The Prime Number Theorem with Error Term

Jason Tang and Richard Chen

Mentor: Chengyang Shao

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It follows that the n^{th} prime number should be approximately of the magnitude $n \log n$.

Numerical Results of the Prime Number Theorem

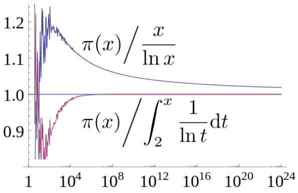

Figure: Ratios for $\pi(x)$ and its approximations; from Wikipedia

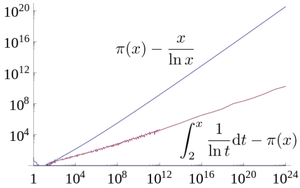

Figure: Absolute error of the approximations of $\pi(x)$; from Wikipedia

Historical Background

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- 1896: Hadamard and de la Vallée Poussin each individually proved properties of Riemann ζ function that completed the proof of the Prime Number Theorem.
- Alternate proofs were found in later years, some much simpler or more elementary.

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$$\vartheta(x) = \sum_{p \leq x} \log p,$$

where the sum runs over all prime numbers less than x.

Chebyshev ψ-function: $\psi(x) = \sum_{n \leq x} \Lambda(n)$.

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By Mobius Inversion, we obtain

$$\vartheta(x) = \sum_{m \leq x} \mu(m) \psi(x^{1/m}).$$

Relation between $\vartheta(x)$ and $\pi(x)$

Abel's Summation Formula: For an arithmetic function $a(n)$, define $A(x) = \sum_{n \le x} a(n)$. Suppose f is continuously differentiable on the interval $[x, y]$ for $0 < x < y$. Then,

$$\sum_{x<n\le y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt.$$

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We take $a(n) = \begin{cases} \log n & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$ and $f(x) = \frac{1}{\log x}$, then

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Furthermore, setting $a(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$ and $f(x) = \log x$ gives

$$\vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} dt.$$

Asymptotic Behaviors of the Chebyshev Functions

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$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 \iff \lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1.$$

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- We can verify these via direct calculation and using Abel's Formula.
- These relations show that the the prime number theorem can be converted to the study of the ψ function as if $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, then $\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$.

Riemann Zeta Function

We now introduce the Riemann zeta function, whose distribution of zeros is connected later to the explicit formula of the Chebyshev ψ function.

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- Observe that since $\sigma > 1$, we may express each term in the right product as an infinite geometric series.
- The product converges absolutely if $\sigma > 1$ so we may use the distribution law. Each term in $\zeta(s)$ can be expressed as a product of terms from the geometric series.

The Logarithmic Derivative:

$$\frac{\zeta'}{\zeta}(s) = \frac{d}{ds} \log \prod_{p\text{: prime}} \frac{1}{1-p^{-s}} = -\sum_{p\text{: prime}} \log(p) \sum_{n=1}^{\infty} p^{-ns} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \qquad (1)$$

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- Heuristically, from this equation we can understand why the logarithmic derivative is related to the ψ function as both can be expressed as a sum using the von Mangoldt function: $\psi(x)$ is nothing but the partial sum of the series $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ when $s = 0$.

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- The precise relation is revealed using *Perron's formula*: roughly speaking,

$$\sum_{n<x} \frac{f(n)}{n^{s_0}} - \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s_0+s) \frac{x^s}{s} ds + \text{Error terms.}$$

Theorem (Perron's Formula)

Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be a Dirichlet series and let σ_a be the absolutely convergent coordinate of F. Let $s_0 = \sigma_0 + it_0$, and let b be a positive number such that $\sigma_0 + b > \sigma_a$. Suppose there is a function $B(\sigma)$ and increasing function $H(\sigma)$ such that $|f(n)| \leq H(n)$ and $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} \leq B(\sigma)$. Then for any half integer $x > 2$ and any $T > 2$,

$$\left| \sum_{n<x} \frac{f(n)}{n^{s_0}} - \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s_0 + s) \frac{x^s}{s} ds \right| \leq \frac{10x^b B(\sigma_0 + b)}{T} + 100 \cdot 2^{b+\sigma_0} x^{1-\sigma_0} H(2x) \frac{\log x}{T}$$

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- We combine the $n < x$ and $n > x$ sums via the Triangle Inequality.

We now apply Perron's Formula to $\psi(x)$.

