On the degenerate Turán problem and its variants

Sunay Joshi

Abstract

Given a family $\mathcal{F}$ of graphs, a central problem in extremal graph theory is to determine the maximum number $\mathrm{ex}(n, \mathcal{F})$ of edges in a graph on n vertices that does not contain any member of $\mathcal{F}$ as a subgraph. The degenerate Turán problem regards the asymptotic behavior of $\mathrm{ex}(n, \mathcal{F})$ for familes $\mathcal{F}$ of bipartite graphs. In this paper, we prove four new theorems regarding the extremal number and its variants. We begin by investigating several notions central to providing lower bounds on extremal numbers, including balanced rooted graphs and the Erdős–Simonovits Reduction Theorem. In addition, we present new lower bounds on the asymmetric extremal number $\mathrm{ex}(m, n, F)$ and the lopsided asymmetric extremal number $\mathrm{ex}^*(m, n, F)$ when F is a blowup of a bipartite graph or a theta graph.

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1 Introduction

Given a graph F, a central problem in extremal graph theory is to determine the maximum number $\mathrm{ex}(n, F)$ of edges in an n-vertex graph that does not contain F as a subgraph. The *Turán problem* regards the asymptotic behavior of the *extremal number* $\mathrm{ex}(n, F)$ as a function of n.

One of the earliest results on the Turán problem was proven in 1907, when Mantel [Man07] showed that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$, where K_r denotes the complete graph on r vertices. (See Figure 1 for an illustration of this fact.) In 1941, Turán generalized Mantel's result in [Tur41] by showing that $\mathrm{ex}(n, K_{r+1}) = \left\lfloor \frac{r-1}{r} \cdot \frac{n^2}{2} \right\rfloor$.

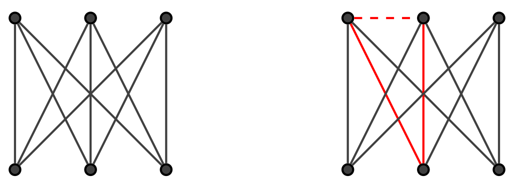

Figure 1: K_3-free graph with $\mathrm{ex}(6, K_3) = 9$ edges (left). K_3 (in red) is formed when an edge is added (right).

A major breakthrough in the study of the extremal number came in 1966, with the proof of the famous Erdős–Stone–Simonovits Theorem [TS66, TS46]. The result states that for any graph F,

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right) \binom{n}{2} + o(n^2),$$

where $\chi(F)$ is the chromatic number of F.

When $\chi(F) > 2$, the first term of the Erdős–Stone–Simonovits formula yields the asymptotic behavior of $\mathrm{ex}(n, F)$ as a quadratic function of n. However, when $\chi(F) = 2$, that is, when F is bipartite, the formula only shows that $\mathrm{ex}(n, F)$ is sub-quadratic. This issue gave rise to the *degenerate Turán problem*, which regards the asymptotic behavior of $\mathrm{ex}(n, F)$ for bipartite F. For a comprehensive account of the degenerate Turán problem, see [FS13].

One key conjecture in the area, put forth by Erdős and Simonovits [Erd81], states that for every rational number $r \in [1, 2]$, there exists a graph F and a positive constant c such that $\mathrm{ex}(n, F) = (c + o(1))n^r$. The following is a weaker version of this conjecture, first stated in [BC18].

Conjecture A (Rational Turán exponents conjecture)**.** *For every rational number $r \in [1, 2]$, there exists a graph F with $\mathrm{ex}(n, F) = \Theta(n^r)$.*

Along the lines of Conjecture A, we say $r \in [1,2]$ is a *Turán exponent* if there exists a graph F with $\text{ex}(n,F) = \Theta(n^r)$. If we relax Conjecture A to allow for a finite family $\mathcal{F}$ of graphs instead of a single graph F, we arrive at the following result of Bukh and Conlon [BC18].

