Asymptotics of Visibility in Three Dimensions

Ezra Erives, Srinivasan Sathiamurthy, Theodore Baas

Mentor: Dr. Zarathustra Brady

MIT PRIMES Conference

May 19, 2018

Line of Sight

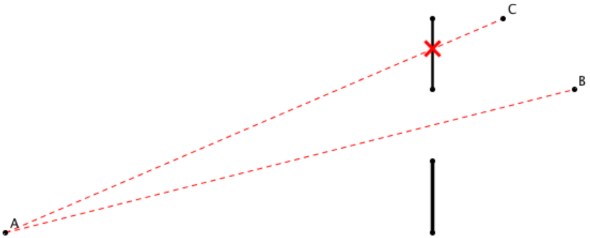


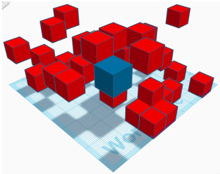
Figure: There exists a line of sight between A and B

Setting The Stage

You reside in a three dimensional grid-world.

1. Only unit cubes with integer coordinates
2. Some cubes are completely filled in, obstructing any line of sight through them. Such cubes will be referred to a **obstructing cubes**
3. A cube is visible from another cube if there exists a sight line connecting the two cubes

Setting The Stage



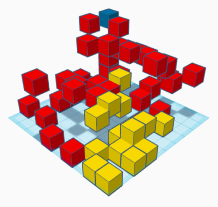


Figure: The cubes that are both obstructing and visible to the blue cube are painted red while the non-visible obstructing cubes are painted yellow.

Visibility on the Plane

The problem of two dimensional visibility has already been studied.

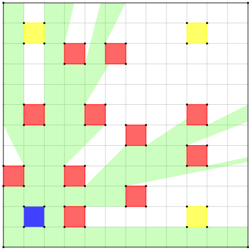


Figure: Visibility as seen from the blue square.

Visibility on the Plane

The problem of two dimensional visibility has already been studied.

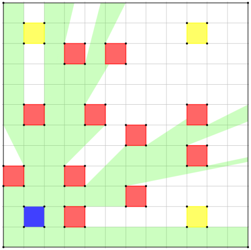


Figure: Visibility as seen from the blue square.

Theorem (Brady, 2010)

Let $P(n)$ be the largest possible number of visible obstructing squares. Then

$$n^{\frac{3}{2}} \leq P(n) \leq 68n^{\frac{3}{2}}$$

Our Question

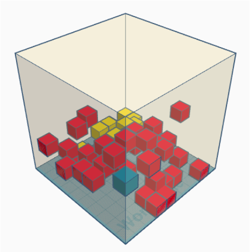


Figure: A set of obstructing cubes within the confines of an $n \times n \times n$ subgrid

Question

Asymptotically, as a function of n, what is the largest number of obstructing cubes visible to the blue cube if the position of the blue cube and the set of obstructing cubes can be altered?

Main Result

Lower Bound

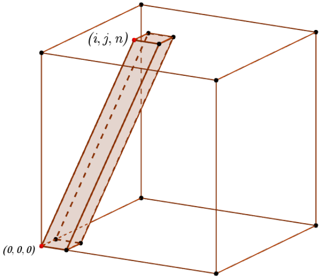
Let $P(n)$ be the largest possible number of visible obstructing cubes. Then

$$\Omega(n^{\frac{8}{3}}) \leq P(n)$$

Two convenient assumptions

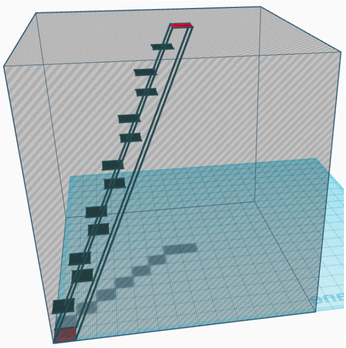
1. We assume that the observer is located at the origin (the corner of the $n \times n \times n$ block).
2. We assume that n is prime.

Parallelepiped Model



1. Consider the parallelepiped joining the unit square at the origin to the unit square with "smallest" vertex (i, j, n) on the top face of the $n \times n \times n$ cube.
2. Visibility is restricted to sight lines that are parallel to the edges of the parallelepiped connecting the corresponding edges of the two squares.

Parallelepiped Model



Additionally, for lower bound, we will only be considering whether or not the upper faces of the obstructing cubes are visible.