- $F(s) = \frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ with $\sigma_a \leq 1$
- $H(n) = \log n$, $B(\sigma) = \frac{10}{\sigma - 1}$, $s_0 = 0$, $b = 1 + \frac{1}{\log x}$

Perron's Formula for half integer $x \geq 2$ and $T \geq 2$:

$$\psi(x) = \sum_{n<x} \Lambda(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds + R(x;T) \quad (2)$$

where $R(x;T) << \frac{x \log^2 x}{T}$.

Definition (Entire Function and Order)

An entire function f is a function holomorphic over all of $\mathbb{C}$. We define

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

If there exists $A, B, \lambda \geq 0$ such that $M_f(r) \leq Ae^{Br^\lambda}$, then we say that f has order $\leq \lambda$. The infimum of all λs such that this inequality hold is called the order of f.

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- Polynomials are entire functions of order zero.
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- e^{e^z} is not of finite order.
- $1/\Gamma$ is an entire function of order one, but the inequality $1/|\Gamma(z)| \leq A + e^{B|z|}$ can never hold for any finite A, B.

Hadamard Factorization Theorem

We now introduce the Hadamard Factorization Theorem, which is necessary for any information concerning the distribution of zeros of $\zeta(s)$.

Theorem (Hadamard Factorization Theorem)

Let f be an entire function of order λ and $f(0) \neq 0$. Let $\{a_n\}$ be the zeros of f, and let

$$E_p(z; a_n) = \left(1 - \frac{z}{a_n}\right) \exp\left[\left(\frac{z}{a_n}\right) + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{a_n}\right)^p\right]$$

for $p = \lfloor \lambda \rfloor$. Then, for some polynomial q of degree less than or equal to p,

$$f(z) = e^{q(z)} \prod_{n=1}^{\infty} E_p(z; a_n),$$

where the infinite product converges absolutely and uniformly on compact subsets of the complex plane to an entire function.

- Proof consists of showing that the product converges and the order of q is bounded
- Uses Jensen's Inequality and Hadamard's corollary in the bounding portion of the proof

Hadamard Factorization Theorem

- Factorization of polynomials:

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- We deduce from these expressions the *partial fraction developments*:

$$\cot(z) = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - (k\pi)^2}$$

$$\frac{\Gamma'}{\Gamma}(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n}\right).$$

Functional Equation of ζ

By direct calculation starting with $\zeta(s)\Gamma(s)$, the ζ function is analytically continued beyond $\sigma > 1$, except for the pole at $s = 1$:

The functional equation for ζ function:

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s). \tag{3}$$

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The ξ function has nice properties, in that the factor $s(s-1)\Gamma(s/2)$ cancels out all the trivial zeros and the pole of $\zeta(s)$, and $\xi(s) = \xi(1-s)$. Therefore, $\xi(s)$ is an entire function of order 1.

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Because $|\xi(s)| \leq A + e^{B|s|}$ cannot hold for finite A, B, the Hadamard Factorization Theorem implies that $\xi(s)$ must have infinitely many zeros.

Consequently, $\zeta(s)$ must have infinitely many nontrivial zeros.

- $\zeta(s)$ has a first order pole at $s = 1$ and residue 1. It is holomorphic for $s \neq 1$.

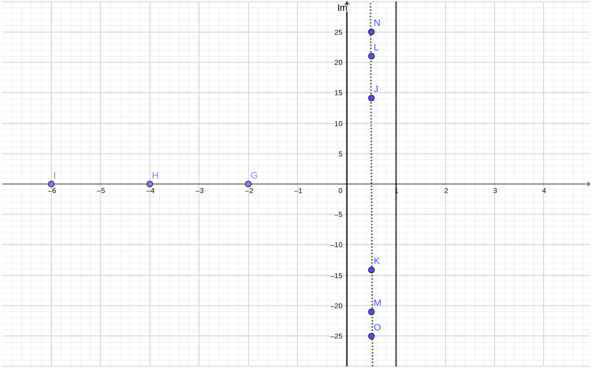

Figure: Zeros of the ζ function

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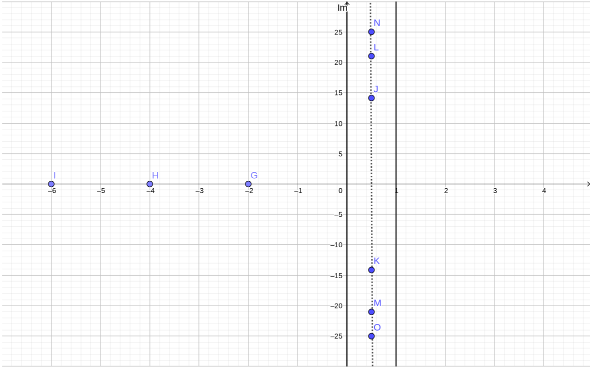
- $\zeta(s)$ has a first order pole at $s = 1$ and residue 1. It is holomorphic for $s \neq 1$.
- $\zeta(s)$ only has zeros at negative even integers for $\sigma < 0$. Additionally, $\zeta(s) \neq 0$ when $\sigma > 1$



