

The Hilbert Series of the Irreducible Quotient of the Polynomial Representation of the Rational Cherednik Algebra of Type A_{n-1} in Characteristic p for $p|n-1$

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Abstract

We study the irreducible quotient $\mathcal{L}_{t,c}$ of the polynomial representation of the rational Cherednik algebra $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$ of type A_{n-1} over an algebraically closed field of positive characteristic p where $p|n-1$. In the $t=0$ case, for all $c \neq 0$ we give a complete description of the polynomials in the maximal proper graded submodule $\ker \mathcal{B}$, the kernel of the contravariant form \mathcal{B} , and subsequently find the Hilbert series of the irreducible quotient $\mathcal{L}_{0,c}$. In the $t=1$ case, we give a complete description of the polynomials in $\ker \mathcal{B}$ when the characteristic $p=2$ and c is transcendental over \mathbb{F}_2 , and compute the Hilbert series of the irreducible quotient $\mathcal{L}_{1,c}$. In doing so, we prove a conjecture due to Etingof and Rains completely for $p=2$, and also for any $t=0$ and $n \equiv 1 \pmod{p}$. Furthermore, for $t=1$, we prove a simple criterion to determine whether a given polynomial f lies in $\ker \mathcal{B}$ for all $n = kp + r$ with r and p fixed.

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1 Introduction

The main object of our study in the current paper is the rational Cherednik algebra of type A_{n-1} , which we will denote by $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$ or simply by $\mathcal{H}_{t,c}(n)$. The Cherednik algebras, also known as Double Affine Hecke Algebras (DAHA), were introduced by Cherednik in [Che93] as a tool in his proof of Macdonald's conjectures about orthogonal polynomials for root systems. Since then Cherednik algebras have appeared in many different mathematical contexts and showed their independent significance. In particular, they are directly linked with exactly solvable models in physics, especially quantum Calogero-Moser systems (see [Eti07]), and quantum KZ equations (see [Che92]). In [Che05], Cherednik gives a more thorough exposition of the applications of DAHA in various mathematical areas, such as harmonic analysis, topology, elliptic

curve theory, Verlinde algebras, Kac-Moody algebras, and more. Another good source on general theory of Cherednik algebras is [EM10].

Representation theory of rational Cherednik algebras over the fields of characteristic zero was well studied, particularly in [Gor03] (in which the Hilbert series of irreducible representations is computed as well).

It is a topic of current research to understand the behaviour of irreducible representations of Cherednik algebras in positive characteristic (for example see [BC13], [DS16], [DS⁺14]). Our work can be seen as a follow up on the article [DS16]. In a similar fashion we restrict ourselves from the general rational Cherednik algebra $\mathcal{H}_{t,c}(\mathfrak{h}, G)$, to the case where $G = S_n$, \mathfrak{h} is a reflection representation of S_n and c is generic, but we also consider the case $t = 0$. In their paper Devadas and Sun have proven the formula for the Hilbert polynomial of the irreducible quotient of the polynomial representation $\mathcal{L}_{t,c}(\text{triv})$ for $p|n$. In our paper we work on the next case $n = kp + 1$. In this case we prove the formula for the Hilbert polynomial of $\mathcal{L}_{t,c}(\text{triv})$ for any pair (p, n) in the case $t = 0$ and for $p = 2$ in the case $t = 1$ and generic c . We also present Conjecture 1.11 due to Etingof and Rains for the Hilbert series in the general case $n = kp + r$, which holds for all of the cases that we, Devadas, and Sun have studied.

Conjecture (Etingof, Rains). Let $n = kp + r$, $0 \leq r < p$, $[k]_z = \frac{1-z^k}{1-z}$, $[k]_z! = [k]_z[k-1]_z \cdots [1]_z$, $Q_r(n, z) = \binom{n-1}{r-1} z^{r+1} + \sum_{i=0}^r \binom{n-r-2+i}{i} z^i$, and c be generic. The Hilbert series for $\mathcal{L}_{t,c}$ is of the form

$$h_{\mathcal{L}_{0,c}}(z) = [r]_z! [p]_z Q_r(n, z) \quad \text{and} \quad h_{\mathcal{L}_{1,c}}(z) = [p]_z^{n-1} [r]_z! [p]_z Q_r(n, z^p).$$

Note that $h_{\mathcal{L}_{1,c}}(z) = [p]_z^{n-1} h_0(z^p)$, which is discussed in [BC13].

In Section 1, we give an overview of the background, terminology, and past results in the representation theory of rational Cherednik algebras, particularly those which are relevant for the case that we work with. In Section 2, we prove Theorem 2.38 which solves the case $t = 0$ and $p|n-1$. In Section 3, we prove Theorem 3.11 which introduces a simple criterion to determine whether a polynomial is in the submodule $\ker \mathcal{B}$, and then prove Theorem 3.17 which solves the case $t = 1$ over a field of characteristic 2 and n odd.

1.1 Preliminaries

We will adopt notation from [BC13].

Fix an algebraically closed field \mathbb{k} of characteristic p for some prime p , and fix a positive integer $n > 1$. Fix $t, c \in \mathbb{k}$. Let S_n be the symmetric group on n elements, and σ_{ij} be the transposition swapping i and j . Consider the n -dimensional permutation representation of S_n , a vector space V spanned by y_1, y_2, \dots, y_n over \mathbb{k} , and its dual space V^* with dual basis x_1, x_2, \dots, x_n . Then consider the subrepresentation $\mathfrak{h} = \text{Span}\{y_i - y_j | i, j \in [n]\}$ over \mathbb{k} and its dual $\mathfrak{h}^* = V^*/(x_1 + x_2 + \cdots + x_n)$. Denote by $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$.

