Extractable Tree-Statistics from the Quasisymmetric Bernardi Polynomial

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Agenda

1. Introduction & Definitions
2. Findings
3. Consequences & Open Questions

Introduction & Definitions

Let $D = (V, A)$ be a directed graph.

Notation

Let $f : V \to \mathbb{N}$ be a coloring of the vertices.

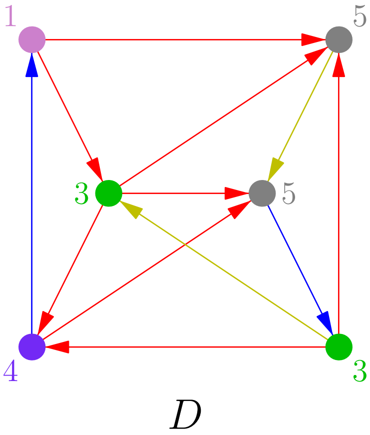
- *Ascents* are the elements of $f_A^> := \{(u, v) \in A | f(v) > f(u)\}$.
- *Descents* are the elements of $f_A^< := \{(u, v) \in A | f(v) < f(u)\}$.

The *Quasisymmetric Bernardi polynomial (QSBP)* is the formal power series

$$B_D(\boldsymbol{x}; y, z) = \sum_{f : V \to \mathbb{N}} \left(\prod_{v \in V} x_{f(v)} \right) y^{|f_A^>|} z^{|f_A^<|}$$

for indeterminates $(x_i)_{i \in \mathbb{N}}$.

- The QSBP is the generating function over all colorings counted by the number of ascents (y), descents (z), and uses of each color (x_i), respectively.
- Motivated by Richard Stanley's Tutte symmetric function.



$$
\begin{aligned}
B_D(\mathbf{x}; y, z) \\
&= B\left((x_1, x_2, \ldots); y, z\right) \\
&= \sum_{f:V\to\mathbb{N}} \left(\prod_{v\in V} x_{f(v)}\right) y^{|f_A^{>}|} z^{|f_A^{<}|} \\
&= \ldots + x_1 x_3^2 x_4 x_5^2 y^8 z^2 + \ldots
\end{aligned}
$$

Introduction & Definitions

Open Question (Stanley, 1995)

Does the Tutte symmetric function distinguish all non-isomorphic trees?

Open Question (Awan & Bernardi, 2018): Analogue for Digraphs

Does the QSBP distinguish all non-isomorphic rooted trees?

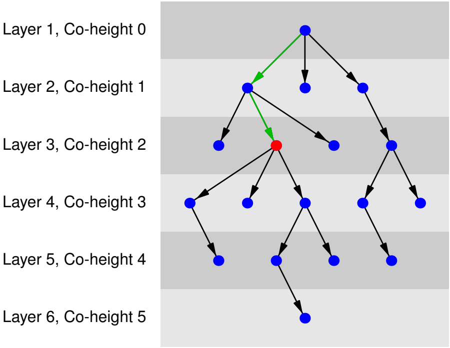
- We want to find information about rooted trees from their QSBP.

Definition

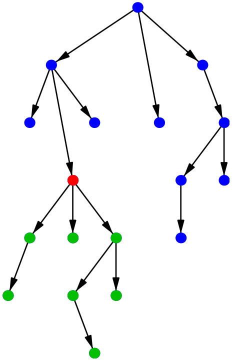
- A *tree-statistic* is a function on the set of rooted trees.
- Tree-statistic S is *extractable* if for all rooted trees T_1, T_2 where $B_{T_1}(\mathbf{x}; y, z) = B_{T_2}(\mathbf{x}; y, z)$, it follows that $S(T_1) = S(T_2)$.

- We want extractable tree-statistics S because if $S(T_1) \neq S(T_2)$, $B_{T_1}(\mathbf{x}; y, z) \neq B_{T_2}(\mathbf{x}; y, z)$.