Theorem 1.1 (Bukh–Conlon)**.** *For every rational number $r \in [1,2]$, there exists a finite family $\mathcal{F}$ of graphs with $\text{ex}(n,\mathcal{F}) = \Theta(n^r)$, where $\text{ex}(n,\mathcal{F})$ denotes the maximum number of edges in an n-vertex graph that does not contain any member of $\mathcal{F}$ as a subgraph.*

Bukh and Conlon's construction of the family $\mathcal{F}$ corresponding to a given rational number r involves a number of novel notions. We say a *rooted graph* (F, R) consists of a graph F and a nonempty subset R of vertices, called roots. Bukh and Conlon utilize a special class of so-called balanced rooted graphs in their analysis.

In order to provide a better understanding of the above notions, we begin this paper by studying certain balanced rooted graphs. We say a rooted graph is *closed* if all of its leaves are roots. We provide a necessary and sufficient condition for a closed rooted graph to be balanced.

Following our discussion of balanced rooted graphs, we return to the study of Turán exponents. Using a reduction theorem of Erdős and Simonovits [ES70], we present a new proof of the Turán exponent of $2 - \frac{2}{2s+1}$ for $s \geq 2$, a result first established by Jiang, Ma, and Yepremyan [JMY18].

In recent years, the study of Turán exponents has been aided by studying certain variants of the extremal number. In 2018, Jiang, Ma, and Yepremyan utilized the asymmetric extremal number $\text{ex}(m,n,\mathcal{F})$ to prove the Turán exponent of 7/5 [JMY18]. The *asymmetric extremal number* $\text{ex}(m,n,F)$ of a graph F is defined as the maximum number of edges in an m-by-n bipartite graph that does not contain F. Regarding this variant, we prove an asymptotic lower bound on the asymmetric extremal number of certain graphs.

We continue this exploration of the asymmetric extremal number by using Bukh and Conlon's random algebraic method to derive a stronger lower bound on the asymmetric extremal number of certain theta graphs.

Lastly, we consider one final class of variants of the extremal number that we refer to as lopsided extremal numbers, introduced by Faudree and Simonovits in [FS83]. As a corollary to our lower bound on the asymmetric extremal number of certain theta graphs, we prove a corresponding lower bound on the lopsided asymmetric extremal number.

The rest of this paper is organized as follows. In Section 2, we study balanced closed rooted graphs. In Section 3, we present our short proof of the Turán Exponent of $2 - \frac{2}{2s+1}$. In Section 4, we present our general lower bound on the asymmetric extremal number. In Section 5, we present a sharper bound in the case of theta graphs, and in Section 6, we present the corresponding bound in the lopsided case.

Throughout the paper, we use the following notational conventions: given a graph G, $V(G)$ denotes its vertex set, $E(G)$ denotes its edge set, $v(G)$ and $e(G)$ denote the number of vertices and respectively edges in G, and given a subset of vertices $S \subseteq V(G)$, $|S|$ denotes the size of S.

2 Balanced Closed Rooted Graphs

In order to present our condition for a closed rooted graph to be balanced, we must clarify a few notions from the introduction.

Definition 2.1. Given a rooted graph (G, R), the *density* $\rho_G(S)$ of a subset $S \subseteq V(G) \setminus R$ is defined as $\frac{e_G(S)}{|S|}$, where $e_G(S)$ denotes the number of edges of G adjacent to at least one vertex of S. If $S = V(G) \setminus R$, we write ρ_G for $\rho_G(S)$. We say that (G,R) is *balanced* if for all nonempty subsets $S \subseteq V(G) \setminus R$, we have $\rho_G(S) \geq \rho_G$.

With these notions in place, we are ready to state and prove the following result.

Theorem 2.1. *Let (G,R) be a closed rooted graph. Then (G,R) is balanced if and only if every closed rooted subgraph of G has density at most that of G.*

Proof of Theorem 2.1. Notice that whether a rooted graph (G,R) is balanced does not depend on the edges between the roots. Without loss of generality, we may assume that there are no edges between the roots, or in other words, the R is an independent set of vertices in G. In particular, $\rho_G = e(G)/(v(G)-|R|)$.

