{i+1}, \ldots, v_{k-1})$ and $X_{r+1} = (v_{k-t}, v_{k-t+1}, \ldots, v_{k+s})$. Since the contribution is nonzero, Theorem 3.14 implies that $X_{r-1} = \{v_{a-1}\}$ reaches X_r. Thus $v_{i+1} = v_{k-t} \in X_{r+1}$. Likewise, X_r reaches X_{r+1}, so $v_{k+s-1} = v_{k-1} \in X_r$. Moreover, there are $2 + (i-1) + (k-j)$ maximal boxes in the maximal cadet sequence X. So, by Theorem 3.14, its contribution is $(-1)^{j-i+1}$. All other maximal cadet sequences contain exclusively maximal boxes of length 1, and thus contribute $(-1)^0$. The result is thus proved. □

Now, we restrict our attention to the trees contributing a nonzero amount.

5. Boxed 1-length and Inefficiency

In this section, we again work with almost transitive arrangements. We define two parameters for any tree in $\mathcal{T}^{(m)}(n)$: the boxed 1-length and the inefficiency. We then reduce the Bernardi formula to the enumeration of trees with boxed 1-length 1 and inefficiency 0.

Definition 5.1. Let $\mathcal{A}_{\mathbf{S}}$ be almost transitive. For a tree $T \in \mathcal{T}^{(m)}(n)$, we denote by $\ell = \ell(T)$ the size of the maximal box containing 1, which we call the *boxed 1-length* of T.

Remark 5.2. In the notation of Proposition 4.7, we have $\ell(T) = j - i$, and the tree T contributes $(-1)^{\ell(T)-1}$.

Definition 5.3. Let $\mathcal{A}_{\mathbf{S}}$ be almost transitive. For a tree $T \in \mathcal{T}^{(m)}(n)$ and for any $j \in \{2, 3, \ldots, n\}$, we call j *inefficient* if j is a left sibling of 1 and $\mathrm{lsib}(j) \notin S^-_{j,1}$. The *inefficiency* of T is the number of inefficient nodes in T, which we denote $E = E(T)$.

We denote by $\mathbf{S}(E_0, \ell_0)$ be the set of trees $T \in \mathcal{T}^{(m)}(n)$ with $E(T) = E_0$ and $\ell(T) = \ell_0$. An example of a tree in $\mathbf{S}(1,2)$ for $n = 6$ when $\mathbf{S}$ is the Ish arrangement is shown in Figure 5.

Remark 5.4. Both $E(T)$ and $\ell(T)$ are bounded above by n, so their sum is bounded above by $2n$.

Lemma 5.5. *Let* $\mathcal{A}_{\mathbf{S}}$ *be almost transitive. Then for all* E_0 *and* ℓ_0 *with* $\ell_0 > 1$ *and* $|\mathbf{S}(E_0, \ell_0)| \neq 0$, *we have*

$$|\mathbf{S}(E_0, \ell_0)| = \sum_{b=E_0+1}^{2n-(\ell_0-1)} |\mathbf{S}(b, \ell_0 - 1)|.$$

Proof. We will demonstrate a bijection between $\mathbf{S}(E_0, \ell_0)$ and $\bigcup_{b=E_0+1}^{2n-(\ell_0-1)} \mathbf{S}(b, \ell_0 - 1)$. An example is shown in Figure 8. We disregard the right siblings (necessarily leaves) of any cadet.

Let $T \in \mathbf{S}(E_0, \ell_0)$. Let u, v be the nodes such that $\mathsf{cadet}(u) = v$ and $\mathsf{cadet}(v) = 1$. We construct a tree $T' \in \bigcup_{b=E_0+1}^{2n-(\ell_0-1)} \mathbf{S}(b, \ell_0 - 1)$ as follows:

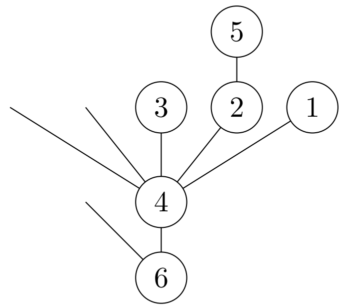

FIGURE 5. A tree in $\mathbf{S}(2,1)$ for $n=6$ and $\mathbf{S}$ as the Ish arrangement.. The maximal cadet sequences are (3), $(2,5)$, and $(6,4,1)$ and the maximal boxes are $\{2\}$, $\{3\}$, $\{5\}$, $\{6,4\}$, and $\{4,1\}$. The boxed 1-length is $\ell=|\{6,4\}|=|\{4,1\}|=2$ and the inefficiency is 1, since 2 is the only inefficient node. Thus this tree contributes $(-1)^{\ell-1}=-1$ to the Bernardi formula.

