Polya Enumeration Theorem

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Groups

Definition (Group)

A group is a set G together with a binary operation $\cdot$ such that the following axioms hold:

- $a \cdot b \in G$ for all $a, b \in G$ (closure)
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$ (associativity)
- $\exists e \in G$ such that for all $a \in G$, $a \cdot e = e \cdot a = a$ (identity)
- for each $a \in G$, $\exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (inverse)

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- In particular they can be functions under function composition
- group in which every element equals power of a single element is called a cylic group
- Ex. $\mathbb{Z}$ is a group under normal addition. The identity is 0 and the inverse of a is $-a$. Group is cyclic with generator 1

Group Actions

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Given a set X and a group G, a group action is a function $\phi : G \times X \to X$ satisfying the following axioms:

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- A group G acting on a set X is written $G \curvearrowright X$.
- In some cases the action of G on X is fairly obvious. Ex. If $G = S_n$ acts on $X = \{1, 2, \ldots, n\}$ then the action is seen to be permutations on n elements.

Orbits

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Let G be a group acting on X. The orbit of $x \in X$ is the set $\{gx \mid g \in G\}$. In other words, the orbit of x is the set of elements of X which can be obtained by composing x with various elements of G.

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- Set of orbits of X under the action of G is denoted X/G, the quotient of the action

Burnside's Lemma

Suppose $G \curvearrowright X$, and let $X^g = \{x \in X \mid gx = x\}$. In other words, X^g represents the set of elements in X fixed by g. Then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

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- This gives a formula for the number of orbits of X under the action of G
- $|X/G|$ represents the number of "distinct elements" of X under the action of G

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The result is

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{1}{8}(4096 + 4 + 16 + 4 + 256 + 4 + 16 + 4) = 550.$$

Polya Enumeration Theorem

Polya Enumeration Theorem (Unweighted)

Let X be a set with group action induced by a permutation group G on X. Let C be a set of colors on X, and let C^X be the set of functions $f : X \to C$. Then

$$|C^X/G| = \frac{1}{|G|} \sum_{g \in G} |C|^{c(g)},$$

where $c(g)$ is the number of cycles of g on X.

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- G must act on C^X to make sense; if q is a coloring and $g \in G$ then $g \cdot q(x) = q(g^{-1}x)$
- This is equivalent to Burnside's lemma because $|C|^{c(g)}$ also counts the number of points fixed by g. To be fixed, each element in a cycle has to have the same color.

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Count the number of graphs of 4 vertices up to isomorphism.

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Definition (Weight)

Suppose that the colors $c \in C$ have weights $w(c) \in \mathbb{Z}_0^+$. Define the weight of a coloring q to be the sum of the weights of the colors used, or

$$w(q) = \sum_{x \in X} w(q(x)).$$

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- weight of a coloring corresponds to number of edges; important for construction of generating function on colorings

Polya Enumeration Theorem

Definition (Cycle Index)

The cycle index of a permutation group G is defined as

$$Z_G(t_1, t_2, \dots) = \frac{1}{|G|} \sum_{g \in G} t_1^{m_1(g)} t_2^{m_2(g)} \dots,$$

where $m_i(g)$ is the number of cycles of length i in the cycle decomposition of g.

The group that acts on graphs with 6 edges is S_4. By inspection, there is 1 element of S_4 with 6 cycles of length 1, 9 elements with 2 one cycles and 2 two cycles, 8 elements with 2 3 cycles, and six elements with a two cycle and a four cycle. Therefore

$$Z_{S_4}(t_1, t_2, t_3, t_4) = \frac{1}{24}(t_1^6 + 9t_1^2 t_2^2 + 8t_3^2 + 6t_2 t_4).$$

Polya Enumeration Theorem

Generating Function for a set of colors

The generating function for a set of colors is

$$f(t) = f_0 + f_1 t + f_2 t^2 + \dots,$$

where f_i is the number of colors with weight i.

The generating function in the graph counting problem is therefore $1 + t$.

Polya Enumeration Theorem

Polya Enumeration Theorem (weighted)

The generating function of the number of colored arrangements by weight is given by

$$F(t) = Z_G(f(t), f(t^2), \dots).$$

Justification: we can show that

$$\sum_{\text{colorings fixed by } g} t^{w(q)} = \prod_i f(t^i)^{m_i(g)}.$$

We can then show that summing the above quantity across all $g \in G$ (and dividing by $|G|$) is the same as $F(t) = Z_G(f(t), f(t^2), \dots)$ through some easy but laborious bashing. Apply Burnside's on the set of colorings of weight i and then combine these for all i to deduce the result.

Polya Enumeration Theorem

On a graph with 4 vertices, we have

$$\begin{aligned} F(t) &= Z_G(1+t, 1+t^2, 1+t^3, 1+t^4) \\ &= \frac{1}{24}((1+t)^6 + 9(1+t)^2(1+t^2)^2 + 8(1+t^3)^2 + 6(1+t^2)(1+t^4)) \\ &= t^6 + t^5 + 2t^4 + 3t^3 + 2t^2 + t + 1. \end{aligned}$$

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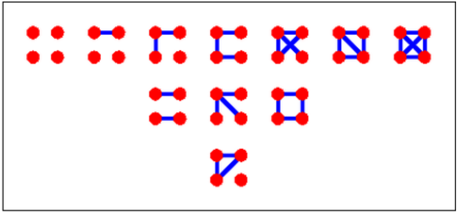
$$= t^6 + t^5 + 2t^4 + 3t^3 + 2t^2 + t + 1.$$

- one graph (K_4) with 6 edges, one (distinct) graph with 5 edges, 2 graphs with 4 edges, 3 graphs with 3 edges, 2 graphs with 2 edges, 1 graph with 1 edge, and 1 graph with no edges.



Polya Enumeration Theorem

Generating Function for a set of colors (multivariate)

Suppose that each color now has multiple weights $w_1(c), w_2(c), \ldots$. The new generating function $f(t_1, t_2, \ldots)$ for the set of colors is

$$f(t_1, t_2, \ldots) = \sum_{m_1, m_2, \cdots \in \mathbb{Z}_0^+} f_{m_1, m_2, \ldots} t_1^{m_1} t_2^{m_2} \ldots,$$

where $f_{m_1, m_2, \ldots}$ is the number of colors with first weight $w_1(c) = m_1$, second weight $w_2(c) = m_2$, etc.

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Polya Enumeration Theorem (multiweighted)

Given a set of colors with multiple weights, a set X, and a permutation group G on X, the generating function of the number of colored arrangements is given by

$$F(t_1, t_2, \dots) = Z_G(f(t_1, t_2, \dots), f(t_1^2, t_2^2, \dots), \dots).$$

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