We shall first prove the forward direction, and then we will prove the reverse statement.

Forward direction: Assume (G,R) is balanced. Let F be a closed rooted subgraph of G with roots $R' := V(F) \cap R$. Consider $S = V(G) \setminus (R \cup V(F))$. As G is balanced, $\rho_G(S) \geq \rho_G$, hence

$$\frac{e_G(S)}{v(G)-|R|-|V(F)\setminus R|} = \frac{e_G(S)}{|S|} \geq \frac{e(G)}{v(G)-|R|} \implies e_G(S) \geq e(G)\cdot\left(1-\frac{v(F)-|R'|}{v(G)-|R|}\right).$$

As R is an independent set, $e(G) = e(F) + e_G(S)$. Thus

$$e(G)-e(F) = e_G(S) \geq e(G)\cdot\left(1-\frac{v(F)-|R'|}{v(G)-|R|}\right) \implies \rho(F) = \frac{e(F)}{v(F)-|R'|} \leq \frac{e(G)}{v(G)-|R|} = \rho(G).$$

Backward direction: Assume (G,R) is a rooted graph such that every closed rooted subgraph has lower density than G. Assume, for the sake of contradiction, that S is the largest subset of $G \setminus R$ such that $\rho_G(S) < \rho_G$. Consider $F = G \setminus S$. First, we claim that F is a closed rooted subgraph of G. To see this, assume for the sake of contradiction that an unrooted $v \in F$ has "in-degree" 1 within F. Then, adding v to S yields $S' = S \cup \{v\}$ with density

$$\rho_G(S') = \frac{e_G(S)+1}{|S|+1}.$$

As G is closed, the "out-degree" of v must be positive. As a result, $e_G(S)$ must be strictly greater than half the sum of the degrees of all vertices in S. Again, since G is closed, each vertex in S must have degree at least 2, and it follows that $e_G(S) > |S|$. Hence, $\rho_G(S') < \rho_G(S) < \rho_G$, contradicting the maximality of $|S|$. It follows that F is closed.

Next, we claim that $\rho_F > \rho_G$. To see this, note that since S has density less than that of G,

$$\frac{e_G(S)}{|S|} < \frac{e(G)}{v(G)-|R|}.$$

As $e_G(S) + e(F) = e(G)$, the above rearranges to

$$\rho_G = \frac{e(G)}{v(G)-|R|} < \frac{e(F)}{v(G)-|R|-|S|} = \rho_F.$$

□

3 Extremal Numbers of Generalized Cubes

In order to present our simplified proof of the Turán Exponent of $2 - \frac{2}{2s+1}$, we need to state the following definition. Given a rooted graph (F,R), F^p denotes the graph consisting of the union of p distinct labelled copies of F, each of which agree on the set of roots R but are otherwise disjoint. Next, we need to state the following crucial lemma, known as the Erdős–Simonovits Reduction Theorem [ES70].

Lemma 3.1 (Erdős–Simonovits Reduction Theorem)**.** *Given a bipartite graph F with bipartition V_1 and V_2, let $\mathsf{L}(F)$ be the graph with two more vertices v_1, v_2 and all edges between v_1 and v_2, v_1 and V_2, v_2 and V_1. If* $\mathrm{ex}(n, F) = O(n^{2-1/\alpha})$ *for some $\alpha \in (1, 2)$, then* $\mathrm{ex}(n, \mathsf{L}(F)) = O(n^{2-1/(1+\alpha)})$.

We are ready to state and prove the following result.