(1) Remove the edge connecting u to its leftmost child that is not a leaf (call this child c), add a node labeled v' as the leftmost child of u, and draw an edge between v' and c so that c is the cadet of v'.
(2) If s_1, s_2, $\dots$, $s_{\mathsf{lsib}(1)}$ are the left siblings of 1 from right to left (including leaves), draw edges from u to $s_1, s_2, \dots, s_{\mathsf{lsib}(1)}$ such that $\mathsf{lsib}(s_k)=k-1$ for all $k\in\{1,2,3,\dots,\mathsf{lsib}(1)\}$.
(3) Delete node v and all of its edges, relabel v' to v, and draw an edge between u and 1. Let this final tree be T'.

Note that since $\mathsf{lsib}(1)+\mathsf{lsib}(v)$ in T is equal to $\mathsf{lsib}(1)$ in T', and $\mathsf{lsib}(w)$ remains unchanged for all $w\notin\{1,v\}$ in the maximal cadet sequence, both of the maximal boxes of size more than one in T only have v removed in T'. Thus, $\ell(T')=\ell(T)-1$.

Since $\mathsf{lsib}(w)$ is the same in T' for all left siblings of 1 in T (or left siblings of v in T'), all the inefficient nodes in T remain inefficient in T'. Since $\mathsf{lsib}(v)$ in T' is the same as $\mathsf{lsib}(1)$ in T, and the nodes v and 1 are in a maximal box in T, we have $\mathsf{lsib}(v)\notin S^-_{j,1}$ in T'. So, v is inefficient in T', meaning that $E(T')\geq E(T)+1$, as desired.

For the reverse direction, starting with a tree $T\in\bigcup_{b=E_0+1}^{2n-(\ell_0-1)}\mathbf{S}(b,\ell_0-1)$, let u,v be the nodes such that $\mathsf{cadet}(u)=1$ and v is the (E_0+1)-th inefficient node (from the left) among the inefficient left siblings of 1. Suppose that $s_1,s_2,\dots,s_{\mathsf{lsib}(v)}$ are the left siblings of v (including leaves). We construct a tree $T'\in\mathbf{S}(E_0,\ell_0)$ as follows:

(1) Delete the edge from u to 1, delete the edge from u to v, delete the edge from v to $c=\mathsf{cadet}(v)$, and draw an edge from u to c.
(2) Delete the edge from u to 1, re-position v and draw an edge from u to v such that $\mathsf{lsib}(1)$ in T is equal to $\mathsf{lsib}(v)$ in the current tree, and draw an edge from v to 1.
(3) Delete the edges from u to $s_1,s_2,\dots,s_{\mathsf{lsib}(v)}$ and draw edges from v to s_1, s_2, $\dots,s_{\mathsf{lsib}(v)}$ such that the left siblings of 1 are s_1, s_2, $\dots$, $s_{\mathsf{lsib}(v)}$ in that order.

Since $\mathsf{lsib}(1)$ in T' is the same as $\mathsf{lsib}(v)$ in T, and v is inefficient in T, we have $\mathsf{lsib}(1)\notin S^-_{j,1}$ in T'. Note that since $\mathsf{lsib}(1)+\mathsf{lsib}(v)$ in T' is equal to $\mathsf{lsib}(1)$ in T, and $\mathsf{lsib}(w)$ remains unchanged for all $w\notin\{1,v\}$ in the maximal cadet sequence, both of the maximal boxes that either contain 1 or have 1 as the child of its last node in T only have v added in T'. Thus, $\ell(T')=\ell(T)+1$.

Since $\mathsf{lsib}(w)$ is the same in T' for all left siblings of 1 in T' (or left siblings of v in T), exactly E_0 (by definition of v) of the inefficient nodes in T remain inefficient in T'. Thus, $E(T')=E_0$.

