

Critical loci of convex domains in the plane

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Abstract

We discuss critical loci of convex symmetric domains in the plane and give a construction which, in particular, furnishes examples of domains having critical locus of fractional Hausdorff dimension.

1 Introduction

Counting lattice points in convex symmetric domains is a classical problem, which dates back to Gauss and belongs to *geometry of numbers*, the branch of number theory that studies number-theoretical problems by the use of geometric methods. Geometry of numbers in its proper sense was pioneered by Minkowski, see [Mi] or the books [Ca, GL] for a comprehensive introduction to the subject.

A *lattice* Λ in \mathbb{R}^n is the set of all integer linear combinations of n linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$; in other words

$$\Lambda = \mathbb{Z}\mathbf{v}_1 + \dots + \mathbb{Z}\mathbf{v}_n = g\mathbb{Z}^n, \quad (1.1)$$

where $g \in \mathrm{GL}_n(\mathbb{Z})$ is the matrix with column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$. The most natural example to consider is the grid of points $\mathbb{Z}^n \subset \mathbb{R}^n$, which is generated by the standard basis of \mathbb{R}^n and corresponds to $g = I_n$, the $n \times n$ identity matrix. Note that $g\mathbb{Z}^n = \mathbb{Z}^n$ if and only if $g \in \mathrm{GL}_n(\mathbb{Z})$, where the latter stands for the group of invertible $n \times n$ matrices with integer entries. Consequently, the space X_n of lattices in \mathbb{R}^n is isomorphic, as a $\mathrm{GL}_n(\mathbb{R})$ -space, to the quotient space $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$.

For any $\Lambda \in X_n$ of the form (1.1) we define its *covolume* $d(\Lambda)$ as the volume of the parallelepiped spanned by $\mathbf{v}_1, \dots, \mathbf{v}_n$. Clearly it does not depend on the choice of the generating set and is equal to the absolute value of the determinant of g .

Let now K be a bounded convex domain in \mathbb{R}^n symmetric about the origin; denote by $V(K)$ the volume of K .

Definition 1.1. A lattice $\Lambda \subset \mathbb{R}^n$ is called *K-admissible* if $\Lambda \cap K = \{0\}$. To K as above we associate a real number $\Delta(K)$, called the *critical determinant* of K , given by

$$\Delta(K) := \inf \{d(\Lambda) : \Lambda \text{ is } K\text{-admissible}\} \quad (1.2)$$

Perhaps one the most fundamental results in geometry of numbers is Minkowski's Convex Body Theorem, see e.g. [Ca, §III.2.2], which states that for K as above, any lattice in \mathbb{R}^n with covolume less than $V(K)/2^n$ must contain a point of K distinct from 0. In other words, for any such K the critical determinant $\Delta(K)$ is positive, and, moreover, one has

$$\Delta(K) \geq \frac{V(K)}{2^n}.$$

This motivates the problem of exhibiting K -admissible lattices with the smallest covolume; those are called *K-critical*.

Definition 1.2. A lattice Λ in \mathbb{R}^n is called *K-critical* if it is a *K*-admissible lattice with $d(\Lambda) = \Delta(K)$. The set of *K*-critical lattices is denoted by $\mathcal{L}(K)$ and is called the *critical locus* of *K*.

Since admissibility is preserved by taking limits, this set is non-empty by a sequential compactness argument due to Mahler, which can be found e.g. in [Ca, §V.4.2]. The critical locus $\mathcal{L}(K)$ can be thought of as a subset of X_n , and we will endow it with the topology induced from $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$. With this topology it is compact, again by Mahler's argument.

It is worthwhile to point out that *K*-critical lattices are in one-to-one correspondence with the densest lattice packings of \mathbb{R}^n by translates of *K*. Indeed, suppose that Λ is *K*-admissible. Then it is easy to see that the collection of sets

$$\{2\mathbf{v} + K : \mathbf{v} \in \Lambda\}$$

is pairwise disjoint. And, conversely, if such a collection of sets is pairwise disjoint for some Λ , then Λ must be *K*-admissible. Minimizing the covolume of Λ over all admissible lattices thus corresponds to maximizing the relative area covered by the collection of the above sets.