Theorem 3.2 (Jiang–Ma–Yepremyan)**.** *For all $2 \le s \le p$,* $\mathrm{ex}(n, H_s^p) = O(n^{2-2/(2s+1)})$*, where H_s is the rooted graph formed when corresponding vertices of two copies of $K_{1,s}$ (the complete bipartite graph with 1 and s vertices on each part of the bipartition) are connected by a matching, and H_s is rooted at the leaves of the two copies of $K_{1,s}$.*

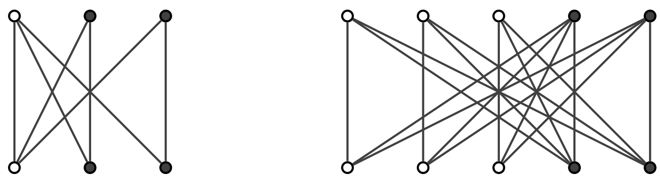

Figure 2: H_2 (left) and H_2^3 (right), where the vertices in black are the roots.

In Figure 2, we display the graphs H_2 and H_2^3.

Proof of Theorem 3.2. We proceed by induction on s.
Base case: For $s = 2$, consider the theta graph θ_3^p, and construct $\mathsf{L}(\theta_3^p)$ according to Lemma 3.1. By Lemma 3.1, since $\mathrm{ex}(n, \theta_3^p) = O(n^{1+1/3})$, it follows that $\mathrm{ex}(n, \mathsf{L}(\theta_3^p)) = O(n^{2-2/5})$. As H_2^p is a subgraph of $\mathsf{L}(\theta_3^p)$, it follows that $\mathrm{ex}(n, H_2^p) = O(n^{2-2/5})$, as desired.
Inductive step: As H_s^p is a subgraph of $\mathsf{L}(H_{s-1}^p)$, it follows from Lemma 3.1 that $\mathrm{ex}(n, H_s^p) \le \mathrm{ex}(n, \mathsf{L}(H_{s-1}^p)) = O(n^{2-2/(2s+1)})$, as desired. The induction is complete. □

4 Asymmetric Extremal Numbers of Bipartite Graphs

The following corollary of Bukh and Conlon's construction of the family $\mathcal{F}$ in Theorem 1.1 is essential to proving our asymptotic lower bound on the asymmetric extremal number. (We shall present the exact construction in Section 5.)

Theorem 4.1 (Bukh–Conlon)**.** *Let (F, R) be a balanced rooted graph with a unrooted vertices and b rooted vertices. Then for sufficiently large p,* $\mathrm{ex}(n, F^p) = \Omega(n^{2-a/b})$*, where $\frac{a}{b}$ equals $\frac{1}{\rho_F}$.*

We are ready to state and prove our lower bound.

Theorem 4.2. *Let (F, R) be a balanced rooted bipartite graph with a unrooted vertices and b edges, and let $m \le n$. Then for sufficiently large p,*

$$\mathrm{ex}(m, n, F^p) = \Omega(mn^{1-a/b}).$$

Proof of Theorem 4.2. By Theorem 4.1, for any n, for sufficiently large p, there exists an F^p-free subgraph G of $K_{n,n}$ with $\Omega(n^{2-1/\rho_F})$ vertices. Let the bipartition of G be (A, B). Randomly sample a set of m vertices from A, and consider the induced subgraph G' of G. For each vertex $v \in A$, the expected degree of v in G'

is equal to $\deg_{G'}(v) = \frac{m}{n} \cdot \deg_G(v)$. By the linearity of expectation, it follows that the expected number of edges in G' is equal to

$$|E(G')| = \sum_{v \in A} \frac{m}{n} \cdot \deg_G(v) = \frac{m}{n} \cdot |E(G)| = \Omega(mn^{1-1/\rho_F}).$$

Thus, there exists an F^p-free m-by-n bipartite graph G' with at least $\Omega(mn^{1-1/\rho_F})$ edges, as claimed. □

5 Asymmetric Extremal Numbers of Theta Graphs

We begin by presenting the definition of a theta graph. The *theta graph* θ_k^p is defined as the union of p paths of length k whose end-vertices are the same, but which are otherwise disjoint.

Note that θ_k^p is identical to $(\theta_k^1)^p$, where θ_k^1 denotes a path of length k. For an example of a theta graph, see the graph of θ_3^3 in Figure 3.

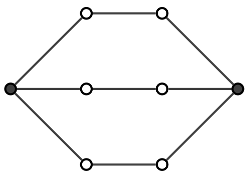

Figure 3: The theta graph θ_3^3 rooted at its end-vertices (in black).