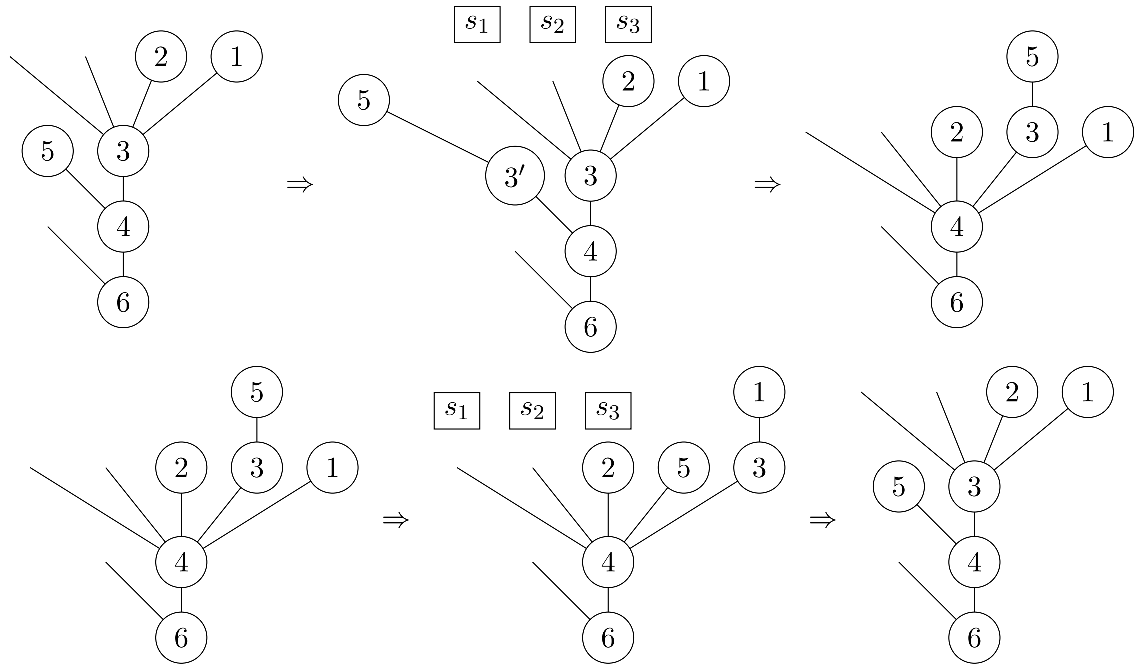

FIGURE 6. An example of the forward and reverse directions of the sign reversing involution between $\mathbf{S}(1,3)$ and $\mathbf{S}(2,2)$ ($n = 6$) in the case where $\mathcal{A}_{\mathbf{S}}$ is the Ish arrangement.

.

It is straightforward to check that these are indeed inverses of each other, so there is a bijection

$$\mathbf{S}(E_0, \ell_0) \leftrightarrow \bigcup_{b=E_0+1}^{2n-(\ell_0-1)} \mathbf{S}(b, \ell_0 - 1).$$

□

An example of the bijection used to prove Lemma 5.5 is shown in Figure 6.

Theorem 5.6. *Let $\mathcal{A}_{\mathbf{S}}$ be almost transitive. Then the regions of $\mathcal{A}_{\mathbf{S}}$ are equinumerous with the trees in $\mathbf{S}(0,1)$. That is, $r_{\mathbf{S}} = |\mathbf{S}(0,1)|$.*

Proof. For $\ell_0 > 1$, the $E_0 = 0$ case of Proposition 4.7 gives a sign reversing involution

$$\mathbf{S}(0, \ell) \leftrightarrow \bigcup_{k=1}^{2n-(\ell-1)} \mathbf{S}(k, \ell - 1).$$

Taking this over all $\ell_0 > 1$, there is a sign reversing involution

$$\bigcup_{\ell_0=2}^{n} \mathbf{S}(0, \ell_0) \leftrightarrow \bigcup_{\ell_0=2}^{n} \bigcup_{k=1}^{2n-(\ell_0-1)} \mathbf{S}(k, \ell_0 - 1),$$

leaving the trees in $\mathbf{S}(0,1)$, each of which contribute 1 by Proposition 4.7. This gives a sum of $|\mathbf{S}(0,1)|$. □