Definition 1.1. The *rational Cherednik algebra of type A_{n-1}* , or $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$, is the quotient of $\mathbb{k}S_n \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

- $[x_i, x_j] = 0$,
- $[y_i - y_j, y_\ell - y_k] = 0$,
- $[y_i - y_j, x_i] = t - c\sigma_{ij} - c \sum_{k \neq i} \sigma_{ik}$,
- $[y_i - y_j, x_k] = c\sigma_{ik} - c\sigma_{jk}$ for $k \neq i, j$.

Remark. One can also work with V and V^* instead of \mathfrak{h} and \mathfrak{h}^* , to define $\mathcal{H}_{t,c}(S_n, V)$. For $p \nmid n$, the Hilbert series of $\mathcal{L}_{t,c}$ (defined in Definition 1.10) are related via

$$\begin{aligned} h_{\mathcal{L}_{0,c}(S_n, V)}(z) &= h_{\mathcal{L}_{0,c}(S_n, \mathfrak{h})}(z), \\ h_{\mathcal{L}_{1,c}(S_n, V)}(z) &= (1 + z + \cdots + z^{p-1}) h_{\mathcal{L}_{1,c}(S_n, \mathfrak{h})}(z). \end{aligned}$$

Consider $S\mathfrak{h}$, the symmetric algebra of \mathfrak{h} , which we can think about as the subalgebra in the algebra of polynomials in y_i , generated by the differences $y_i - y_j$ for distinct i, j . Consider also $S\mathfrak{h}^*$ the symmetric algebra of \mathfrak{h}^* , which we can think about as the algebra of polynomials in x_i modulo the relation $(x_1 + \cdots + x_n)$; i.e., $S\mathfrak{h}^* \cong \mathbb{k}[x_1, \dots, x_n]/(x_1 + \cdots + x_n)$.

In [DS16], the PBW theorem is stated for $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$.

Theorem 1.2 (PBW¹). *We have the decomposition $\mathcal{H}_{t,c}(S_n, \mathfrak{h}) \simeq S\mathfrak{h} \otimes_{\mathbb{k}} \mathbb{k}[S_n] \otimes_{\mathbb{k}} S\mathfrak{h}^*$ as vector spaces.*

We can introduce a \mathbb{Z} grading on $\mathcal{H}_{t,c}$ by setting $\deg y = -1$ for $y \in \mathfrak{h}$, $\deg x = 1$ for $x \in \mathfrak{h}^*$, and $\deg \sigma = 0$ for $\sigma \in S_n$.

Since $\mathcal{H}_{t,c}(S_n, \mathfrak{h}) \cong \mathcal{H}_{at,ac}(S_n, \mathfrak{h})$ for any $a \in \mathbb{k}^\times$, it suffices to study the cases $t = 0$ and $t = 1$.

Definition 1.3. For parameters t, c , the *Dunkl operator* is defined as

$$D_{y_i} = t\partial_{x_i} - c \sum_{k \neq i} (x_i - x_k)^{-1} (1 - \sigma_{ik}) \in \text{End}(S\mathfrak{h}^*).$$

Remark. Define $D_{y_i - y_j} = D_{y_i} - D_{y_j}$. This uniquely extends to a homomorphism $S\mathfrak{h} \rightarrow \text{End}(S\mathfrak{h}^*)$, since the D_{y_i} commute.

Define a structure of an $\mathcal{H}_{t,c}$ -representation on $S\mathfrak{h}^*$ by sending $y_i - y_j \mapsto D_{y_i - y_j}$, $\sigma \mapsto \sigma$ (with the natural action on $S\mathfrak{h}^*$), and $x_i \mapsto x_i$ (acting by multiplication). The Dunkl operators satisfy the same commutator relations given in Definition 1.1, which means that this is indeed a representation (see [EM10], Proposition 2.14 and Theorem 2.15).

1.2 Verma Modules

There is another way to define a polynomial representation of $\mathcal{H}_{t,c}$, and that is via Verma modules.

Consider the trivial representation \mathbb{k} of $\mathbb{k}S_n \ltimes S\mathfrak{h}$; S_n acts by 1 and $y_i - y_j$ acts by 0.

Definition 1.4. The *Verma module* is the induced $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$ -module

$$\mathcal{M}_{t,c}(S_n, \mathfrak{h}, \mathbb{k}) = \mathcal{H}_{t,c}(S_n, \mathfrak{h}) \otimes_{\mathbb{k}S_n \ltimes S\mathfrak{h}} \mathbb{k}.$$

We will refer to it as $\mathcal{M}_{t,c}$.

Proposition 1.5. The Verma module $\mathcal{M}_{t,c}$ is isomorphic to $S\mathfrak{h}^*$ as graded vector spaces.

Proof. We have

$$\begin{aligned} \mathcal{M}_{t,c} &= \mathcal{H}_{t,c}(S_n, \mathfrak{h}) \otimes_{\mathbb{k}S_n \ltimes S\mathfrak{h}} \mathbb{k} = \mathbb{k}S_n \ltimes (S\mathfrak{h} \otimes S\mathfrak{h}^*) \otimes_{\mathbb{k}S_n \ltimes S\mathfrak{h}} \mathbb{k} \\ &\implies f(\mathbf{x})\sigma q(\mathbf{y}) \otimes 1 = f(\mathbf{x}) \otimes \sigma q(\mathbf{y})1 = f(\mathbf{x}) \otimes q(0), \end{aligned}$$

where f, q are polynomials, \mathbf{x} and \mathbf{y} are vectors (x_1, \dots, x_n) and (y_1, \dots, y_n) , and $\sigma \in S_n$. So, by the PBW theorem for Cherednik algebras, $\mathcal{M}_{t,c}$ has a basis of elements of $S\mathfrak{h}^*$ (polynomials in x_i). \square

Remark. The Verma module $\mathcal{M}_{t,c}$ has a grading by degree, setting $\deg x_i = 1$, similar to that of $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$.

