On Khovanov homology, Bar-Natan's perturbation, and Conway mutation

Maximillian Guo[1]
Mentor: Jianfeng Lin[2]

[1]Sycamore High School

[2]MIT

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What is a Knot?

A *knot* is an embedding of a circle in $\mathbb{R}^3$. A *link* is composed of several knotted loops tangled together.

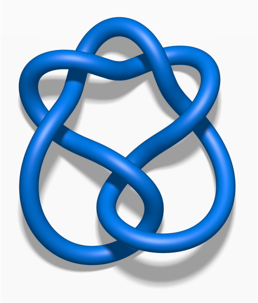

Figure: A knot (Source: Google Images)

Knot diagrams

A *knot diagram* is a projection of a knot onto a plane. Where the knot diagram crosses itself is called a *crossing*.

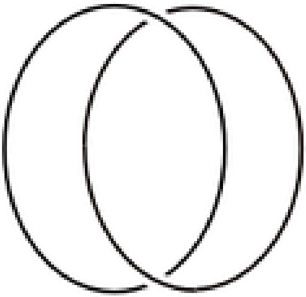

Figure: Knot diagram of the Hopf Link (Source: Google Images)

Equivalent Knots

Two knots are *isotopic* to each other if one can deform one continuously into the other one.

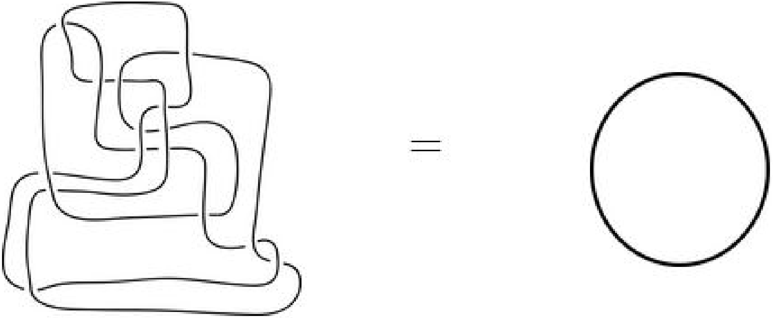

Figure: Thistlethwaite knot is the unknot (Source: Google Images)

Reidemeister Moves

Reidemeister's Theorem

Two knot (or link) diagrams represent the same knot if and only if one diagram can be transformed to the other through a series of Reidemeister moves.

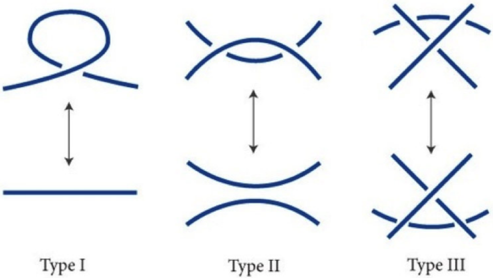

Figure: The three Reidmeister Moves (Source: Google Images)

Knot invariants

A *knot invariant* is a quantity defined for each knot which is the same for equivalent knots.

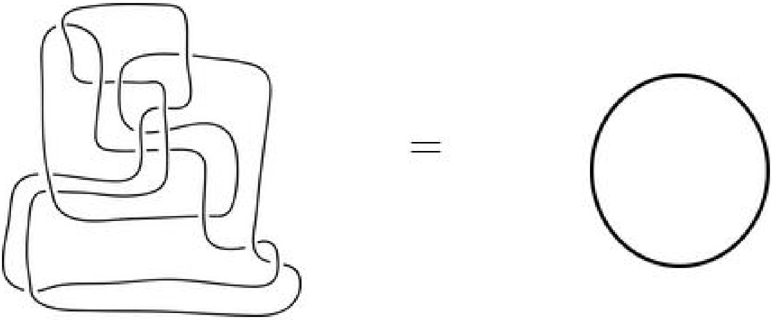

Figure: Thistlethwaite knot and unknot, again

Khovanov and Bar-Natan Homology

In our project, we work with the following two knot invariants: the *Khovanov Homology*, and a related quantity, the *Bar-Natan Homology*.

Theorem (Khovanov)

The Khovanov Homology of the knot is a knot invariant.

The *Bar-Natan Homology* of a knot is defined in a similar way as the Khovanov Homology.

Theorem (Bar-Natan)

The Bar-Natan Homology of a knot is a knot invariant.