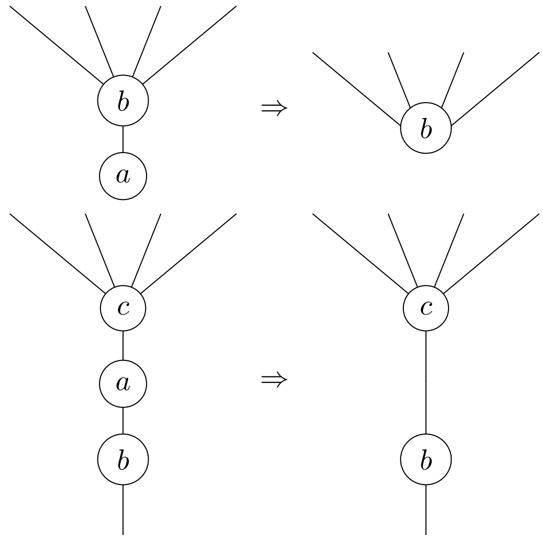

FIGURE 7. Extracting the node labeled a.

6. Nested Ish Arrangements

In this section, we show directly that for any non-negative nested Ish arrangement the number of trees in $\mathbf{S}(0,1)$ is equal to the known counting formula (Equation 1).

Lemma 6.1. *Let $\mathcal{A}_{\mathbf{S}}$ be a non-negative nested Ish arrangement. Consider the set $\mathfrak{T}(\mathbf{S})$ of rooted, labeled trees with n nodes and some number of unlabeled leaves such that*

(1) The node 1 has $m+1$ children.
(2) Every other node has one child.
(3) For k a node, $\mathsf{lsib}(k) \in S^-_{k,1}$

Then $|\mathfrak{T}(\mathbf{S})| = \prod_{k=2}^{n}(n+1+|S_{1,k}|-k)$.

Proof. Let $\mathfrak{S}(\mathbf{S})$ be the set of sequences $a_2,\dots,a_n$ where for all k,

$$a_k \in \mathfrak{A}_k := \{(0,i) \mid i \in \{k+1,k+2,\dots,n\}\} \cup \{(1,s) \mid s \in S^-_{k,1}\} \cup \{(2,0)\}.$$

We observe that

$$|\mathfrak{S}(\mathbf{S})| = \prod_{k=2}^{n}((n-k)+|S^-_{k,1}|+1) = \prod_{k=2}^{n}(n+1+|S_{1,k}|-k).$$

We will construct a bijection between $\mathfrak{T}(\mathbf{S})$ and $\mathfrak{S}(\mathbf{S})$.

For a vertex v of degree at most 2 in a tree, define *extracting* v as follows:

(1) If v has degree 1 and is not a child of the node 1, remove it from the tree.
(2) If v has degree 2, remove it from the tree and draw an edge between its neighbors.
(3) Otherwise, replace v with a leaf.

An example is shown in Figure 7.

Let $T_1 = T \in \mathfrak{T}(\mathbf{S})$. We associate T to a sequence $a_2,\dots,a_n$ in $\mathfrak{S}(\mathbf{S})$ as follows: For each $k \in \{2,3,4,\dots,n\}$ in increasing order, do the following:

(1) From the tree T_{k-1}, define

$$a_k = \begin{cases} (2,0) & \text{if } \mathsf{child}(k) = 1 \text{ in } T_{k-1} \\ (1, \mathsf{lsib}(k)) & \text{if } \mathsf{parent}(k) = 1 \text{ in } T_{k-1} \\ (0, \mathsf{child}(k)) & \text{if } k \text{ is below 1 and } \mathsf{child}(k) \neq 1 \text{ in } T_{k-1} \\ (0, \mathsf{parent}(k)) & \text{if } k \text{ is above 1 and } \mathsf{parent}(k) \neq 1 \text{ in } T_{k-1} \end{cases}$$

(2) Extract k from T_{k-1} to yield T_k.

To show this is well-defined, we will show that for $i \in \{2, \ldots, n\}$ and $k \geq i$, we have $\mathsf{lsib}(k) \in S^-_{k,1}$ for the tree T_{i-1}. This is automatic for $i = 2$.

