Agent-based Models for Conservation Equations

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Conservation Equations

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x} = 0$$

$$\rho_t + f_x = 0$$

ρ : density

f : flux

Possible Usage

- Cars
- Blood
- Electric Charge

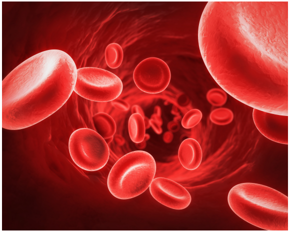
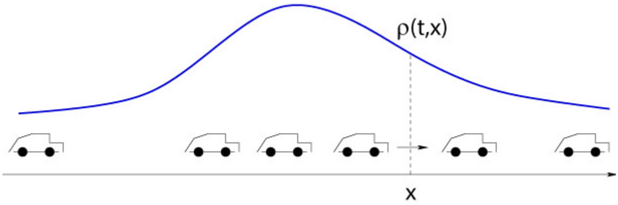


Figure: Red Blood Cells

Figure: Traffic Flow

Density

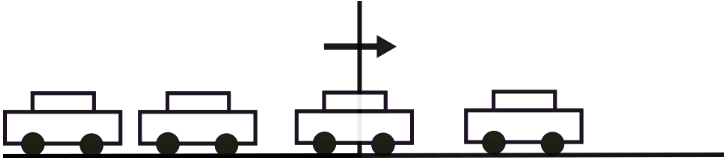
$\rho(x,t)$: Density defined as mass per unit length.
Example: $\rho = \frac{cars}{length}$



Flux

$f(x,t)$: Flux defined as the amount of mass passing through x per unit time.

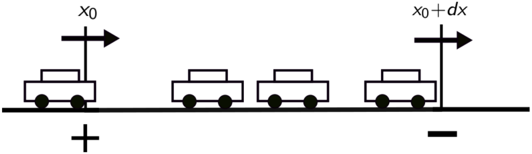
$$f = \frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta x}\frac{\Delta x}{\Delta t} = \rho v$$



Derivation of the Conservation Equation

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} \rho(x,t)dx = f(x_0,t) - f(x_0+\Delta x, t)$$

$$\rho_t = -f_x$$



Constitutive Laws

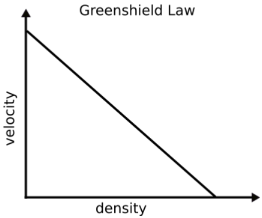
We need to relate flux with density.

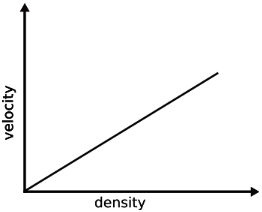
- Greenshield's Law

$$v = v_m(1 - \frac{\rho}{\rho_m})$$
$$f = v_m(1 - \frac{\rho}{\rho_m})\rho$$
$$\rho_t + (v_m(1 - \frac{\rho}{\rho_m})\rho)_x = 0$$

- Burger's Equation

$$v(\rho) = \frac{1}{2}\rho$$
$$f(\rho) = \frac{1}{2}\rho^2$$
$$\rho_t + (\frac{1}{2}\rho^2)_x = 0$$





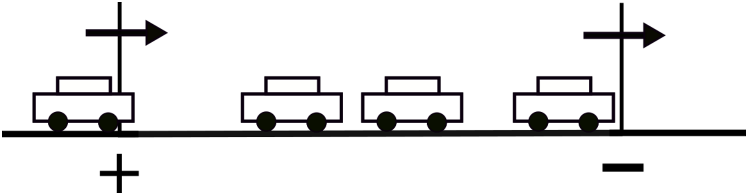
Finite Volume Method

Let flux $f(x,t)$ at $x = x_0$ and $t = t_0$ be written as $f_{x_0}^{t_0}$

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} \rho(x,t)dx = f(x_0,t) - f(x_0+\Delta x, t)$$

$$\frac{\Delta x}{\Delta t}(\overline{\rho}_x^{t+1} - \overline{\rho}_x^{t}) = f_{x-\frac{1}{2}} - f_{x+\frac{1}{2}}$$

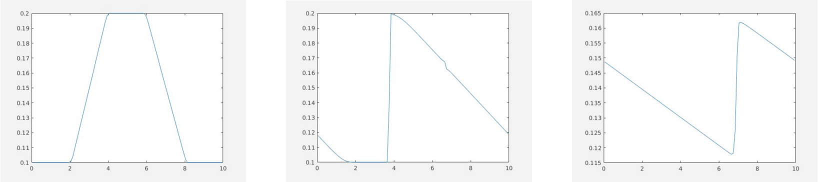
$$\overline{\rho}_x^{t+1} = \overline{\rho}_x^{t} + \frac{\Delta t}{\Delta x}(f_{x-\frac{1}{2}} - f_{x+\frac{1}{2}})$$



Example: Upwind Method

$$f(\overline{\rho}_x, \overline{\rho}_{x+1}) = \frac{1}{2}\left(f(\overline{\rho}_x) + f(\overline{\rho}_{x+1}) - a(\overline{\rho}_{x+1} - \overline{\rho}_x)\right)$$

Where $a = \left|\frac{f(\overline{\rho}_x) - f(\overline{\rho}_{x+1})}{\overline{\rho}_x - \overline{\rho}_{x+1}}\right|$



at t_0 at t_1 at t_2

Figure: Screenshots of the numerical solution. Horizontal axis: position. Vertical axis: density