This shows that $\mathcal{M}_{t,c} \cong S\mathfrak{h}^*$ as graded vector spaces, but they are also isomorphic as representations. We have the map $y_i \rightarrow D_{y_i}$, since the action of D_{y_i} and y_i are given by the same relations. We also have the following identification:

Proposition 1.6. We have an isomorphism $\mathcal{H}_{t,c}(S_n, \mathfrak{h})^{opp} \cong \mathcal{H}_{t,c}(S_n, \mathfrak{h}^*)$.

Definition 1.7. The contravariant form $\mathcal{B} : \mathcal{M}_{t,c}(S_n, \mathfrak{h}, \mathbb{k}) \times \mathcal{M}_{t,c}(S_n, \mathfrak{h}^*, \mathbb{k}) \rightarrow \mathbb{k}$ is a bilinear form satisfying the following properties:

- It is S_n -invariant: for $\sigma \in S_n$, then $\mathcal{B}(\sigma f, \sigma q) = \mathcal{B}(f, q)$.

¹PBW stands for Poincare-Birkhoff-Witt, and this case (for Cherednik algebras) is a generalization of the famous theorem for Lie algebras.

- For $x \in \mathfrak{h}^*$, $f \in \mathcal{M}_{t,c}(\mathfrak{h})$, $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)$, then $\mathcal{B}(xf, q) = \mathcal{B}(f, D_x(q))$.
- For $y \in \mathfrak{h}$, $f \in \mathcal{M}_{t,c}(\mathfrak{h})$, $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)$, then $\mathcal{B}(f, yq) = \mathcal{B}(D_y(f), q)$.
- The form is zero on elements of different degrees; i.e., if $f \in \mathcal{M}_{t,c}(\mathfrak{h})_i$ and $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)_j$ for $i \neq j$, then $\mathcal{B}(f, q) = 0$.
- If $f \in \mathcal{M}_{t,c}(\mathfrak{h})_0$ and $q \in \mathcal{M}_{t,c}(\mathfrak{h}^*)_0$, then $\mathcal{B}(f, q) = f \cdot q$.

Effectively, this contravariant form defines a bilinear form $\mathcal{B} : S\mathfrak{h} \times S\mathfrak{h}^* \rightarrow \mathbb{k}$ satisfying $\mathcal{B}(1, 1) = 1$, $\mathcal{B}(1, x_i) = 0$, and $\mathcal{B}(f(y), q(x)) = \mathcal{B}(1, D_{f(y)}(q(x))) = [x^0]f(D_y)q(x)$ where $[x^0]$ denotes the constant term when $f(D_y) \in S\mathfrak{h}$ acts on $q(x) \in S\mathfrak{h}^*$.

Definition 1.8. Define an $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$ representation $\mathcal{L}_{t,c} = \mathcal{M}_{t,c} / \ker \mathcal{B}$, where $\ker \mathcal{B} = \{x \in S\mathfrak{h}^* | \mathcal{B}(y, x) = 0 \ \forall \ y \in S\mathfrak{h}\}$.

Note that $\ker \mathcal{B}$ is a subrepresentation and due to the properties of \mathcal{B} , it is also an ideal in the algebra of polynomials.

Definition 1.9. Define the *Baby Verma module* $\mathcal{N}_{t,c}(S_n, \mathfrak{h}, \mathbb{k})$ as follows:

- If $t = 1$, then $\mathcal{N}_{1,c} = \mathcal{M}_{1,c} / \left((S\mathfrak{h}^*)^{S_n} \right)_+^p \mathcal{M}_{1,c}$, or $S\mathfrak{h}^*$ modulo the ideal generated by the S_n -invariant polynomials of positive degree raised to the p^{th} power.
- If $t = 0$, then $\mathcal{N}_{0,c} = \mathcal{M}_{0,c} / \left((S\mathfrak{h}^*)^{S_n} \right)_+ \mathcal{M}_{0,c}$, or $S\mathfrak{h}^*$ modulo the ideal generated by the S_n -invariant polynomials of positive degree.

It follows that $\mathcal{L}_{t,c} = \mathcal{N}_{t,c} / \ker \mathcal{B}$, because $\left((S\mathfrak{h}^*)^{S_n} \right)_+^p \mathcal{M}_{1,c} \subset \ker \mathcal{B}$.

We have the following statements from, e.g., [BC13]:

1. $\left((S\mathfrak{h}^*)^{S_n} \right)_+$ is finitely generated over \mathbb{k} . (Fundamental theorem on symmetric polynomials)
2. All $\mathcal{N}_{t,c}$ (and thus $\mathcal{L}_{t,c}$) are finite dimensional.
3. $\ker \mathcal{B}$ is a maximal proper graded submodule of $\mathcal{M}_{t,c}$.
4. $\mathcal{L}_{t,c}$ is irreducible.

Definition 1.10. We define the Hilbert series of an \mathbb{N} -graded module M to be $h_M(z) = \sum_{i \geq 0} \dim M[i] z^i$, where $M[i]$ is the i^{th} graded component of M .

The quotient $\mathcal{L}_{t,c}$ inherits the grading from $\mathcal{M}_{t,c}$, hence we assign to it the Hilbert series $h_{\mathcal{L}_{t,c}}(z) = \sum_{i \geq 0} \dim \mathcal{L}_{t,c}[i] z^i$. In the general case, Etingof and Rains present the following (yet unpublished) conjecture for the Hilbert series. Let $n = kp + r$, $0 \leq r < p$,

$$[k]_z = \frac{1 - z^k}{1 - z}, \quad [k]_z! = [k]_z [k-1]_z \cdots [1]_z, \quad Q_r(n, z) = \binom{n-1}{r-1} z^{r+1} + \sum_{i=0}^r \binom{n-r-2+i}{i} z^i.$$

Conjecture 1.11 (Etingof, Rains). The Hilbert series for $\mathcal{L}_{t,c}$, with c generic, is of the form

$$h_{\mathcal{L}_{0,c}}(z) = [r]_z! [p]_z Q_r(n, z) \quad \text{and} \quad h_{\mathcal{L}_{1,c}}(z) = [p]_z^{n-1} [r]_z! [p]_z! Q_r(n, z^p).$$

Remark. In the case $t = 0$, we merely need $c \neq 0$ to be generic.