Parallelepiped Model

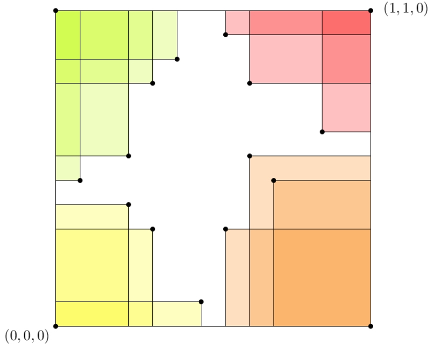


Figure: Taking the projections of the obstructing squares.

Now we project these squares onto the bottom unit square.

Partially Ordered Sets

A partially ordered set consists of the following:

1. A set P.
2. A relation "$<$"; the partial order on P.
3. For any $a, b \in P$, either $a < b$, $a > b$, or a and b are incomparable.

Partially Ordered Sets

A partially ordered set consists of the following:

1. A set P.
2. A relation "$<$"; the partial order on P.
3. For any $a, b \in P$, either $a < b$, $a > b$, or a and b are incomparable.

Example:

Take $\mathbb{Z}^3$ as our set. Let our relation be defined as follows. $(v_1, v_2, v_3) < (w_1, w_2, w_3)$ if and only if $v_i < w_i$ for $i \in (1, 2, 3)$.

- $(1, 2, 3) < (4, 5, 4)$ and $(1, 1, 2) > (0, 0, 0)$.
- $(1, 2, 3)$ and $(2, 2, 2)$ are incomparable.

Partially Ordered Sets

Let P be a partially ordered set.

1. A **chain** is a subset of P all of whose elements are comparable to each other.
   Example: $\{(3,4,5),(6,8,10),(7,24,25)\}$ is a chain of size three.
2. An **antichain** is a subset of P none of whose elements are comparable to each other
   Example: $\{(3,4,5),(2,8,10),(7,2,8)\}$ is an antichain of size three.

1. The **width** of P is the size of P's largest antichain

A Result of Brightwell

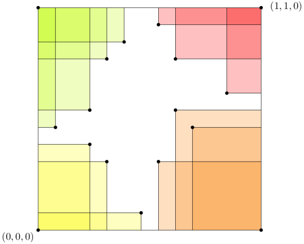
Let $P_k(n)$ denotes a random partially ordered set of n k-tuples, and let $W_k(n)$ be the width of such a set.

Theorem (Brightwell, 1992)

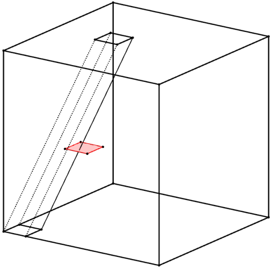
There exists a constant C such that, for each fixed k, almost every P_k(n) satisfies:

$$\left(\frac{1}{2}\sqrt{k} - C\right)n^{1-\frac{1}{k}} \leq W_k(n) \leq \frac{7}{2}kn^{1-\frac{1}{k}}$$

Partial Order For Lower Bound



(a) Projection



(b) Preimage of one of the red projections

The corner of the red square whose pre-image had z-coordinate k is the point $\left(1-\left\{\frac{k\cdot i}{n}\right\},1-\left\{\frac{k\cdot j}{n}\right\}\right)$. Can be represented by the ordered triple

$$\left(1-\left\{\frac{k\cdot i}{n}\right\},1-\left\{\frac{k\cdot j}{n}\right\},k\right)$$

Partial Order For Lower Bound

Consider the following poset:

$$P = \{(k \cdot i \pmod{p}, k \cdot j \pmod{p}, k) \mid 1 \leq k \leq p - 1)\}$$

Our relation is the same as in the example.

We are interested in constructing an antichain of maximal size.

Lemma

There exists an antichain of P with size at least $n^{\frac{2}{3}}$.

Our Lower Bound

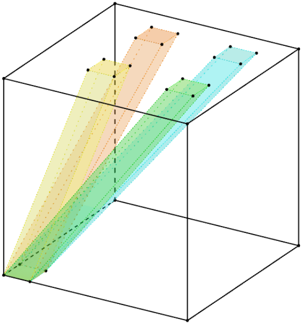


Figure: Summing over all of the parallelepipeds

Lower Bound

In an $n \times n \times n$ cube, the largest possible number of obstructing cubes visible is at least $\Omega(n^{\frac{8}{3}})$.

Future Work

1. Generalize the lower bound proof to higher dimensions
2. Prove our upper bound conjecture
3. Find an upper bound for non-restricted visibility

Far future

Study our problem's associated visibility graph structures

Acknowledgements

We would like to thank the following people:

1. Our mentor, Dr. Zarathustra Brady
2. Our parents
3. Dr. Tanya Khovanova
4. Dr. Slava Gerovitch and Prof. Pavel Etingof
5. MIT PRIMES