Another motivation for studying critical loci of convex symmetric domains comes from Diophantine approximation. Take $k, \ell \in \mathbb{N}$ and let $n = k + \ell$. One says that a $k \times \ell$ matrix A is *Dirichlet-improvable* if there exists $c < 1$ such that the system of inequalities

$$\|A\mathbf{q} - \mathbf{p}\| \leq \frac{c}{T^{\ell/k}} \quad \text{and} \quad \|\mathbf{q}\| \leq T \tag{1.3}$$

has a nonzero solution $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^k \times \mathbb{Z}^\ell$ for all sufficiently large T (here $\|\cdot\|$ stands for the supremum norm on \mathbb{R}^k and \mathbb{R}^ℓ). Note that Dirichlet's Theorem, see [S, Theorem II.1E], asserts that the system (1.3) always has a nonzero integer solution if $c = 1$. On the other hand, the set of Dirichlet-improvable matrices has Lebesgue measure zero [DS]. It was essentially shown in [DS], see [KWe, KWa] for generalizations, that A is Dirichlet-improvable if and only if a certain one-parameter trajectory of the lattice

$$\begin{pmatrix} I_k & A \\ 0 & I_\ell \end{pmatrix} \mathbb{Z}^n \in X_n$$

stays away from the critical locus $\mathcal{L}(K)$, where K is the unit ball with respect to the supremum norm on \mathbb{R}^n . In this case a precise description of $\mathcal{L}(K)$ is given by the Hajós-Minkowski Theorem [F, Chapter 3]. See [KR, KY] for more recent work on the subject.

From now on let us restrict our attention to the case $n = 2$. Critical loci of planar domains have been systematically studied by Mahler in a series of papers written in the 1940s, see Section 2 for a list of examples. However all examples of sets $\mathcal{L}(K)$ previously constructed by Mahler and others were either finite sets or finite unions of closed curves, and there does not exist a precise description of compact subsets of X_2 which can arise as critical loci of convex symmetric bounded domains of \mathbb{R}^2 . (The situation for $n > 2$ is even further from being understood.) For instance, after studying Examples 2.2–2.5 below, one may pose the following natural question:

Is it possible for some K to have critical locus homeomorphic to a Cantor set?

We give a construction that shows that this is indeed possible (see Theorem 1.7 below). The approach is to start with an irreducible domain K (Definition 1.3 below) which is not a parallelogram, and then perturb it suitably. The concept of irreducibility, introduced by Mahler in [Ma1], is as follows:

Definition 1.3. A convex symmetric bounded domain K in \mathbb{R}^2 is said to be *irreducible* if each convex symmetric bounded domain $H \subsetneq K$ has $\Delta(H) < \Delta(K)$. K is said to be *reducible* if it is not irreducible, that is, if there exists $H \subsetneq K$ with $\Delta(H) = \Delta(K)$.

Here and hereafter H, K will always denote convex symmetric bounded domains in \mathbb{R}^2 . We remark that Mahler actually defines a stronger notion of irreducibility; namely, according to his definition K is irreducible if all bounded symmetric *star domains* $S \subsetneq K$ have $\Delta(S) < \Delta(K)$. However it is actually equivalent to the one given in Definition 1.3; this follows from [Ma2, Theorem 1], see a footnote to Lemma 2.8 below.

Among the examples in Section 2 below, the square, disc and their linear transforms are irreducible. Another example of an irreducible domain is given in [Ma3, Section 12], see (3.3). Mahler studied these extremal domains and proved a number of results. For example, the theorem stated below shows that irreducible domains are ubiquitous in the following sense:

Theorem 1.4 ([Ma2], Theorem 1). *Every $K \subset \mathbb{R}^2$ contains an irreducible H with $\Delta(H) = \Delta(K)$.*

In Section 2, capitalizing on this and other results of Mahler, we prove

Theorem 1.5. *If K is irreducible but not a parallelogram, then there is a continuous map $\phi : [0, 1] \rightarrow X_2$ which descends to a homeomorphism of the circle with $\mathcal{L}(K)$.*

In the case of a parallelogram, as made explicit in Example 2.3, the critical locus is topologically the wedge of two circles.