Definition 1.12. A *singular polynomial* is a polynomial $f \in S\mathfrak{h}^*$ which lies in the simultaneous kernel of all Dunkl operators $D_{y_i - y_j}$, i.e. $D_{y_i - y_j} f = 0$ for all i, j .

The singular polynomials generate a submodule lying in $\ker \mathcal{B}$, thus (in positive characteristic) we would like to find such generators to understand $\ker \mathcal{B}$. This would allow us to understand $\mathcal{L}_{t,c}$.

1.3 Characteristic 0

The singular polynomials for characteristic 0 are known; see, for example, [EM10].

Proposition 1.13. If $\text{char } \mathbb{k} = 0$ and $c = \frac{r}{n}$ for some r not divisible by n , then the singular polynomials for $t = 1$ are $\text{Res}_\infty \left[\frac{dz}{z-x_j} \prod_{i=1}^n (z-x_i)^c \right]$ for $j = 1, 2, \dots, n-1$. (See [CE03], Proposition 3.1, for original reference, or [DS⁺14], Proposition 1.2.)

The lowest-weight irreducible representations of the rational Cherednik algebra associated to S_n in characteristic 0 are studied in [Gor03], and he computes their Hilbert series.

1.4 The case where $p|n$, by Devadas and Sun

In [DS16], Devadas and Sun found the Hilbert polynomial for the representation of the Cherednik algebra $\mathcal{L}_{1,c}$ where $p|n$.

Define the polynomials

$$g(z) = \prod_{j=1}^n (1 - x_j z) \quad \text{and} \quad F(z) = \sum_{m=0}^{p-1} \binom{c}{m} (g(z) - 1)^m.$$

Then for $i = 1, 2, \dots, n-1$, define $f_i = [z^p] \frac{F(z)}{1-x_i z}$.

Devadas and Sun showed that the polynomials f_i are singular, linearly independent and homogeneous degree p . They also show that if $I_c = \langle f_1, \dots, f_{n-1} \rangle \subset M_{t,c}$, then $\mathcal{M}_{t,c}/I_c$ is a complete intersection for generic c . In doing so, they show that for generic c , the Hilbert series of $\mathcal{L}_{1,c} = \mathcal{M}_{1,c}/I_c$ is $h(z) = \left(\frac{1-z^p}{1-z} \right)^{n-1}$.

1.5 Some results from Balagovic and Chen

In [BC13], the following Hilbert series are described.

Proposition 1.14. The Hilbert polynomial for $\mathcal{N}_{1,c}$ is $h_{\mathcal{N}_{1,c}}(z) = \frac{(1-z^{2p})(1-z^{3p})\dots(1-z^{np})}{(1-z)^{n-1}}$ while the Hilbert polynomial for $\mathcal{N}_{0,c}$ is $h_{\mathcal{N}_{0,c}}(z) = \frac{(1-z^2)(1-z^3)\dots(1-z^n)}{(1-z)^{n-1}}$.

Proposition 1.15. The Hilbert polynomial for $\mathcal{L}_{1,c}$ is $h_{\mathcal{L}_{1,c}}(z) = \left(\frac{1-z^p}{1-z} \right)^{n-1} h(z^p)$ for some polynomial h with nonnegative integer coefficients.

1.6 Main Results

We find the Hilbert series for $\mathcal{H}_{t,c}(S_n, \mathfrak{h})$ over fields \mathbb{k} of characteristic $p|n-1$. The main theorems are Theorem 2.38 (which generalizes Theorem 2.12), Theorem 3.11, and Theorem 3.17. Theorem 2.38 states that the Hilbert series for $t = 0$ is $h_{\mathcal{L}_{0,c}}(z) = \left(\frac{1-z^p}{1-z} \right) (1 + (n-2)z + z^2)$. Theorem 3.11 gives a simple, computation-based criterion for whether a given polynomial $f \in \ker \mathcal{B}$ for all $n \equiv 1 \pmod{p}$: one only needs to check this condition for small (and finitely many) n . Theorem 3.17 states that the Hilbert series for $t = 1$ and $p = 2$ is $h_{\mathcal{L}_{1,c}}(z) = (1+z^2)(1+z)^{n-1}(1+(n-2)z^2+z^4)$.

2 The case $t = 0$

Note that in this case, the Dunkl operator is just

$$D_{y_i - y_j} = -c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} + c \sum_{\ell \neq j} \frac{1 - \sigma_{j\ell}}{x_j - x_\ell},$$

so the parameter c does not matter (so long as it is nonzero, in which case the representation is trivial) and we may assume that $c = 1$.

There is a basis of $\mathcal{M}_{t,c}$ consisting of elements of $\mathbb{F}_p[x_1, x_2, \dots, x_n]$, and hence we assume all coefficients are from \mathbb{F}_p .

We proceed degree by degree and analyze each subspace $\mathcal{M}_{0,c}[i]$ starting from $i = 0$ and going up. We will find some polynomials which constitute a subspace $J[i] \subset \ker \mathcal{B}$ and then find bases of $\mathcal{M}_{0,c}/J[i]$. We compute the action of the Dunkl operators to explicitly show that these are not in $\ker \mathcal{B}$, hence $J[i] = \mathcal{B}[i]$.

2.1 Characteristic $p = 2$

We will first examine the case when the characteristic is 2. Frequently, we will make the substitution $x_n = -(x_1 + x_2 + \dots + x_{n-1}) = x_1 + x_2 + \dots + x_{n-1}$.

Proposition 2.1. For $i \neq j$, the polynomials $x_i^2 + x_i x_j + x_j^2$ for $i \neq j$ are singular.