Agent-based Models

- $x^{i+1} = x^i + v(\rho_L, \rho_R)\Delta t$
- The density $\rho(x,t)$ is approximated as $\frac{\Delta m}{x_{j+1} - x_j}$

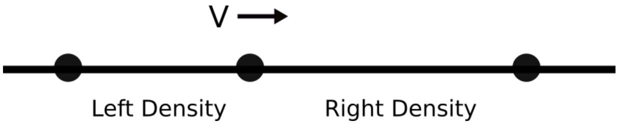


Figure: Agent Model

Example: Greenshields Law

$\rho_t + (v_m(1 - \frac{\rho}{\rho_m})\rho)_x = 0$
$v = v_m(1 - \frac{\rho}{\rho_m})$

$$v(u_L, u_R) = \begin{cases} v_{max} & u_L \geq u_R \\ v_{min} & u_L \leq u_R \end{cases}$$

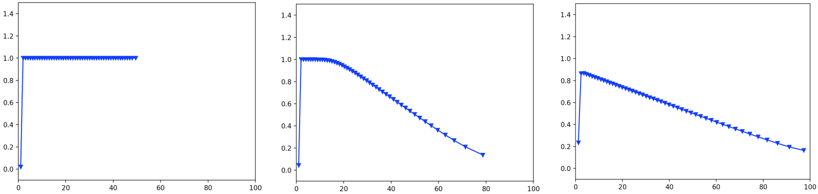


Figure: Each agent correspond to a point. X axis: location. Y axis: density

Video Link

Finite Volume Method for Specific Volume

- σ:the specific volume, defined as $\frac{\Delta x}{\Delta m}$, equals to $\frac{1}{\rho}$.
- Consider the amount of distance between two particles:

$$
\begin{aligned}
\Delta x^{i+1} &= \Delta x^{i} + v_R \Delta t - v_L \Delta t \\
\frac{\Delta x^{i+1}}{\Delta m} &= \frac{\Delta x^{i}}{\Delta m} + \frac{\Delta t}{\Delta m}((-v_L) - (-v_R)) \\
\overline{\sigma}^{i+1} &= \overline{\sigma}^{i} + \frac{\Delta t}{\Delta m}((-v_L) - (-v_R))
\end{aligned}
$$

- Equals to the finite volume formula ($\overline{\rho}_x^{t+1} = \overline{\rho}_x^{t} + \frac{\Delta t}{\Delta x}(f_{x-\frac{1}{2}} - f_{x+\frac{1}{2}})$) for specific volume σ where the flux is $-v$.

Mass Function and Its Inverse

Mass function: $m(x,t) = m(x,0) + \int_0^t -f(x,a)da$ where $m(x,0)$ is the initial mass value at x

- $\frac{\partial m}{\partial t} = -f(x,t)$
- $\frac{\partial m}{\partial x} = \rho(x,t)$
- $x(m,t)$ is defined as $m^{-1}(m,t)$
- $\frac{\partial x(m,t)}{\partial m} = \frac{1}{\rho}$

$$\frac{\partial m^{-1}(m(x,t),t)}{\partial t}|_x = \frac{\partial m^{-1}(m,t)}{\partial t}|_m + \frac{\partial m^{-1}(m,t)}{\partial m}\frac{\partial m}{\partial t}$$
$$\frac{\partial x(m,t)}{\partial t} = \frac{f}{\rho} = v$$

Conservation of Specific Volume

$$\sigma_t|_m + (-v)_m = 0$$

- The conservation equation's finite volume formula is $\overline{\sigma}^{i+1} = \overline{\sigma}^i + \frac{\Delta t}{\Delta m}((-v_L) - (-v_R))$
- Agent-based model can be viewed as a finite volume method for the specific volume where the the total distance that passes each cell wall is recorded.
- Any finite volume method has its agent-based version.
- If a finite volume method converges to a solution for the specific volume conservation equation, its agent-based model converges to a solution for the original conservation equation.

Vector Conservation Equation

$\vec{\rho}_t + \vec{f}_t = 0$

Example:

$$\begin{cases} v_1 = v_{m_1}\left(1 - \frac{\rho_1+\rho_2}{\rho_m}\right) \\ v_2 = v_{m_2}\left(1 - \left(\frac{\rho_1+\rho_2}{\rho_m}\right)^2\right) \end{cases}$$

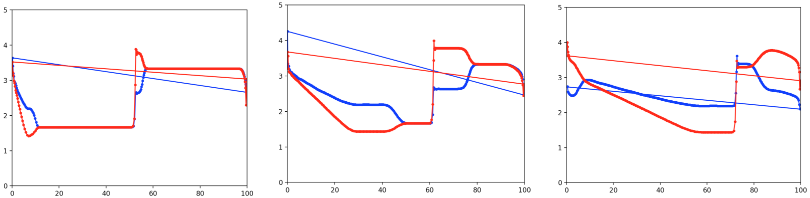


Figure: Red Dots: ρ_1 agents. Blue Dots: ρ_2 agents.

Future Goals

- Comparing agent-based solver with other solvers
- Vector conservation equations
- Conservation equation in 2-D space
- Adapt to source and sink terms.

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References

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