Resolutions

Given any crossing, we can define a 0-smoothing (or 0-resolution) and a 1-smoothing. When all the crossings have been resolved, the result is a *resolution* or *smoothing*.

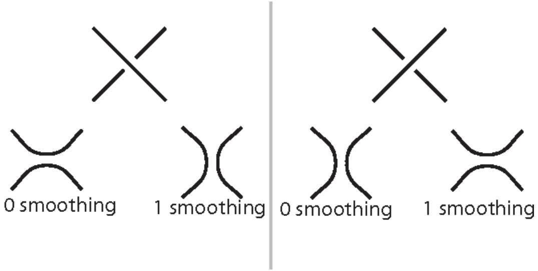

Figure: 0 and 1-smoothings/resolutions (Source: Google Images)

Resolution Cube

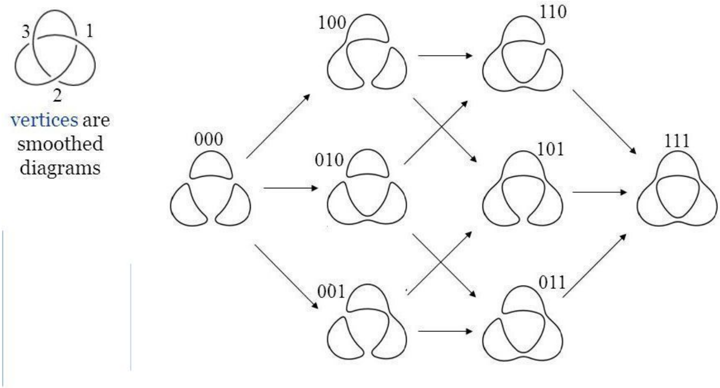

Figure: Resolution cube for the left handed trefoil (Source: Google Images)

Definition cont.

j\i	-3	-2	-1	0
1				
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}/2\mathbb{Z}$		
-9	$\mathbb{Z}$			

j\i	-3	-2	-1	0
1				$\mathbb{F}$
-1				$\mathbb{F}$
-3				
-5		$\mathbb{F}$		
-7				
-9	$\mathbb{F}$			

Figure: The Khovanov and Bar-Natan homologies of the left-handed trefoil, respectively (Sources: [1] and [2])

Conway mutation

Draw a circle around some part of the knot diagram so the circle intersects the knot diagram at exactly 4 points. Now any one of these is a *Conway mutation*:

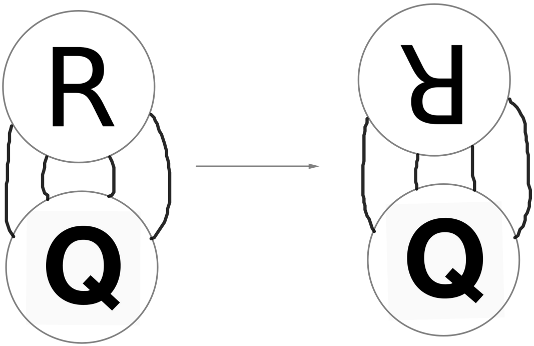

Figure: A Conway mutation (Created with Geogebra)

Examples of Conway mutation

A famous example of knots related by Conway mutation:

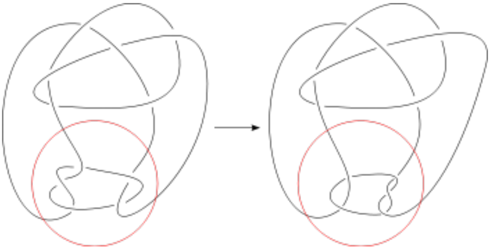

Figure: The Kinoshita-Teresaka Knot and a mutation (Source: Wikipedia)

Invariance under Conway mutation

Theorem (Bloom, Wehrli, 2009)

The Khovanov Homology of a knot is invariant under Conway mutation.

Our Main Result

The Bar-Natan Homology of a knot is also invariant under Conway mutation.

Future Research

1. The Khovanov Homology and Bar-Natan Homologies are *graded vector spaces*, but our proof ignores the gradings aspect.

$$V = \bigoplus_{i,j \in \mathbb{Z}} V_{ij}$$

2. Our proof shows an existence of an isomorphism, but does not construct one.

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References

Pictures taken from Google Images, Wikipedia, and Geogebra
[1] Dan Jones. *An Introduction to Khovanov Homology.*
[2] Francesco Lin. *Khovanov homology in characteristic two and involutive monopole Floer homology.*