Proof. It suffices to prove that the action of the Dunkl operators $D_{y_1-y_r}$ for $r = 2, 3, \dots, n$ on $f = x_1^2 + x_1 x_2 + x_2^2$ results in 0. If $r = 2$, then

$$D_{y_1-y_2} f = \left[\sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} + \sum_{k \neq 2} \frac{1 - \sigma_{2k}}{x_2 - x_k} \right] (x_1^2 + x_1 x_2 + x_2^2) = \sum_{k \neq 1, 2} [-(x_1 + x_2 + x_k) + (x_1 + x_2 + x_k)] = 0.$$

If $r \neq 2$ then we see that the first sum is the same, $\sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} f = (n-2)x_1 + (n-2)x_2 - (x_1 + x_2) = 0$. The second sum, $\sum_{k \neq r} \frac{1 - \sigma_{kr}}{x_r - x_k} f$, is 0 whenever $k \neq 1, 2$. But for $k = 1, 2$ we obtain $-2(x_1 + x_2 + x_r) = 0$. Hence all $D_{y_1-y_r} f = 0$. \square

Proposition 2.2. The dimension of $\mathcal{L}_{0,c}[0]$ is 1.

Proof. All elements are constants. \square

Proposition 2.3. The dimension of $\mathcal{L}_{0,c}[1]$ is $n - 1$.

Proof. The basis consists of x_1, x_2, \dots, x_{n-1} after the substitution for x_n . Suppose a singular polynomial existed $f = \sum_{i < n} a_i x_i$. Note that $D_{y_i-y_j} x_i = 1$ while $D_{y_i-y_j} x_k = 0$. Then $D_{y_i-y_n} f = a_i = 0$, so $f = 0$. \square

Proposition 2.4. The dimension of $\mathcal{L}_{0,c}[2]$ is $n - 1$.

Proof. After the substitution for x_n , we see that $\dim \mathcal{M}_{0,c}[2] = n - 1 + \binom{n-1}{2}$, with a basis given by x_i^2 and $x_i x_j$ for $i, j < n$. The singular polynomials $x_i^2 + x_i x_j + x_j^2$ for $i, j \leq n - 1$ (Proposition 2.1) are all linearly independent (each contains a unique $x_i x_j$), hence they span a space of dimension $\binom{n-1}{2}$. Subtracting the dimensions shows that $\dim \mathcal{L}_{0,c}[2] \leq n - 1$. Now suppose there existed another singular polynomial f . Substitute for x_n and then remove all $x_i x_j$ terms by adding in $x_i^2 + x_i x_j + x_j^2$. This new polynomial is $g = x_1^2 + \dots + x_C^2$ after a permutation of indices for some $1 \leq C < n$. But

$$-D_{y_1} g = \sum_{k > C} \frac{1 - \sigma_{1k}}{x_1 - x_k} g = (n - C)x_1 + x_{C+1} + x_{C+2} + \dots + x_n = Cx_1 + x_2 + \dots + x_C.$$

Then note that

$$D_{y_n} g = \sum_{k \neq n} \frac{1 - \sigma_{nk}}{x_n - x_k} g = (x_1 + \dots + x_C) + Cx_n.$$

Thus $D_{y_n-y_1} g = (C+1)x_1 + Cx_n = x_1 + C(x_2 + \dots + x_{n-1})$, which is never 0, so $\dim \mathcal{L}_{0,c}[2] = n - 1$. \square

Proposition 2.5. The dimension of $\mathcal{L}_{0,c}[3]$ is 1.

Proof. First, consider some of the possible polynomials in $\ker \mathcal{B}$. The polynomials which come from the singular polynomials are of the form $x_i^3 + x_i^2 x_j + x_i x_j^2$ and $x_i^2 x_k + x_i x_j x_k + x_j^2 x_k$. We will now build up the polynomials in $\ker \mathcal{B}[3]$.

Lemma 2.6. The polynomials $x_i^3 + x_j^3 \in \ker \mathcal{B}$.

Proof. We have $(x_i + x_j)(x_i^2 + x_i x_j + x_j^2) = x_i^3 + x_j^3 \in \ker \mathcal{B}$. \square

Next, we split into two cases.

Lemma 2.7. *If $n \equiv 3 \pmod{4}$, then $x_i^3 \in \ker \mathcal{B}$.*

Proof. Consider the sum $S = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} (x_i^3 + x_i^2 x_j + x_i x_j^2)$. Clearly $S \in \ker \mathcal{B}$ and $X = x_1^3 + \dots + x_n^3 = (\sum_{h < n} x_h)^3 + \sum_{k < n} x_k^3 \in \ker \mathcal{B}$. Note that $S = x_1^3 + x_3^3 + x_5^3 + \dots + x_{n-2}^3 + X$. Therefore $x_1^3 + x_3^3 + x_5^3 + \dots + x_{n-2}^3 \in \ker \mathcal{B}$. But this is a sum of $\frac{n-1}{2}$ cubes, which is odd. Using Lemma 2.6 to remove $\frac{n-3}{2}$ of the cubes, we find that $x_i^3 \in \ker \mathcal{B}$. \square