Introduction & Definitions



$h_v = 2$



$w_v = |S_v| = 8$

Introduction & Definitions

Definition

- Rooted tree $T = (V, A)$, vertex $v \in V$, given $(a_u)_{u \in V}$ a vertex-statistic, we define P_v^a to denote the multiset

$$\{a_u \mid u \in S_v\}$$

- P_T^a means $P_{v_T}^a$, where v_T is the root of T.

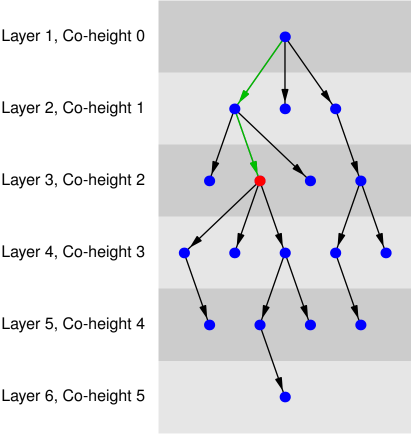
Definition

- *Co-height profile* of a tree T is P_T^h,
- *Weight profile* is P_T^w.

Definition

We say e.g., the *co-height profile profile* of a tree T is $P_T^{P^h}$.

Introduction & Definitions



$h_v = 2$

$$P_v^h = \{2,3,3,3,4,4,4,5\}$$

$$P_T^h = \{0,1,1,1,2,2,2,2,3,3,3,3,3,4,4,4,4,5\}$$

$$P_v^{P^h} = \{\{2,3,3,3,4,4,4,5\}, \{3,4\},\{3\},\{3,4,4,5\}, \{4\},\{4,5\},\{4\}, \{5\}\}$$

$$P_T^{P^h} = \{\{0,1,1,1,2,2,2,2,3,3,3,3,3,4,4,4,4,5\}, \{1,2,2,2,3,3,3,4,4,4,5\},\{1\},\{1,2,3,3,4\}, \{2\},\{2,3,3,3,4,4,4,5\},\{2\},\{2,3,3,4\}, \{3,4\},\{3\},\{3,4,4,5\},\{3,4\},\{3\}, \{4\},\{4,5\},\{4\},\{4\} \{5\}\}$$

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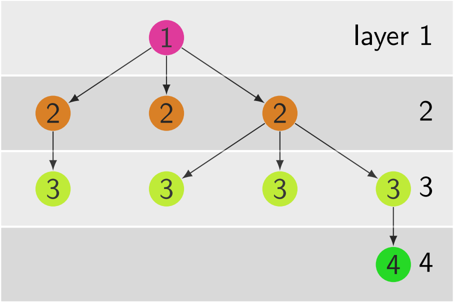
Findings

Theorem

For a rooted tree T, the coheight profile P_T^h is extractable.

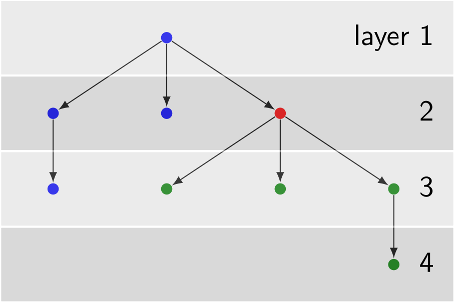


$$x_1^1 x_2^3 x_3^4 x_4^1 y^{|A|} z^0 \longrightarrow P_T^h = \{\underbrace{0}_{1}, \underbrace{1, 1, 1}_{3}, \underbrace{2, 2, 2, 2}_{4}, \underbrace{3}_{1}\}$$

Findings

Theorem

The coheight profile profile $P_T^{P^h}$ is extractable.