The proof of our lower bound on the asymmetric extremal number of certain theta graphs will closely follow the proof of Lemma 1.2 in [BC18]. Before we proceed with the result, we must explicitly state Bukh and Conlon's construction in Theorem 1.1. Given a rooted graph (F, R) and a positive integer p, the *blowup* $\mathcal{F}^p$ denotes the family of all possible unions of p distinct labelled copies of F, each of which agree on the set of roots R [BC18]. In [BC18], Bukh and Conlon prove that for any balanced rooted graph (F, R), $\text{ex}(n, \mathcal{F}^p) = \Theta(n^{2-1/\rho_F})$.

We may begin proving the necessary lemmas for the proof of our lower bound. The following probabilistic lemma, taken from [BC18], will be used throughout the proof.

Lemma 5.1. *Suppose that $q > \binom{m}{2}$ and $d \geq m-1$. Then if $f : \mathbb{F}_q^t \to \mathbb{F}_q$ is a random polynomial of degree at most d and if $x_1, \ldots, x_m$ are m distinct points in $\mathbb{F}_q^t$,*

$$\mathbb{P}[f(x_i) = 0 \textit{ for all } 1 \leq i \leq m] = q^{-m}.$$

The following inequality, also crucial to our proof, generalizes an inequality stated in [BC18] for the special case of paths.

Lemma 5.2. *Let (F, R) denote the path of length k, rooted at its endpoints. If $H \in \mathcal{F}^p$ is bipartite with bipartition (A, B), where $|A| = u$ and $|B| = v$ and $u \leq v$, then for all $1 \leq t \leq k$, u and v satisfy the inequality*

$$tu + \left(2k - 2 - \frac{k-2}{k}t\right) v \leq (k-1)e(H).$$

Proof. By induction on p, it suffices to show that if S is a connected subset of the unrooted vertices of F (that is, a subpath), then

$$tu' + \left(2k - 2 - \frac{k-2}{k}t\right) v' \leq (k-1)e(S),$$

where u', v' ($u' \leq v'$) denote the number of vertices on each side of the bipartition of S, and $e(S)$ denotes the number of edges in F adjacent to at least one vertex of S.

Let x be the number of vertices in S. Then $e(S) = x + 1$, $u' = \lfloor x/2 \rfloor$, and $v' = \lceil x/2 \rceil$. We now split into cases based on the parity of x.

Case 1: x is even.

Let $x = 2y$. Then, by symmetry, it remains to show that

$$ty + \left(2k - 2 - \frac{k-2}{k}t\right) y \leq (k-1)(2y+1),$$

or equivalently,

$$\left(2k - 2 + \frac{2t}{k}\right) y \leq (k-1)(2y+1).$$

Rearranging, this is equivalent to

$$2k - 2 + \frac{2t}{k} \leq 2k - 2 + \frac{(k-1)}{y} \iff 2ty \leq k(k-1),$$

which follows from $t \leq k$ and $y \leq \frac{k-1}{2}$.

Case 2: x is odd.

Let $x = 2y + 1$. It suffices to show that

$$ty + \left(2k - 2 - \frac{k-2}{k}t\right)(y+1) \leq (k-1)(2y+2),$$

or equivalently, that

$$t\frac{y}{y+1} + 2k - 2 - \frac{k-2}{k}t \leq 2k - 2.$$

Rearranging, this is equivalent to $\frac{y}{y+1} \leq \frac{k-2}{k}$, which follows from the fact that $y \leq \frac{k-2}{2}$.

□

Finally, the proof utilizes elements of affine algebraic geometry. Specifically, we will need the following lemma stated in [BC18], a corollary of the Lang–Weil bound.

