Bases for Quotients of Symmetric Polynomials

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Basic Ideas

Definition

R is a commutative ring with unity.

Definition

Ring of polynomials:

$$R[x] = \left\{ r_0 + r_1 x + \cdots + r_d x^d \;\middle|\; r_j \in R, r_d \neq 0 \right\}$$

d is the degree of the polynomial.

Example

- $2 - x + x^2 \in \mathbb{Z}[x]$, degree 2
- $\pi + 2x^2 - ix^5 \in \mathbb{C}[x]$, degree 5

More Indeterminates

Definition

k indeterminates $x_1, \ldots, x_k$:

$$R[x_1, \ldots, x_k] = R[x_1] \cdots [x_k] = \left\{ \sum_{\substack{\text{Finitely many terms} \\ j_1, \ldots, j_k \geq 0}} r_{j_1, \ldots, j_k} x_1^{j_1} \cdots x_k^{j_k} \,\middle|\, r_{j_1, \ldots, j_k} \in R \right\}$$

$\max(j_1 + \cdots + j_k \mid r_{j_1, \ldots, j_k} \neq 0)$ is the degree of the polynomial.

Example

- $x_1 x_2 + x_1 x_2 x_3 + 2x_1^5 x_3^2 \in \mathbb{Z}[x_1, x_2, x_3]$, degree 7
- $\pi + 2x_1^2 - i x_1 x_2 + x_4^3 \in \mathbb{C}[x_1, x_2, x_3, x_4]$, degree 3

Symmetric Polynomials

Definition

$\boldsymbol{S}$ is the subset of $R[x_1, \ldots, x_k]$ of polynomials that remain unchanged when indeterminates are permuted.

Example

If $k = 2$, then

$$x_1 + x_2 \in \boldsymbol{S}$$

since $x_2 + x_1 = x_1 + x_2$.

Example

If $k = 3$, then

$$x_1 + x_2 \notin \boldsymbol{S}$$

since $x_2 + x_3 \neq x_1 + x_2$, but

$$x_1 + x_2 + x_3 \in \boldsymbol{S}$$

Partitions

Definition

A **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is a decreasing sequence of positive integers, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$. The **Young diagram** of λ is the left-aligned grid of boxes with λ_i boxes in the ith row.

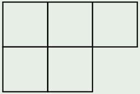
$\mathbf{Par}_{\boldsymbol{k}}$ is the set of partitions with $\ell(\lambda) \leq k$. $\mathbf{Par}_{\boldsymbol{k,n-k}}$ is the set of partitions whose Young diagram fits inside of box of height k and length $n-k$.

The conjugate of λ, $\boldsymbol{\lambda'}$, is the partition whose Young diagram is the reflection of the Young diagram of λ across the main diagonal.

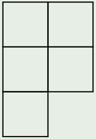
Example

Let $\lambda = (3,2)$. Then $\lambda \in \mathrm{Par}_2$, $\lambda \in \mathrm{Par}_{2,3}$, $\lambda \notin \mathrm{Par}_{2,2}$, $\lambda' = (2,2,1)$.

λ :



λ' :



Note that $\lambda' \in \mathrm{Par}_k \iff \lambda_1 \leq k$.

Homogeneous Symmetric Polynomials

Definition

$$h_i = \sum_{\substack{j_1+\cdots+j_k=i \\ j_1,\ldots,j_k \geq 0}} x_1^{j_1} \cdots x_k^{j_k}$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$$

Example

If $k = 2$:

$$h_0 = 1$$

$$h_3 = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$$

$$h_{(2,1)} = h_2 h_1 = (x_1^2 + x_1 x_2 + x_2^2)(x_1 + x_2) = x_1^3 + 2x_1^2 x_2 + 2x_1 x_2^2 + x_2^3$$

Theorem (*Enumerative Combinatorics Vol. 2*)

$\{h_\lambda \mid \lambda' \in Par_k\}$ is a basis for $\boldsymbol{S}$ over R

Schur Polynomials

Definition

Let $\ell(\lambda) \leq k$. Then

$$s_\lambda = det(h_{\lambda_i + j - i})_{i,j=1}^{\ell(\lambda)}$$

Example

If $k = 2$:

$$\begin{aligned} s_{(2,1)} &= \begin{vmatrix} h_2 & h_0 \\ h_3 & h_1 \end{vmatrix} \\ &= h_2 h_1 - h_3 h_0 \\ &= (x_1^2 + x_1 x_2 + x_2^2)(x_1 + x_2) - (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) \\ &= x_1^2 x_2 + x_1 x_2^2 \end{aligned}$$

Theorem (*Enumerative Combinatorics Vol. 2*)

$\{s_\lambda \mid \lambda \in Par_k\}$ is a basis for $\boldsymbol{S}$ over R.