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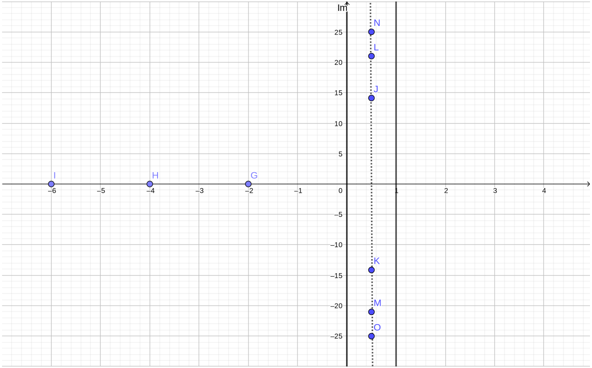
- $\zeta(s)$ only has zeros at negative even integers for $\sigma < 0$. Additionally, $\zeta(s) \neq 0$ when $\sigma > 1$
- Zeros in the strip $0 \leq \sigma \leq 1$ are called nontrivial zeros. They are symmetric with respect to the real axis and the vertical line $\sigma = 1/2$. They will be denoted $\rho = \beta + i\gamma$.



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$$\frac{\zeta'}{\zeta}(s) = D - \frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\frac{1}{s+2n} - \frac{1}{2n}\right) + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

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We can now bound the first sum and thus obtain the following:

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + O\left(\frac{1}{\lambda(s)} + \log(|s|+2)\right),$$

where $\lambda(s) = \min_{n \geq 1} |s+2n|$.

Formula for ζ'/ζ

By the equation from the previous slide, we have an estimate for the sum over all the zeros. Therefore, we obtain the following:

Theorem (Asymptotic Formula for ζ'/ζ)

The following asymptotic formula holds for any $s \in \mathbb{C}$:

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{\rho : |\gamma - t| \leq 1} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + O\left(\frac{1}{\lambda(s)} + \log(|s|+2) \right).$$

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We now introduce an important corollary.

Corollary

For every $T \geq 2$, there exists $T' \in [T, T+1]$ such that $\frac{\zeta'}{\zeta}(\sigma + iT') << \log^2 |\sigma + iT|$ for every $\sigma \in \mathbb{R}$.

Theorem (Zero-free region of $\zeta(s)$; due to de la Vallée Poussin)

(1) $\zeta(1+it) \neq 0$ *for any real number* t*;*
(2) There is a constant $A > 0$ *such that* $\zeta(s)$ *is zero-free for* $\sigma \geq 1 - \frac{A}{\log t}, t \geq 2$*, shown as the shaded region in the following figure.*

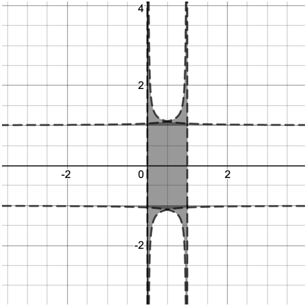

Figure: Zero-free region of $\zeta(s)$

Zero Free Region

- Let $s = \sigma + it$ with $\sigma > 1$, $\rho = \beta + i\gamma$. Then,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{p}\sum_{n=1}^{\infty}\frac{\log p}{p^{n\sigma}}(\cos(nt\log p) - i\sin(nt\log p)).$$

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- We utilize the identity $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0$. Letting $nt\log p = \theta$, and taking real part,

$$\begin{aligned}&-3\mathrm{Re}\frac{\zeta'}{\zeta}(\sigma) - 4\mathrm{Re}\frac{\zeta'}{\zeta}(\sigma+it) - \mathrm{Re}\frac{\zeta'}{\zeta}(\sigma+2it)\\ &=\sum_{p}\sum_{n=1}^{\infty}\frac{\log p}{p^{n\sigma}}(3 + 4\cos(nt\log p) + \cos(2nt\log p)) \geq 0.\end{aligned}$$