Lemma 2.8. *If $n \equiv 1 \pmod{4}$, then $x_i^3 \in \ker \mathcal{B}$.*

Proof. Consider S as in Lemma 2.7, but removing the terms $x_i^3 + x_i^2 x_j + x_i x_j^2$ for $i, j \in \{n-1, n-2, n-3\}$. Denote this by T . Then adding the terms

$$\begin{aligned} & (x_{n-3}^2 x_{n-4} + x_{n-3}^2 x_{n-2} + x_{n-4} x_{n-3} x_{n-2}) + (x_{n-3}^2 x_{n-4} + x_{n-3}^2 x_{n-1} + x_{n-3} x_{n-4} x_{n-1}) \\ & + (x_{n-2}^2 x_{n-4} + x_{n-2}^2 x_{n-3} + x_{n-4} x_{n-2} x_{n-3}) + (x_{n-2}^2 x_{n-4} + x_{n-2}^2 x_{n-1} + x_{n-4} x_{n-2} x_{n-1}) \\ & + (x_{n-1}^2 x_{n-4} + x_{n-1}^2 x_{n-3} + x_{n-4} x_{n-1} x_{n-3}) + (x_{n-1}^2 x_{n-4} + x_{n-1}^2 x_{n-2} + x_{n-4} x_{n-1} x_{n-2}) \\ & = x_{n-3}^2 x_{n-2} + x_{n-3}^2 x_{n-1} + x_{n-2}^2 x_{n-3} + x_{n-2}^2 x_{n-1} + x_{n-1}^2 x_{n-2} + x_{n-1}^2 x_{n-3}, \end{aligned}$$

we obtain $x_1^3 + x_3^3 + x_5^3 + \dots + x_{n-4}^3 + X \in \ker \mathcal{B}$. Once again, this yields an odd number of cubes, hence $x_i^3 \in \ker \mathcal{B}$. \square

Lemma 2.9. *The polynomials $x_i x_j x_k \in \ker \mathcal{B}$.*

Proof. Consider

$$\begin{aligned} & (x_j^2 x_i + x_j^2 x_k + x_i x_j x_k) + (x_j^2 x_i + x_j^2 x_l + x_j x_i x_l) + (x_k^2 x_i + x_k^2 x_j + x_i x_j x_k) + \\ & (x_k^2 x_i + x_k^2 x_l + x_i x_k x_l) + (x_l^2 x_i + x_l^2 x_j + x_i x_j x_l) + (x_l^2 x_i + x_l^2 x_k + x_i x_k x_l) \\ & = x_j^2 x_k + x_j^2 x_l + x_k^2 x_j + x_k^2 x_l + x_l^2 x_k + x_l^2 x_j \end{aligned}$$

and

$$\begin{aligned} & (x_i x_j x_k + x_i^2 x_j + x_k^2 x_j) + (x_i x_j x_k + x_i^2 x_k + x_j^2 x_k) + (x_i x_j x_k + x_j^2 x_i + x_k^2 x_i) \\ & = x_i x_j x_k + x_i^2 x_j + x_i^2 x_k + x_j^2 x_i + x_j^2 x_k + x_k^2 x_i + x_k^2 x_j. \end{aligned}$$

Both are in the kernel. Subtracting yields $x_i x_j x_k$, which is also in the kernel. \square

Lemma 2.10. *All monomials of the form $x_i^2 x_j$ are equivalent modulo $\ker \mathcal{B}$.*

Proof. Fix a monomial $x_a^2 x_b$. Then $x_a^2 x_b + x_a x_b x_k + x_k^2 x_b \in \ker \mathcal{B}$, which implies that $x_a^2 x_b + x_k^2 x_b \in \ker \mathcal{B}$ for all k (since $x_a x_b x_k \in \ker \mathcal{B}$ from Lemma 2.9). Thus $x_a^2 x_b = x_k^2 x_b$ in $\mathcal{L}_{0,c}$. Similarly, $x_b^3 + x_b^2 x_k + x_b x_k^2 \in \ker \mathcal{B}$, so $x_b^2 x_k = x_b x_k^2$ in $\mathcal{L}_{0,c}$. From these two equalities, we find that $x_a^2 x_b = x_i^2 x_j$ for all a, b, i, j . \square

All terms are either of the form x_i^3 or $x_i^2 x_j$ or $x_i x_j x_k$. Using Lemmas 2.7, 2.8, 2.9, and 2.10, we conclude that $\dim \mathcal{L}_{0,c}[3] \leq 1$. But we can easily see that the dimension is not zero. If it were, then some $x_i^2 x_j$ would necessarily be in $\ker \mathcal{B}$. Without loss of generality, suppose $x_1^2 x_2 \in \ker \mathcal{B}$. But then $D_{y_3-y_4} x_1^2 x_2 = x_2 x_3 - x_2 x_4 \notin \ker \mathcal{B}$, so this is impossible. \square

Proposition 2.11. *The dimension of $\mathcal{L}_{0,c}[j]$ is 0 for $j > 3$.*

Proof. We show that $\ker \mathcal{B}[4]$ contains all polynomials. It suffices to check for the appearance of all monomials of the form $x_i^4, x_i^3 x_j, x_i^2 x_j^2, x_i^2 x_j x_k, x_i x_j x_k x_\ell$ in $\ker \mathcal{B}$. We can obtain any monomial with at least 3 terms from $x_i x_j x_k \in \ker \mathcal{B}[3]$ (2.9). We also can obtain any monomial of the form x_i^4 or $x_i^3 x_j$ because $x_i^3 \in \ker \mathcal{B}[3]$ (2.7, 2.8). It remains to show that we can obtain all monomials of the form $x_i^2 x_j^2$. But we know that $x_i^2 x_j + x_i x_j^2 \in \ker \mathcal{B}[3]$ (2.10). Multiplying by x_i yields $x_i^3 x_j + x_i^2 x_j^2 \in \ker \mathcal{B}[4]$, and since $x_i^3 x_j \in \ker \mathcal{B}[4]$, then so is $x_i^2 x_j^2$, and we find that $\dim \mathcal{L}_{0,c}[4] = 0$. By the properties of the contravariant form (particularly that $\ker \mathcal{B}$ is an ideal), the Proposition follows as well. \square

Combining Propositions 2.2, 2.3, 2.4, 2.5, and 2.11, we conclude with the Hilbert series.

Theorem 2.12. *The Hilbert series for $\mathcal{L}_{0,c}$ when $p = 2$ is $h_{\mathcal{L}_{0,c}}(z) = (1+z)(1+(n-2)z+z^2)$.*

Proof. By expanding, we may compare coefficients and verify that they match. \square

2.2 Characteristic p is odd

We will now study the case where the characteristic p is odd. We will frequently use the substitution $x_n = -x_1 - x_2 - \cdots - x_{n-1}$.