Motivation

- $R = \mathbb{Z}$
  Cohomology ring of the Grassmannian,

$$H^*(Gr(k,n)) \cong \boldsymbol{S}/\langle h_{n-k+1}, \ldots, h_n \rangle$$

- $R = \mathbb{Z}[q]$
  Quantum cohomology ring of the Grassmannian,

$$QH^*(Gr(k,n)) \cong \boldsymbol{S}/\langle h_{n-k+1}, \ldots, h_{n-1}, h_n + (-1)^k q \rangle$$

Theorem (*Postnikov*)

$$\{s_\lambda \mid \lambda \in Par_{k,n-k}\}$$

is a basis (over R) for both quotients; that is, every member of S can written uniquely as

$$\textit{some member of the ideal} + \sum c_\lambda s_\lambda, \quad c_\lambda \in R,\ \lambda \in Par_{k,n-k}$$

Quotients with h_i's

Theorem (Grinberg)

Let $a_i \in R$. Then

$$\{s_\lambda \mid \lambda \in Par_{k,n-k}\}$$

is a basis for

$$\boldsymbol{S}/\langle h_{n-k+1} - a_1, \ldots, h_n - a_k \rangle$$

Quotients with h_i's

Example

If $k = 2$, $n = 4$:

$$\{s_\emptyset, s_{(1)}, s_{(1,1)}, s_{(2)}, s_{(2,1)}, s_{(2,2)}\}$$
$$= \{1, x_1 + x_2, x_1x_2, x_1^2 + x_1x_2 + x_2^2, x_1^2x_2 + x_1x_2^2, x_1^2x_2^2\}$$

is a basis for

$$\boldsymbol{S}/\langle h_3 - a_1, h_4 - a_2\rangle$$
$$= \boldsymbol{S}/\langle x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 - a_1, x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 - a_2\rangle$$

For instance:

$$x_1^4 + x_2^4 = -(x_1 + x_2)(h_3 - a_1) + 2(h_4 - a_2) + 2a_2s_\emptyset - a_1s_{(1)}$$

Power Sums

Definition

$$p_i = x_1^i + \cdots + x_k^i$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$$

Example

If $k = 2$:

$$p_3 = x_1^3 + x_2^3$$

$$p_{(2,1)} = p_2 p_1 = (x_1^2 + x_2^2)(x_1 + x_2) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$$

Theorem (*Enumerative Combinatorics Vol. 2*)

If $\mathbb{Q} \subseteq R$*, then* $\{p_\lambda \mid \lambda' \in Par_k\}$ *is a basis for* $\boldsymbol{S}$ *over* R

Quotients with p_i's

Theorem (W)

Let $\mathbb{Q} \subseteq R$. Then

$$\{s_\lambda \mid \lambda \in Par_{k,n-k}\}$$

is a basis for

$$\boldsymbol{S}/\langle p_{n-k+1}, \ldots, p_n \rangle$$

Example

If $k = 2$, $n = 4$:

$$\{s_\emptyset, s_{(1)}, s_{(1,1)}, s_{(2)}, s_{(2,1)}, s_{(2,2)}\}$$

$$= \{1, x_1 + x_2, x_1x_2, x_1^2 + x_1x_2 + x_2^2, x_1^2x_2 + x_1x_2^2, x_1^2x_2^2\}$$

is a basis for

$$\boldsymbol{S}/\langle p_3, p_4 \rangle = \boldsymbol{S}/\langle x_1^3 + x_2^3, x_1^4 + x_1^4 \rangle$$

For instance:

$$x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 = (x_1 + x_2)p_3 + s_{(2,2)}$$

Future Directions

- $a_i \notin R$ for both h_i's and p_i's.
- Writing Pieri's rule in the basis of the quotients:

$$h_i s_\lambda = \sum_{\substack{\mu/\lambda \text{ has } i \text{ squares} \\ \text{across } i \text{ columns}}} s_\mu = \sum_{\mu \in \mathrm{Par}_{k,n-k}} c_{\lambda,\mu} s_\mu$$

- What is $\boldsymbol{S}$ mod other ideals of symmetric polynomials?
- Which other ideals of $\boldsymbol{S}$ give the same basis when modded out?
- s_λ and p_λ are related by representation theory; is this usable?

Thank You

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- My parents

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