Corollary 1.6. *If K is not a parallelogram, $\mathcal{L}(K)$ is a closed subset of an embedded circle $\mathcal{L}(H)$, the critical locus of some irreducible non-parallelogram H .*

This simple corollary of Theorems 1.4 and 1.5 is proved in Section 2, and our main theorem is a partial converse to it:

Theorem 1.7. *If $H \subset \mathbb{R}^2$ is strictly convex¹ and irreducible, so that $\mathcal{L}(H)$ is an embedded circle inside X_2 , then each non-empty closed subset of $\mathcal{L}(H)$ is the critical locus of some domain $K \supset H$.*

In particular, one can choose a closed subset of $\mathcal{L}(H)$ homeomorphic to the middle-third Cantor set, or to any other closed fractal subset of $[0, 1]$.

Note the conclusion of the above theorem does not hold for parallelograms. Indeed, if H is a parallelogram, one can consider any subset of $\mathcal{L}(H)$ homeomorphic to the transverse intersection of two line segments; clearly it could never be embedded in a circle, and therefore, as Corollary 1.6 would imply, could not be a critical locus of any K . At the end of Section 3 we discuss the further possibility of the strict convexity assumption being removed for non-parallelogram irreducible domains.

As for the rest of the article, Section 2 gives the basic theorems on irreducible domains and critical lattices and examples to motivate the study, and Section 3 contains the proof of our main result.

2 Preliminaries and examples

The following fundamental and intuitive theorem on admissible and critical lattices is taken from [Ca, §V.8.3] where one may also find a formal proof. It shows that critical lattices are realised by inscribed parallelograms in the domains.

¹The strict convexity of H means that the line joining two distinct points on ∂H cannot intersect ∂H at any other points.

Theorem 2.1. *Let Λ be K -critical, and let C be the boundary of K . Then one can find three pairs of points $\pm \mathbf{p}_1, \pm \mathbf{p}_2, \pm \mathbf{p}_3$ of the lattice on C . Moreover these three points can be chosen such that*

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 \quad (2.1)$$

and any two vectors among $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ form a basis of Λ .

Conversely, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ satisfying (2.1) are on C , then the lattice generated by \mathbf{p}_1 and \mathbf{p}_2 is K -admissible. Furthermore no additional (excluding the six above) point \mathbf{p}_4 of Λ is on C unless K is a parallelogram.

In light of this theorem, we may discuss the critical loci for the following examples of domains $K \subset \mathbb{R}^2$.

Example 2.2. When K is the unit disc, one has that $\Delta(K) = \sqrt{3}/2$, and the set of critical lattices is given by

$$\left\{ k \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \mathbb{Z}^2 : k \in \text{SO}(2) \right\}, \quad (2.2)$$

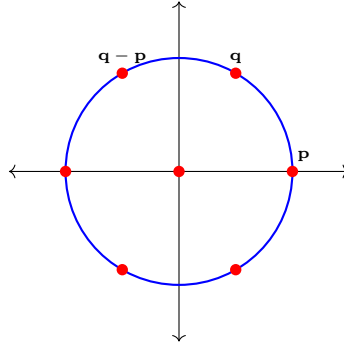
which is homeomorphic to a circle.

More precisely, the map $\phi : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{R}) / \text{GL}_2(\mathbb{Z})$ of the form

$$\phi(t) := \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix} \mathbb{Z}^2$$

descends to a homeomorphism $\mathbb{R}/(\frac{\pi\mathbb{Z}}{3}) \simeq \mathcal{L}(K)$.

This is also an example of an irreducible domain; For if H is a subset of K with $\Delta(H) = \Delta(K)$, then all K -critical lattices are H -admissible and thus also H -critical. Then the first part of Theorem 2.1 shows that each K -critical lattice $\phi(t)$ above must contain three pairs of points on ∂H . Since $H \subset K$, those three pairs of points must coincide exactly with the set $\phi(t) \cap \partial K$, thus showing that the boundaries of H and K coincide.



(2.3)

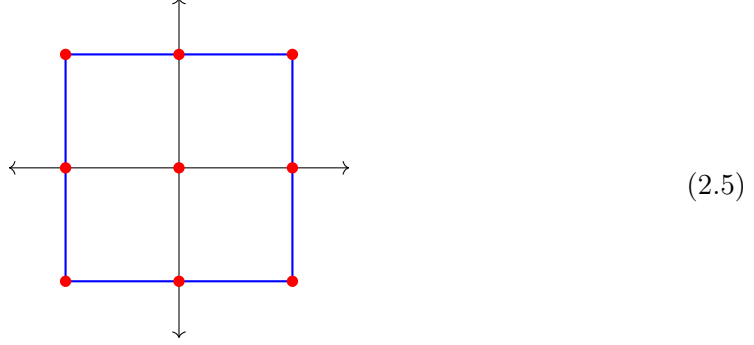
Points \mathbf{p} and \mathbf{q} generate a critical lattice.
All other critical lattices are obtained by rotating this one.