Lemma 5.3. *Suppose W and D are varieties over $\overline{\mathbb{F}}_q$ of complexity at most M that are defined over $\mathbb{F}_q$. Then one of the following holds for all q sufficiently large in terms of M:*

- $|W(\mathbb{F}_q) \setminus D(\mathbb{F}_q)| \geq q/2$, *or*
- $|W(\mathbb{F}_q) \setminus D(\mathbb{F}_q)| \leq c_M$, *where c_M depends only on M.*

We are ready to state and prove our lower bound.

Theorem 5.4. *Let $k \in \mathbb{N}$, and let $q > \binom{6k^2}{2}$ be a prime power. Let $m = q^t$ and $n = q^{2k-2-\frac{k-2}{k}t}$, where $t \in \mathbb{N}$ and $t \leq k$. Then for sufficiently large p,*

$$\mathrm{ex}(m, n, \theta_k^p) = \Omega(m^{\frac{k+2}{2k}} n^{\frac{1}{2}}).$$

Proof of Theorem 5.4. Let (F, R) denote the path of length k, rooted at its endpoints. Let $R = \{u_1, u_2\}$ and $V(F) \setminus R = \{v_1, \ldots, v_{k-1}\}$. Let $s = 6k$ and $d = sk$. Consider the bipartite parent graph $G = (M, N)$, where M and N represent $\mathbb{F}_q^t$ and $\mathbb{F}_q^{2k-2-\frac{k-2}{k}t}$, respectively. We randomly sample $k-1$ polynomials $f_1, \ldots, f_{k-1} \colon \mathbb{F}_q^t \times \mathbb{F}_q^{2k-2-\frac{k-2}{k}t} \to \mathbb{F}_q$ of degree at most d, and draw an edge between $x, y \in V(G)$ if $f_i(x, y) = 0$ for all $1 \le i \le k-1$.

By Lemma 5.1, the probability that a given edge xy is in $E(G)$ is $q^{-(k-1)}$. Hence the expected number of edges in G is

$$\mathbb{E}\left[|E(G)|\right] = q^{-(k-1)} \cdot mn = q^{(k-1)+\frac{2t}{b}} = m^{\frac{k+2}{2k}} n^{\frac{1}{2}}.$$

Suppose that $\{w_1, w_2\}$ are fixed vertices in G, and let C be the set of all copies of F in G whose roots u_1, u_2 correspond to w_1, w_2, respectively.

We will bound $\mathbb{E}\left[|C|^s\right]$ from above. Note that $|C|^s$ counts the number of ordered collections of s copies of F rooted at $\{w_1, w_2\}$. Since the number of edges m in a given collection H of s copies of F rooted at $\{w_1, w_2\}$ is at most $sk = 6k^2$ and since $q > \binom{6k^2}{2}$, by Lemma 5.1, it follows that the probability of any such collection H is at most $q^{-(k-1)m}$.

Define $\mathcal{F}_{\le}^s = \bigcup_{i=1}^s \mathcal{F}^i$. Given $H \in \mathcal{F}_{\le}^s$, let $N_s(H)$ denote the number of ordered collections of s copies of F rooted at $\{w_1, w_2\}$ whose union is H. For H bipartite, if there are u, v vertices on each side of the bipartition of H, it follows that $N_s(H) = O_s(m^u n^v + m^v n^u)$. Hence

$$\begin{aligned}
\mathbb{E}\left[|C|^s\right] &= \sum_{H \in \mathcal{F}_{\le}^s} N_s(H) q^{-(k-1)e(H)} \\
&= \sum_{H \in \mathcal{F}_{\le}^s} O_s(m^u n^v + m^v n^u) q^{-(k-1)e(H)} \\
&= O_s\left(\sum_{H \in \mathcal{F}_{\le}^s} (m^u n^v + m^v n^u) q^{-(k-1)e(H)} \right) \\
&= O_s\left(\sum_{H \in \mathcal{F}_{\le}^s} q^{tu+\left(2k-2-\frac{k-2}{k}t\right)v-(k-1)e(H)} + q^{\left(2k-2-\frac{k-2}{k}t\right)u+tv-(k-1)e(H)} \right).
\end{aligned}$$