Assume the property is true for $i \in \{2, \ldots, n-1\}$ and consider the tree T_{i-1}. Note that if i is below 1 or $\mathsf{parent}(i) \neq 1$, then extracting i does not change the number of left siblings of any other node. Thus suppose $\mathsf{parent}(i) = 1$. If i has degree 1 then again extracting i does not change the number of left siblings of any other node. If i has degree 2, denote $\mathsf{child}(i)$ the child of i in T_{i-1}. Then extracting i makes $\mathsf{lsib}(\mathsf{child}(i))$ in T_i equal to $\mathsf{lsib}(i)$ in T_{i-1}, while all other nodes have their number of left siblings unchanged. Now since $\mathsf{child}(i) > i$ and $\mathcal{A}_{\mathbf{S}}$ is a non-negative nested Ish arrangement, we know $S^-_{i,1} \subseteq S^-_{\mathsf{child}(i),1}$. Thus the desired property is true for T_i. By induction, we conclude that each tree in $\mathfrak{T}(\mathbf{S})$ is mapped to a well-defined sequence in $\mathfrak{S}(\mathbf{S})$. See Figure 8 for an example of this map.

Now, we construct the inverse of this mapping from $\mathfrak{T}(\mathbf{S})$ to $\mathfrak{S}(\mathbf{S})$. Let $a_2, a_3 \ldots, a_n$ be a sequence in $\mathfrak{S}(\mathbf{S})$.

Start with the tree T_n consisting of a node labeled 1 with $m+1$ children (all of which are leaves). For each $k \in \{n, n-1, \ldots, 2\}$ in decreasing order, construct T_{k-1} from T_k as follows:

(1) If $a_k = (2, 0)$, define T_{k-1} to be the tree such that $\deg(k) \in \{1, 2\}$, $\mathsf{cadet}(k) = 1$, and, when k is extracted, the resulting tree is T_k.
(2) If $a_k = (1, i)$, define T_{k-1} to be the tree such that $\deg(k) \in \{1, 2\}$, $\mathsf{lsib}(k) = i$, $\mathsf{parent}(k) = 1$, and, when k is extracted, the resulting tree is T_k.
(3) If $a_k = (0, i)$, define T_{k-1} to be the tree such that $\deg(k) \in \{1, 2\}$, $\mathsf{cadet}(k) = i$ if the node i is above 1, $\text{parent}(k) = i$ if the node i is below 1, and, when k is extracted, the resulting tree is T_k.

Note that the final tree $T = T_1$ is in $\mathfrak{T}(\mathbf{S})$. It remains to show these operations are inverse to one another. To see this, observe that for all k, extracting k in T_{k-1} as in the second algorithm yields T_k. Moreover, the value of a_k in the given sequence satisfies

$$a_k = \begin{cases} (2,0) & \text{if } \mathsf{child}(k) = 1 \text{ in } T_{k-1} \\ (1, \mathsf{lsib}(k)) & \text{if } \mathsf{parent}(k) = 1 \text{ in } T_{k-1} \\ (0, \mathsf{child}(k)) & \text{if } k \text{ is below 1 and } \mathsf{child}(k) \neq 1 \text{ in } T_{k-1} \\ (0, \mathsf{parent}(k)) & \text{if } k \text{ is above 1 and } \mathsf{parent}(k) \neq 1 \text{ in } T_{k-1} \end{cases},$$

which is identical to the definition in the first algorithm. This implies that the two given algorithms are inverses of each other. Thus, there is a bijection between $\mathfrak{T}(\mathbf{S})$ and $\mathfrak{S}(\mathbf{S})$, as desired. □

An example of the construction of a sequence in $\mathfrak{S}(\mathbf{S})$ from a tree in $\mathfrak{T}(\mathbf{S})$, where $\mathbf{S}$ is the Ish arrangement with $n = 4$, is shown in Figure 8.

Combing Lemmas 6.1 and 4.3 with Theorem 5.6, we recover the known counting result of [1] for the number of regions of a non-negative nested Ish arrangement.