Example 2.3. When K is the square with side-length 2 and sides perpendicular to the coordinate axes, Theorem 2.1 again shows that the critical lattices are given by

$$\left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2 : t \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \mathbb{Z}^2 : t \in \mathbb{R} \right\}, \quad (2.4)$$

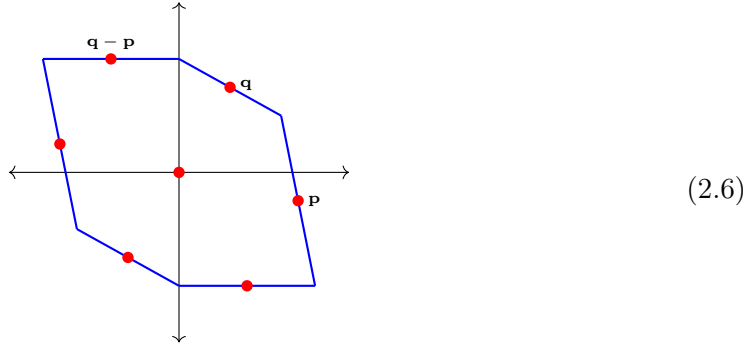
which is topologically a wedge of two circles. In this case too, K is irreducible by the same

application of Theorem 2.1 as in the case of the disc.



Shearing the standard lattice along each axis gives rise to all the critical lattices.

Example 2.4. When K is a hexagon, there is exactly one critical lattice. This lattice is spanned by the two vectors on the midpoints of adjacent sides. Moreover, in this case one has $V(K) = 4\Delta(K)$. This follows as a corollary of Theorem 2.1, and a proof can be found in [Ca, §V.8.4]. It is also known [Ma3, Theorem 5] that when K is a $2n$ -gon with $n \geq 3$, there can only be a finite number of critical lattices.



The lattice generated by \mathbf{p} and \mathbf{q} is the only critical lattice.

Example 2.5. If K is the unit ball of the L^p norm on \mathbb{R}^2 ($1 < p < \infty$, $p \neq 2$), then, depending on p , the critical locus comprises either of one or two lattices. For a historical account and a proof of this (Minkowski's conjecture), see the paper [GGM].

Remark 2.6. If $H = gK$ for some $g \in \text{GL}_2(\mathbb{R})$, then the critical loci of H and K are related by $\mathcal{L}(H) = g\mathcal{L}(K)$. Also, irreducibility is preserved under transformation by g .

We now collect the results needed to make Theorem 1.5 more precise. Moreover, the parameterization of the critical locus arising will be crucial to our result.

Lemma 2.7 ([Ma2], Lemma 6). *Suppose that K is not a parallelogram, Λ is K -critical, and let $\mathbf{p}_i : i = 1, \dots, 6$ be the points of Λ contained in $C = \partial K$, labelled in a counter-clockwise order. Let A_i denote open segment of C between \mathbf{p}_i and \mathbf{p}_{i+1} . If Λ' is another K -critical lattice distinct from Λ , then each A_i contains exactly one point of Λ' .*

Lemma 2.8 ([Ma2], Lemma 9²). *Assume K is not a parallelogram and is irreducible. Then for each $\mathbf{p} \in C = \partial K$ there is exactly one critical lattice of K containing \mathbf{p} .*

Proof of Theorem 1.5. Fix Λ , any critical lattice for K , and let C denote the boundary of K . Let \mathbf{p}_i denote the six points of $C \cap \Lambda$ oriented counter-clockwise. By convexity,

²Even though the statement of Lemma 2.8 is taken almost verbatim from Mahler's paper, our meaning is changed in light of the different definitions of irreducibility. Logically speaking, one can use [Ma2, Theorem 1] to show that the two definitions of irreducibility are equivalent.

one can parameterize the closed segment of C between \mathbf{p}_1 and \mathbf{p}_2 by a continuous map $\mathbf{p} : [0, 1] \rightarrow \mathbb{R}^2$.