By Lemma 5.2, $tu + \left(2k-2-\frac{k-2}{k}t\right)v - (k-1)e(H) \le 0$. Furthermore, as $t \le k$, the Rearrangement Inequality implies that $\left(2k-2-\frac{k-2}{k}t\right)u + tv - (k-1)e(H) \le 0$ as well. It follows that

$$\mathbb{E}\left[|C|^s\right] = O_s\left(\sum_{H \in \mathcal{F}_{\le}^s} 2 \right) = O_s(1).$$

Finally, by Markov's Inequality,

$$\mathbb{P}[|C| \ge c] = \mathbb{P}[|C|^s \ge c^s] \le \frac{O_s(1)}{c^s}.$$

Next, we must provide bounds on $|C|$. The argument in [BC18] must be modified as follows. Given an arbitrary copy of F rooted at w_1, w_2 in G, let $(x_1, \ldots, x_{k-1})$ denote its set of unrooted vertices. For $1 \le i \le k-1$, let

$$f(v_i) = \begin{cases} g(w_1) & \text{if } d(v_i, u_1) \text{ is even,} \\ g(w_2) & \text{if } d(v_i, u_1) \text{ is odd,} \end{cases}$$

where $g(w_i)$ denotes the size of the bipartition of G that w_i lies in, and d denotes the distance function. It follows that for all copies of F in C, $(x_1, \ldots, x_{k-1}) \in \overline{\mathbb{F}}_q^L$, where $L = \sum_{i=1}^{k-1} f(v_i)$. The argument from [BC18] can now be applied, with the algebraic variety $X(\mathbb{F}_q)$ defined as a subset of $\overline{\mathbb{F}}_q^L$. From Lemma 5.3, it follows that there exists a constant c_F, depending only on F, such that either $|C| \leq c_F$ or $|C| \geq q/2$. Thus, from the above,

$$\mathbb{P}[|C| > c_F] = \mathbb{P}[|C| \geq q/2] \leq \frac{O_s(1)}{(q/2)^s}.$$

To finish, we call a pair of vertices (w_1, w_2) *bad* if there are more than c_F copies of F rooted such that u_1, u_2 correspond to w_1, w_2, respectively. Let B be the random variable equal to the number of bad sequences. Then, as $s = 6k$ and $q > \binom{6k^2}{2}$,

$$\mathbb{E}[B] \leq (m+n)^2 \cdot \frac{O_s(1)}{(q/2)^s} = O_s(q^{4k-s}) = O_s(q^{-2k}).$$

We remove a vertex from each bad sequence to yield a $\mathcal{F}^p$-free graph G' with at least

$$m^{\frac{k+2}{2k}} n^{\frac{1}{2}} - \mathbb{E}[B]\, n = m^{\frac{k+2}{2k}} n^{\frac{1}{2}} - O_s(1) = \Omega(m^{\frac{k+2}{2k}} n^{\frac{1}{2}})$$

edges. Hence there exists an $\mathcal{F}^p$-free m-by-n bipartite graph G' with $\Omega(m^{\frac{k+2}{2k}} n^{\frac{1}{2}})$ edges, implying that $\mathrm{ex}(m, n, F^p) \geq \mathrm{ex}(m, n, \mathcal{F}^p) \geq \Omega(m^{\frac{k+2}{2k}} n^{\frac{1}{2}})$, as desired.

□

6 Lopsided Asymmetric Extremal Numbers of Theta Graphs

We begin by defining the lopsided asymmetric extremal number.

Definition 6.1. Let F be a bipartite graph with a fixed proper 2-coloring in red and blue. Then the *lopsided asymmetric extremal number* $\mathrm{ex}^*(m, n, F)$ denotes the maximum number of edges in a bipartite graph G (with m red vertices on one part and n blue vertices on the other) that does not contain a copy of F whose blue and red vertices are in the blue and red classes of G, respectively. We write $\mathrm{ex}^*(n, F)$ for $\mathrm{ex}^*(n, n, F)$, and call this the *lopsided extremal number.*

In general, $\mathrm{ex}^*(m, n, F)$ is larger than $\mathrm{ex}(m, n, F)$. As a result, Theorem 5.4 implies a corresponding lower bound on $\mathrm{ex}^*(m, n, \theta_{k,p})$, and the following result is true.