Proposition 2.13. The dimension of $\mathcal{L}_{0,c}[0]$ is 1.

Proof. All elements are constants. \square

Proposition 2.14. The dimension of $\mathcal{L}_{0,c}[1]$ is $n-1$.

Proof. After the substitution for x_n , we may assume some $f = \sum_{i < n} a_i x_i$ is singular. Note that $D_{y_n} f = \sum a_i$ and $-D_{y_1} f = (n-1)a_1 - (a_2 + a_3 + \cdots + a_{n-1}) = a_1 - \sum_{i < n} a_i$. Then $D_{y_n - y_i} = a_i = 0$, hence $f = 0$ and all x_i for $i < n$ span $\mathcal{L}_{0,c}[1]$. \square

Proposition 2.15. For distinct $i, j, k \in [n]$, the polynomials $(x_j - x_k)(x_i - x_j - x_k)$ are singular.

Proof. Without loss of generality, assume $i = 1, j = 2$, and $k = 3$, and denote $f = (x_2 - x_3)(x_1 - x_2 - x_3)$. Then it suffices to check the action of the Dunkl operators $D_{y_1 - y_r}$ for $r = 2, 3, 4, \dots, n$.

Notice that the cases $r = 2$ and $r = 3$ are the same, because the polynomial $(x_2 - x_3)(x_1 - x_2 - x_3)$ is invariant under the operator $-\sigma_{23}$, and since the Dunkl operator is linear, they yield the same result.

First, note that

$$\begin{aligned} -D_{y_1} f &= (x_1 + x_2 - x_3) + (x_2 - x_1 - x_3) + (x_2 - x_3) \sum_{s \neq 1, 2, 3} \frac{x_1 - x_s}{x_1 - x_s}, \\ &= x_1 + x_2 - x_3 + x_2 - x_1 - x_3 + (n-3)(x_2 - x_3) = (n-1)(x_2 - x_3) = 0. \end{aligned}$$

Now we check the action of D_{y_2} . We find that

$$\begin{aligned} -D_{y_2} f &= (x_3 - x_1 - x_2) + 2(x_1 - x_2 - x_3) + \sum_{s \neq 1, 2, 3} (x_1 - x_2 - x_k), \\ &= (n-1)(x_1 - x_2) - \sum_{s \in [n]} x_s = 0. \end{aligned}$$

Finally, it remains to check $D_{y_r} f$ for $r > 3$. Note that this leaves

$$D_{y_r} f = (x_3 - x_2) + (x_r + x_2 - x_1) + (x_1 - x_3 - x_r) = 0.$$

Thus $D_{y_i} f = 0$ for all i , which implies that $(x_j - x_k)(x_i - x_j - x_k)$ is singular. \square

Proposition 2.16. The following is a basis for the degree 2 singular polynomials:

- $(x_1 - x_i)(x_2 - x_1 - x_i)$ for $i = 3, 4, \dots, n-1$

- $(x_j - x_2)(x_i - x_j - x_2)$ for each unordered pair (i, j) where $i, j \leq n - 1$, with the order of i and j chosen arbitrarily.

Proof. Perform the substitution for x_n and consider only indices between 1 and $n - 1$ inclusive. Notice that this does not affect singular polynomials which depend on x_n , since those are in fact a combination of singular polynomials without an x_n :

$$\sum_{j, k \neq i \leq n-1} (x_j - x_k)(x_i - x_j - x_k) = \sum_{j, k \neq i \leq n-1} (x_i - x_j)(x_k - x_i - x_j) = 0.$$

Lemma 2.17. *We can obtain all singular polynomials of the form $(x_j - x_2)(x_i - x_j - x_2)$ for $3 \leq i, j \leq n - 1$ using the aforementioned basis.*

Proof. Suppose for a given (unordered) pair (i, j) the (ordered) polynomial $(x_j - x_2)(x_i - x_j - x_2)$ is part of the basis. Then we obtain the alternate polynomial via

$$(x_i - x_2)(x_j - x_i - x_2) = (x_1 - x_j)(x_2 - x_1 - x_j) - (x_1 - x_i)(x_2 - x_1 - x_j) - (x_j - x_2)(x_i - x_j - x_2).$$

□

Lemma 2.18. *We can obtain all singular polynomials containing an x_2 using the aforementioned basis.*

Proof. Lemma 2.17 yields all singular polynomials containing an x_2 but not an x_1 . So now assume $j = 1$. We show that we can obtain the polynomials $(x_j - x_2)(x_1 - x_j - x_2)$ and $(x_1 - x_2)(x_j - x_1 - x_2)$. But

$$(x_1 - x_2)(x_j - x_1 - x_2) = (x_j - x_2)(x_1 - x_j - x_2) + (x_1 - x_j)(x_2 - x_1 - x_j),$$

and two of those polynomials are already in the basis, hence all three are generated by the basis. □

It suffices to note that

$$(x_j - x_k)(x_i - x_j - x_k) = (x_j - x_2)(x_i - x_j - x_2) - (x_k - x_2)(x_i - x_k - x_2).$$

□

Proposition 2.19. The dimension of $\mathcal{L}_{0,c}[2]$ is n .

Proof. From the basis in Proposition 2.16, we know that $\dim \mathcal{L}_{0,c}[2] \leq \dim \mathcal{M}_{0,c}[2] - |\text{basis}| = \frac{n(n-1)}{2} - \frac{n^2-3n}{2} = n$. To show equality, we show that $D_{y_1-y_2}x_i^2$ for $i = 1, 2, \dots, n - 1$ and $D_{y_1-y_2}x_1x_2$ are all linearly independent, showing that those n polynomials generate all of $\mathcal{L}_{0,c}[2]$. (Clearly, these n polynomials generate the entire subspace: first, we can obtain all polynomials of the form x_i^2 . From the singular polynomials which contain a term $x_k(x_i - x_j)$, we only need a single monomial of the form x_ix_j to generate all of the other monomials of the form x_ax_b . This covers every possible monomial of degree 2.)

