Bounds on the Maximal Cardinality of an Acute Set in a Hypercube

Sathwik Karnik

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Outline

1. Introduction
2. Discrete Acute Set Problem
3. Bounds in Discrete Acute Set Problem
4. Future Work
5. Questions?

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Problem Statement

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- **Question:** What is the maximal cardinality (size, denoted as $f(d)$) of an acute set in $\mathbb{R}^d$?
- In other words, what is the maximal cardinality of a subset of $\mathbb{R}^d$ such that for any $x, y, z \in S$, $\langle x - y, z - y \rangle > 0$?

Points form an Acute Set

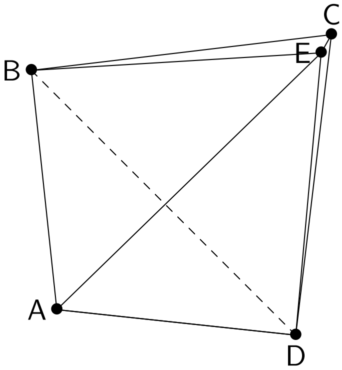

Figure 1: $S = \{(0,0,0),(0,1,0.25),(0.75,0.75,-0.75),(1,0,0.25),(1,0.97,0)\}$

Points do not form an Acute Set

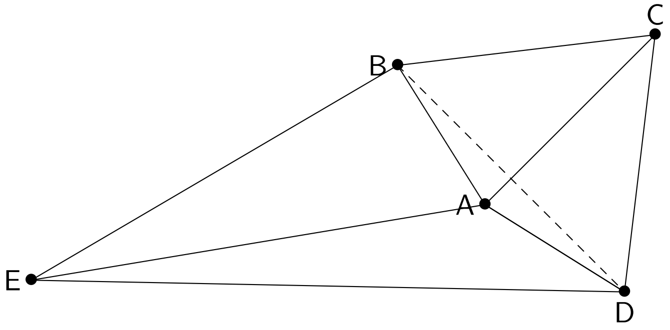

Figure 2: $S = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0),(-2,-1/3,0)\}$

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- **April 2017:** Zakharov improved this bound to $f(d) \geq 2^{d/2}$
- **September 2017:** Gerencsér and Harangi showed that $f(d) \geq 2^{d-1}+1$, thus determining the growth rate of $f(d)$

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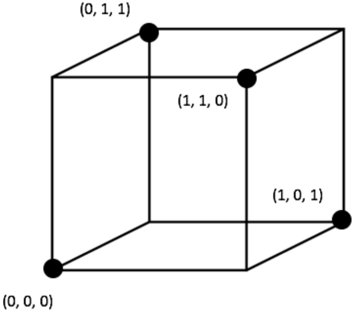

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Improved Lower Bound for $h(d)$

- $\{(0, 0, \ldots, 0), (1, 1, \ldots, 1, 0), \ldots, (0, 1, \ldots, 1)\}$ form an acute set with $d + 1$ points (called a *simplex*), resulting in $h(d) \geq d + 1$

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- By concatenating points in an acute set in $\{0,1\}^d$ $(S = \{v_0, v_1, \ldots, v_{h(d)-1}\})$ to form points in $\{0,1\}^{3d}$, we find that $h(3d) \geq (h(d))^2$, which results in a bound of $h(d) \geq 2^{2^{\lfloor \log_3 d \rfloor}}$

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- Through a similar concatenation of points in an acute set in $\{0, 1\}^d$ and two points in $\{0, 1\}^3$ to form points in $\{0, 1\}^{d+6}$, we find that $h(d + 6) \geq 4h(d)$, which results in a bound of $h(d) \geq 2^{d/3}$, which is stronger for larger dimensions

Concatenation Example

- Let $v_0 = (0, 0, 0)$, $v_1 = (1, 1, 0)$, $v_2 = (1, 0, 1)$, and $v_3 = (0, 1, 1)$ be the points in an acute set in the 3-dimensional cube.

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- The point (v_0, v_0, v_0) represents the point $(0, 0, 0, 0, 0, 0, 0, 0, 0)$ in the 9-dimensional hypercube.

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- The point (v_0, v_0, v_0) represents the point $(0,0,0,0,0,0,0,0,0)$ in the 9-dimensional hypercube.
- Here are 16 points in $\{0,1\}^9$ that form an acute set:

$$(v_0, v_0, v_0), (v_0, v_1, v_1), (v_0, v_2, v_2), (v_0, v_3, v_3)$$
$$(v_1, v_0, v_1), (v_1, v_1, v_2), (v_1, v_2, v_3), (v_1, v_3, v_0)$$
$$(v_2, v_0, v_2), (v_2, v_1, v_3), (v_2, v_2, v_0), (v_2, v_3, v_1)$$
$$(v_3, v_0, v_3), (v_3, v_1, v_0), (v_3, v_2, v_1), (v_3, v_3, v_2)$$

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- In general, when we concatenate 3 points to form an acute set, observe that no two points of the three points are in the same position in other concatenated points.

Improved Upper Bound for $h(d)$

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- Further improvement:
  - Consider a point P in the acute set and all points diagonally opposite on a 2-face
  - The maximum average number of points in the acute set on a face is $1 + \frac{2}{d}$, and there are $(d-1) \cdot d \cdot 2^{d-3}$ 2-faces in a hypercube
  - After considering overcount, $h(d) \leq \left(1 + \frac{2}{d}\right) \cdot 2^{d-2}$

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Combinatorial Interpretation

- A combinatorial interpretation of the acute set problem is that for any three points x, y, and z in the acute set, there exists three positions in these points so that one of the positions is $\{0, 0, 1\}$ or $\{1, 1, 0\}$, another is $\{0, 1, 0\}$ or $\{1, 0, 1\}$, and the other is $\{1, 0, 0\}$ or $\{0, 1, 1\}$.

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- Example of an Acute Set:

$$(1, 1, 1)$$

$$(0, 0, 1)$$

$$(1, 0, 0)$$

- Example of Points Not Forming an Acute Set:

$$(1, 0, 0, 1)$$

$$(0, 0, 1, 0)$$

$$(0, 0, 1, 0)$$

Future Work

- Potential combinatorial generalization: for any k points $v_1, v_2, \ldots, v_k$, there exists k positions such that there exists one of $\{0, 0 \ldots, 0, 1\}$ or $\{1, 1, \ldots, 1, 0\}$, $\{0, 0 \ldots, 0, 1, 0\}$ or $\{1, 1, \ldots, 1, 0, 1\}$, $\ldots$, and one of $\{1, 0 \ldots, 0, 0\}$ or $\{0, 1, 1, \ldots, 1\}$. What is the maximal size of such a set?

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- Does the geometric interpretation of the discrete acute set problem generalize, as well?

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- Does the geometric interpretation of the discrete acute set problem generalize, as well?
- In other words, is it true that, given the combinatorial interpretation, that any two $k-1$ dimensional hyperplanes in the set of points form an acute angle?

Acknowledgements

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- Professor Larry Guth
- Dr. Tanya Khovanova
- MIT PRIMES program
- My parents

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