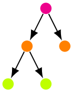


Coheight profile of v: $P_v^h = \{1, 2, 2, 2, 3\}$

Coheight profile profile: $P_T^{P^h} = \{\{0, 1, 1, 1, 2, 2, 2, 2, 3\}, \{1, 2\}, \{1\}, \{1, 2, 2, 2, 3\}, \{2\}, \{2\}, \{2\}, \{2, 3\}, \{3\}\}$

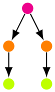
Findings

T_1



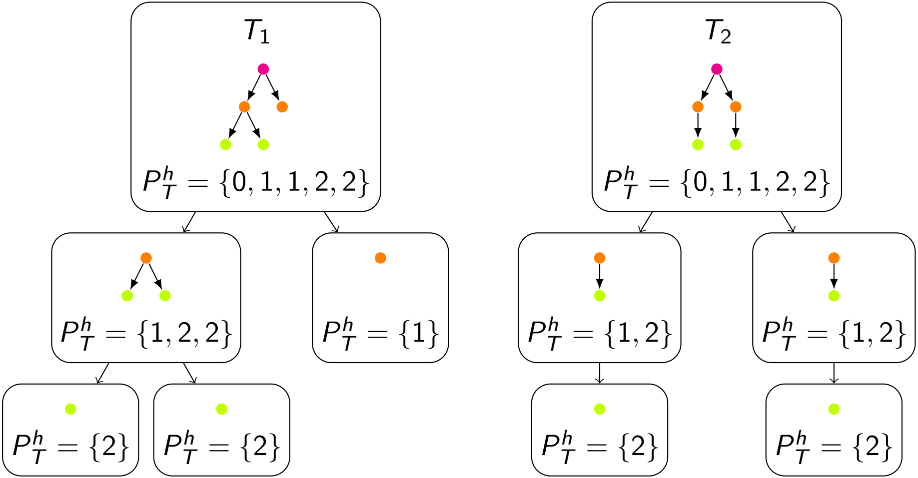
$P_T^h = \{0, 1, 1, 2, 2\}$

T_2



$P_T^h = \{0, 1, 1, 2, 2\}$

Findings


T_1
$P_T^h = \{0, 1, 1, 2, 2\}$
$P_T^h = \{1, 2, 2\}$
$P_T^h = \{1\}$
$P_T^h = \{2\}$
$P_T^h = \{2\}$
T_2
$P_T^h = \{0, 1, 1, 2, 2\}$
$P_T^h = \{1, 2\}$
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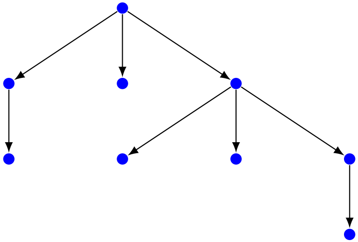
Consequences

Corollary

We can extract:

1. the number of leaves in each layer.
2. the outdegree distribution of each layer.
3. the *height profile* of each layer.
4. the *weight profile* of each layer.



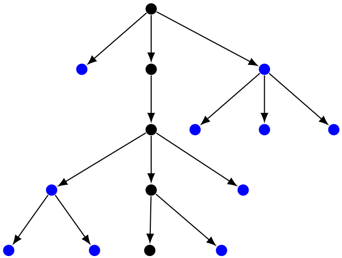
Consequences: 2-Caterpillars

Definition

An *n-caterpillar* tree is a rooted directed tree in which all vertices are at most distance n from a central path.

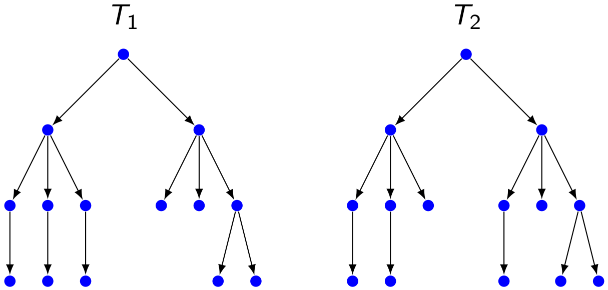
Corollary

We can distinguish between all 2-caterpillar trees.



Indistinguishable Trees

The previous theorem cannot distinguish these two trees:



By computer evidence, they do not have the same QSBP, but they have the same coheight profile profiles.

Open Questions

Main Question

Does the QSBP distinguish between all rooted directed trees?

1. Does the QSBP distinguish between all rooted directed trees with 4 layers?
2. For what $n > 2$ can the QSBP distinguish between all n-caterpillars?
3. Under what conditions will two trees T_1, T_2 share the same coheight profile profile?

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