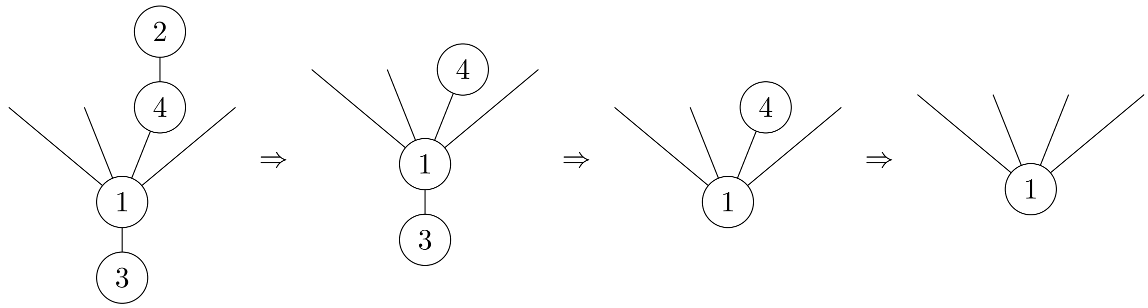

FIGURE 8. In the case of the Ish arrangement with $n = 4$, the tree on the left (in $\mathfrak{T}(\mathbf{S})$) corresponds to the sequence $(0, 4), (2, 0), (1, 2)$ by the bijection in the proof of Lemma 6.1 via the sequence of extractions shown.

Theorem 6.2. *Let $\mathcal{A}_{\mathbf{S}}$ be a non-negative nested Ish arrangement. Then the number of regions of $\mathcal{A}_{\mathbf{S}}$ is given by*

$$r_{\mathbf{S}} = \prod_{k=2}^{n} (n + 1 + |S_{1,k}| - k).$$

In particular, if $\mathcal{A}_{\mathbf{S}}$ is the (n-dimensional) Ish arrangement, then the number of regions is given by the Cayley formula: $r_{\mathbf{S}} = (n + 1)^{n-1}$.

Proof. By Theorem 5.6 and Lemma 4.3, we need only show that $|\mathbf{S}(0, 1)| = \prod_{k=2}^{n}(n+1+|S_{1,k}|-k)$. Observe that we have the following bijection between $\mathbf{S}(0, 1)$ and $\mathfrak{T}(\mathbf{S})$.

- Map trees from $\mathbf{S}(0, 1)$ to T by swapping the label of 1 with the label of its parent (or doing nothing if 1 does not have a parent), and removing all leaves of the nodes 2, 3, $\ldots, n$.
- Map trees from T to $\mathbf{S}(0, 1)$ by swapping the label of 1 with the label of its cadet (or doing nothing if all of its children are leaves) and adding right siblings as leaves such that each node has n children.

Lemma 6.1 then implies the result. □

As an example, for $\mathcal{A}_{\mathbf{S}}$ the Ish arrangement with $n = 4$, the tree $T \in \mathfrak{T}(\mathbf{S})$ shown in Figure 8 would have the nodes 1 and 4 interchanged by this bijection.

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References

1. Takuro Abe, Daisuke Suyama, and Shuhei Tsujie, *The freeness of Ish arrangements*, J. Combin. Theory, Series A **146** (2017), 169–183, DOI 10.1016/j.jcta.2016.09.008.
2. Drew Armstrong, *Hyperplane arrangements and diagonal harmonics*, J. Comb. **4** (2013), no. 2, 157–190, DOI 10.4310/JOC.2013.v4.n2.a2.
3. Drew Armstrong and Brendon Rhoades, *The Shi arrangement and the Ish arrangement*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1509–1528, DOI 10.1090/S0002-9947-2011-05521-2.
4. Olivier Bernardi, *Deformations of the braid arrangement and trees*, Adv. Math. **335** (2018), 466–518, DOI 10.1016/j.aim.2018.07.020.
5. Ira M. Gessel, Sean Griffin, and Vasu Tewari, *Schur positivity and labeled binary trees*, Adv. Math. **356** (2019), no. 7, DOI 10.1016/j.aim.2019.106814.

6. Emily Leven, Brendon Rhoades, and Andrew Timothy Wilson, *Bijections for the Shi and Ish arrangements*, European J. Combin. **39** (2014), 1–23, DOI 10.1016/j.ejc.2013.12.001.
7. Alexander Postnikov and Richard P. Stanley, *Deformations of Coxeter hyperplane arrangements*, J. Combin. Theory, Series A **91** (2000), no. 1-2, 544–597, DOI 10.1006/jcta.2000.3106.
8. Jian-Yi Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Math., vol. 1179, Springer-Verlag, 1986.
9. Richard P. Stanley, *An introduction to hyperplane arrangements*, Geometric Combinatoricss (E. Miller, V. Reiner, and B. Sturmfels, eds.), IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence RI, 2007, pp. 389–496.
10. ———, *Enumerative combinatorics, Volume I*, 2 ed., Cambridge Stud. Adv. Math., no. 49, Cambridge University Press, Cambridge, 2011.

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