Theorem 6.1. *Let $k \in \mathbb{N}$, and let $q > \binom{6k^2}{2}$ be a prime power. Let $m = q^t$ and $n = q^{2k-2-\frac{k-2}{k}t}$, where $t \in \mathbb{N}$ and $t \leq k$. Then for sufficiently large p,*

$$\mathrm{ex}^*(m, n, \theta_k^p) = \Omega(m^{\frac{k+2}{2k}} n^{\frac{1}{2}}).$$

7 Conclusion and Future Work

In conclusion, our work provides new insights on the behavior of the extremal number and its variants. In recent months, there has been significant activity on Conjecture A; see [Jan19b] and [CJL19]. A current focus is on studying the extremal number of so-called subdivisions of graphs [GJN19] [Jan18] [Jan19a]. This technique has led to recent discoveries of new sets of Turán exponents [JQ19a] [JQ19b]. In the light of our proof of Theorem 3.2, it would be interesting to explore further applications of the Erdős–Simonovits Reduction Theorem in simplifying related bounding arguments and progressing towards Conjecture A. In addition, one could consider extending our lower bound in Theorem 4.2 to the case that $m \ll n$, as well as generalizing Theorems 5.4 and 6.1 to cover all possible pairs (m, n). Lastly, given the applicability of extremal graph theory in information security [PRUW13], it would be interesting to explore the benefits our results could provide to the fields of coding theory and cryptography.

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References

[BC18] Boris Bukh and David Conlon. Rational exponents in extremal graph theory. *J. Eur. Math. Soc. (JEMS)*, 20(7):1747–1757, 2018. `arXiv:math/1506.06406[math.CO]`.

[CJL19] David Conlon, Oliver Janzer, and Joonkyung Lee. More on the extremal number of subdivisions, 2019.

[Erd81] P. Erdős. On the combinatorial problems which i would most like to see solved. *Combinatorica*, 1(1):25–42, Mar 1981.

[ES70] Paul Erdős and Miklós Simonovits. Some extremal problems in graph theory. *Combinatorial theory and its applications, I*, pages 377 – 390, 1970.

[FS83] Ralph J. Faudree and Miklós Simonovits. On a class of degenerate extremal graph problems. *Combinatorica*, 3(1):83–93, Mar 1983.

[FS13] Zoltán Füredi and Miklós Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdös centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 169–264. János Bolyai Math. Soc., Budapest, 2013.

[GJN19] Andrzej Grzesik, Oliver Janzer, and Zoltán Lóránt Nagy. The turán number of blow-ups of trees, 2019.

[Jan18] Oliver Janzer. Improved bounds for the extremal number of subdivisions, 2018.

[Jan19a] Oliver Janzer. The extremal number of longer subdivisions, 2019.

[Jan19b] Oliver Janzer. The extremal number of the subdivisions of the complete bipartite graph, 2019.

[JMY18] Tao Jiang, Jie Ma, and Liana Yepremyan. On Turán exponents of bipartite graphs. *arXiv e-prints*, page arXiv:1806.02838, Jun 2018.

[JQ19a] Tao Jiang and Yu Qiu. Many turan exponents via subdivisions, 2019.

[JQ19b] Tao Jiang and Yu Qiu. Turan numbers of bipartite subdivisions, 2019.

[Man07] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60 – 61, 1907.

[PRUW13] Monika Polak, Urszula Romańczuk, Vasyl Ustimenko, and Aneta Wróblewska. On the applications of extremal graph theory to coding theory and cryptography. *Electronic Notes in Discrete Mathematics*, 43:329 – 342, 2013.

[TS46] Paul Turán and Arthur Harold Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087 – 1091, 1946.

[TS66] Paul Turán and Miklós Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51 – 57, 1966.

[Tur41] Paul Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48:436 – 452, 1941.