From Lemma 2.8 above, we see that each $\mathbf{p}(t)$ belongs to a unique K -critical lattice Λ_t . Let $\mathbf{q}(t)$ denote the point of $\Lambda_t \cap C$ coming after $\mathbf{p}(t)$ going counter-clockwise in angle. We claim that the function \mathbf{q} is continuous. If not, then for some t we would find a neighborhood U of $\mathbf{q}(t)$ with a converging sequence $t_n \rightarrow t$ but $\mathbf{q}(t_n) \notin U$. Continuity of \mathbf{p} implies $\mathbf{p}(t_n) \rightarrow \mathbf{p}(t)$. Without loss of generality, we can assume $t_n < t$ so that applying Lemma 2.7 to a lattice Λ containing a point in $U \cap C$ between $\mathbf{q}(t)$ and each $\mathbf{q}(t_n)$ gives a contradiction. Thus we can set

$$\phi(t) = [\mathbf{p}(t) \ \mathbf{q}(t)] \mathbb{Z}^2,$$

which, in light of Lemma 2.7, descends to the required homeomorphism. \square

Remark 2.9. Mahler proved (cf. Theorem 3 in [Ma3]) that for such K , the boundary C is a \mathcal{C}^1 submanifold of \mathbb{R}^2 . Thus the preceding proof can be modified to show that $\mathcal{L}(K)$ is a \mathcal{C}^1 submanifold of the space of lattices.

Proof of Corollary 1.6. Suppose that K is not a parallelogram. By Theorem 1.4, K contains an irreducible H with $\Delta(K) = \Delta(H)$. It follows that we have the containment $\mathcal{L}(K) \subset \mathcal{L}(H)$. Moreover we claim that H cannot be a parallelogram.

For if this were the case, as made explicit in Example 2.3 above, $\Delta(H)$ would have to equal $\frac{V(H)}{4}$. However, we also have that $V(H) < V(K)$ and that $\frac{V(K)}{4} \leq \Delta(K)$. Together these give

$$\Delta(H) = \frac{V(H)}{4} < \frac{V(K)}{4} \leq \Delta(K),$$

a contradiction.

Since H is not a parallelogram, Theorem 1.5 applies and we are done. \square

3 Proof of Theorem 1.7

The proof of our theorem will be based on the following simple observation:

Lemma 3.1. *If $H \subset K$ and one of the H -critical lattices is also K -admissible, then $\mathcal{L}(K)$ is exactly the set of K -admissible lattices in $\mathcal{L}(H)$.*

Proof. Since $H \subset K$, we have that $\Delta(H) \leq \Delta(K)$. Further, the existence of an H -critical and K -admissible lattice shows that we have the equality $\Delta(H) = \Delta(K)$.

First, let Λ be K -critical. It is then also H -admissible with $d(\Lambda) = \Delta(K) = \Delta(H)$. Thus it is H -critical and, by definition, also K -admissible.

For the other containment, let $\Lambda \in \mathcal{L}(H)$ be K -admissible. Since $\Delta(H) = \Delta(K)$, Λ must in fact be K -critical. \square

Proof of Theorem 1.7. Take H , not a parallelogram and irreducible, and say $Z \subset \mathcal{L}(H)$ is non-empty and compact. Let $\Lambda \in Z$. Let C denote the boundary of H , and, as before, let us label the six points of $\Lambda \cap C$ as $\mathbf{p}_1, \dots, \mathbf{p}_6$ (ordered by angle). Let $\mathbf{p}(t)$ denote a continuous parameterization of the segment C from \mathbf{p}_1 to \mathbf{p}_2 by the interval $[0, 1]$ (see diagram (3.2) below), and let $\mathbf{q}(t)$ denote the point of Λ coming after $\mathbf{p}(t)$ (going counter-clockwise).

We use the parameterization $\phi(t) := [\mathbf{p}(t) \ \mathbf{q}(t)] \mathbb{Z}^2$ for $\mathcal{L}(H)$. We can now assume that our non-empty, closed set is the image under ϕ of some compact $Q \subset [0, 1]$ with $\{0, 1\} \subset Q$. Set