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- If we let $\sigma \to 1^+$, by using the asymptotic formula attained previously we obtain that for any non-trivial zero $\rho = \beta + i\gamma$,

$$1 - \beta \geq \frac{A}{\log\gamma}.$$

Explicit Formula for $\psi(x)$

We start with our expression derived from Perron's Formula:

$$\psi(x) = \sum_{n<x} \Lambda(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds + O(\frac{x \log^2 x}{T}).$$

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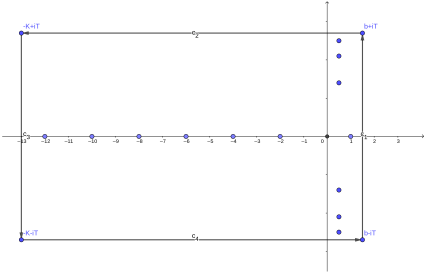

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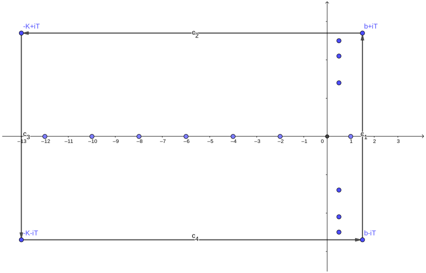

By the residue theorem,

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds$$
$$= -\frac{1}{2\pi i}\left(\int_{c_2} + \int_{c_3} + \int_{c_4}\right) -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds + x - \frac{\zeta'}{\zeta}(0) + \sum_{\rho:|\gamma|\leq T}\frac{x^\rho}{\rho} + \sum_{n=1}^{(K-1)/2}\frac{x^{-2n}}{2n}.$$

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From a corollary above, we can always find a $T' \in [T, T+1]$ that has $\frac{\zeta'}{\zeta}(\sigma + iT') = O(\log^2 T)$.

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From a corollary above, we can always find a $T' \in [T, T+1]$ that has $\frac{\zeta'}{\zeta}(\sigma + iT') = O(\log^2 T)$. Thus, we can bound the integrals from the previous expression, arriving at the following explicit formula.

Explicit formula of prime numbers:

$$\psi(x) = x - \frac{\zeta'}{\zeta}(0) + \sum_{\rho:|\gamma|\leq T}\frac{x^\rho}{\rho} + \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) + O\left(\frac{x\log^2 T}{T\log x} + \frac{x\log^2 x}{T}\right)$$

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- Because of our restrictions on the zero-free region,

$$\begin{aligned}
\left|\sum_{\rho:|\gamma|\leq T}\frac{x^\rho}{\rho}\right| &\leq xe^{-\frac{A\log x}{\log T}}\sum_{\rho:|\gamma|\leq T}\frac{1}{|\rho|}\\
&<< xe^{-\frac{A\log x}{\log T}}\sum_{k=1}^{[T]+1}\sum_{\rho:k<|\gamma|\leq k+1}\frac{N(k+1)-N(k)}{k}\\
&<< xe^{-\frac{A\log x}{\log T}}\sum_{k=1}^{[T]+1}\frac{\log k}{k}\\
&<< xe^{-\frac{A\log x}{\log T}}\log^2 T.
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Proof of the Prime Number Theorem

- Take $T = e^{\sqrt{\log x}} + O(1)$. We can thus adjust our explicit formula for $\psi(x)$ to be

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- Finally,

$$\begin{aligned}
\pi(x) &= \frac{\vartheta(y)}{\log x} + \int_2^x \frac{\vartheta(y)}{y\log^2 y}dy \\
&= \frac{x}{\log x} + \int_2^x \frac{1}{\log^2 y}dy + \frac{O(xe^{-c\sqrt{\log x}})}{\log x} \\
&\quad + \int_2^x \frac{O(ye^{-c\sqrt{\log y}})}{y\log^2 y}dy \\
&= \mathsf{Li}(x) + O(xe^{-c\sqrt{\log x}}).
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Expression for $\pi(x)$

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Applications

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- Additionally, the explicit formula for $\psi(x)$ suggests that the distribution of zeros of $\zeta(s)$ is equivalent to the distribution of prime numbers.

- Riemann's hypothesis asserts that the nontrivial zeros are always on the line $\mathrm{Re}(s) = 1/2$. If this is true, it follows easily from the explicit formula that

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log^2 x),$$

the optimal result on distribution of prime numbers.

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- Our mentor, Chengyang Shao
- MIT PRIMES
- Our parents