$$[0, 1] \setminus Q = \bigsqcup (a_i, b_i). \tag{3.1}$$

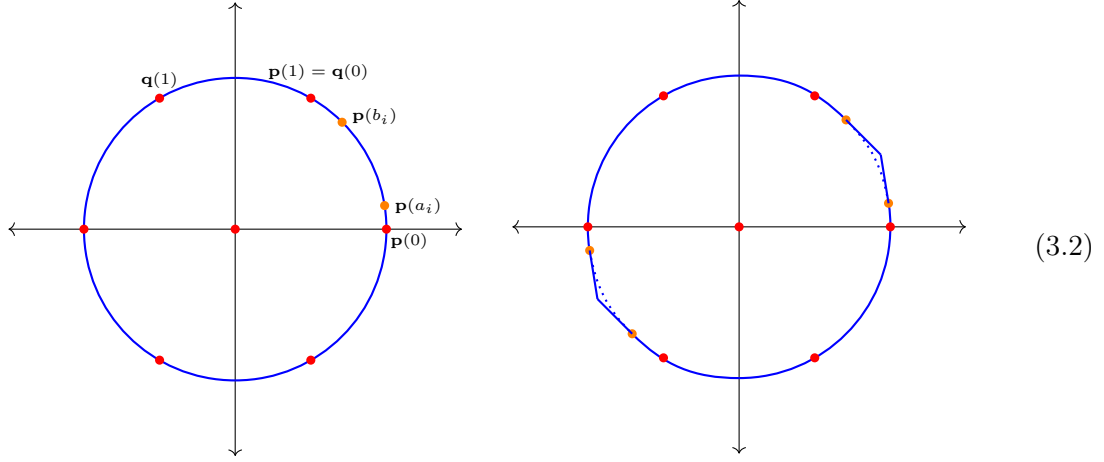
Now, for each i , consider $\mathbf{p}(a_i)$, $\mathbf{p}(b_i)$. In light of Remark 2.9, we can find tangent lines L_1, L_2 to C at these two points. Let T_i be the curvilinear triangle bounded by the three curves L_1, L_2 and $C_i = \{\mathbf{p}(t) : a_i \leq t \leq b_i\}$. We define K_i to be the union $H \cup T_i \cup (-T_i)$ as is illustrated in Diagram (3.2) below. Note that the lines L_1, L_2 are not parallel; one argument is that $-L_1$ is also a tangent line to C by symmetry and strict convexity would prevent L_2 from being parallel to either of these. Moreover, strict convexity also implies that the curve segment C_i is contained in K_i .

We now define K to be the union $\bigcup K_i$. Clearly K is a bounded convex symmetric domain containing H . Moreover, K and H share the boundary points

$$\mathbf{p}(t), \mathbf{q}(t), \mathbf{q}(t) - \mathbf{p}(t)$$

for each $t \in Q$, which shows that the H -critical lattice $\phi(0)$ is K -admissible.

Lemma 3.1 thus applies to show that $\mathcal{L}(K)$ is exactly the set of K -admissible lattices of $\mathcal{L}(H)$. For any $t \in Q$, the H -critical lattice $\phi(t)$ is K -admissible by Theorem 2.1. On the other hand, for $t \notin Q$, $\phi(t)$ is not K -admissible since $C_i \subset K$. This ends the proof. \square

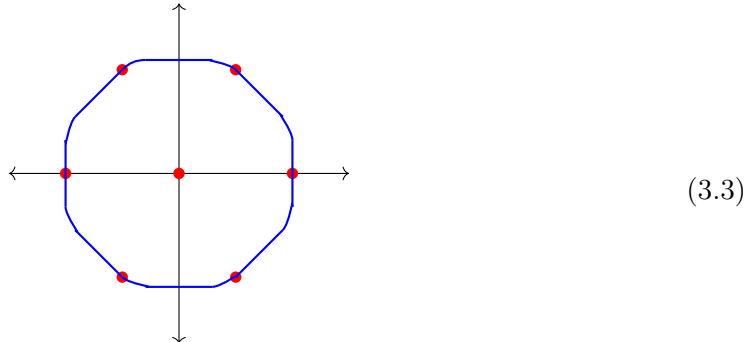


An illustration when H is a disc.

The union of H and two curvilinear regions.

Remark 3.2. Note the construction of above can be modified to ensure that K is \mathcal{C}^1 by smoothing a vertex of each curvilinear triangle.

Our method does not work for non-parallelogram H which are not strictly convex, and it even seems unlikely that the theorem holds in that generality. The interested reader is encouraged to examine the irreducible domain described by Mahler in [Ma3, Section 12].



Mahler's curvilinear octagon with one of its critical lattices.

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