For $r \neq 1, 2$, we have

$$\begin{aligned} D_{y_1-y_2}x_1^2 &= -x_2, \\ D_{y_1-y_2}x_2^2 &= -x_1, \\ D_{y_1-y_2}x_r^2 &= x_1 - x_2, \\ D_{y_1-y_2}x_1x_2 &= x_2 - x_1. \end{aligned}$$

Suppose such a linear combination existed as $f = a_1x_1^2 + a_2x_2^2 + \dots + a_{n-1}x_{n-1}^2 - bx_1x_2$. Then we obtain the relations $a_1 + a_2 = 0$, and $a_1 + a_3 + a_4 + \dots + b = 0$. But by symmetry (using other Dunkl operators $D_{y_1-y_k}$ for $k \neq 2$), we obtain that $a_1 + a_k = 0$. Again by symmetry, we obtain that $a_i + a_j = 0$, which implies that all a_i are 0. Obviously, x_1x_2 is not singular, so we have the conclusion. □

Proposition 2.20. The dimension of $\mathcal{L}_{0,c}[3]$ is n for $p > 3$.

Proof. We perform the substitution for x_n and show that the polynomials x_i^3 for $i < n$ and $x_a^2 x_b$ for fixed $a, b < n$, combined with multiples of the singular polynomials in degree 2, will generate all of $\mathcal{M}_{0,c}[3]$.

We first show that they generate all of $\mathcal{M}_{0,c}[3]$. Note that we have the polynomial $x_k(x_k - x_j)(x_i - x_k - x_j) = -x_k^3 + x_j^2 x_k + x_i x_k^2 - x_i x_j x_k$ from $\ker \mathcal{B}$. Since x_k^3 can be produced, we can remove it, so we have $x_j^2 x_k + x_i x_k^2 - x_i x_j x_k$. Now take the polynomial $x_k^2 x_i + x_j x_i^2 - x_i x_j x_k$, which is simply the permutation $(ijk) \in S_n$ acting on the previous polynomial. Their difference yields $x_k x_j^2 - x_j x_i^2$. Again by S_n action, we can also produce the polynomial $x_l x_j^2 - x_j x_i^2$. Their difference yields

$$(x_k x_j^2 - x_j x_i^2) - (x_l x_j^2 - x_j x_i^2) = x_i^2 x_j - x_l^2 x_j.$$

Thus, if any one term of the form $x_i^2 x_j$ were not in $\ker \mathcal{B}[3]$, using $x_1^2 x_2$ we can produce anything of the form $a^2 b$. Since all terms of the form x_i^3 are already produced, we only need terms of the form $x_i x_j x_k$, which can be easily obtained from $x_k(x_k - x_j)(x_i - x_j - x_k)$. Hence the dimension is at most n .

To show that the dimension is exactly n , we will show that no linear combination of $x_1^3, x_2^3, \dots, x_{n-1}^3, x_1^2 x_2$ is in $\ker \mathcal{B}$. We will perform computations in $\mathcal{L}_{0,c}[2]$ (which was already found in Proposition 2.19) for the $D_{y_1-y_2} x_r^3$, subtracting the $(x_1 - x_2)(x_r - x_1 - x_2)$ polynomial, since that polynomial is in $\ker \mathcal{B}$ and thus is 0 in $\mathcal{L}_{0,c}$. For $r \neq 1, 2$, we have

$$\begin{aligned} D_{y_1-y_2} x_1^3 &= x_1^2 - x_1 x_2 - x_2^2, \\ D_{y_1-y_2} x_2^3 &= x_1^2 + x_1 x_2 - x_2^2, \\ D_{y_1-y_2} x_r^3 &= x_1^2 + x_1 x_r - x_2 x_r + x_2^2 = 2x_1^2, \\ D_{y_1-y_2} x_1^2 x_2 &= x_2^2 - x_1^2. \end{aligned}$$

Suppose a linear combination existed, $f = a_1 x_1^3 + \dots + a_{n-1} x_{n-1}^3 + b x_1^2 x_2$. We must have that $D_{y_1-y_2} f = 0$ in $\mathcal{L}_{0,c}[2]$, in accordance with the action of the Dunkl operators above. This means that $D_{y_1-y_2} f + \sum_{i,j,k} d_{ijk}(x_j - x_k)(x_i - x_j - x_k) = 0$ in $\mathcal{M}_{0,c}[2]$. In particular, any singular polynomial from the summation introduces two $x_i x_j$ monomials, whereas there is only one from the action of $D_{y_1-y_2} f$ (which is $x_1 x_2$). Therefore, by parity, all $d_{ijk} = 0$ (the monomials of the form $x_i x_j$ will never cancel with each other or the $x_1 x_2$ term). Thus we may only concern ourselves with the results from the above computations. From the action of $D_{y_1-y_2}$, we have $a_1 = a_2$. By symmetry $a_1 = a_2 = \dots = a_{n-1}$, so let $a_i = a$. Then $a D_{y_1-y_2} (x_1^3 + \dots + x_{n-1}^3) = 2a(n-2)x_1^2 - 2ax_2^2 = -2a(x_1^2 + x_2^2)$. Comparing the coefficient of x_1^2 and x_2^2 , we find that $a = -a \implies a = 0$ and $b = 0$ as well. Hence no such linear combination exists. \square

Proposition 2.21. The polynomials $x_i^2 x_j - x_i x_j^2 \in \ker \mathcal{B}$.

Proof. Let $f = x_3^2 x_4 - x_3 x_4^2$. Note that $-D_{y_2} f = -x_3^2 + x_4^2 + x_2 x_3 - x_2 x_4 \in \ker \mathcal{B}$. Similarly, $-D_{y_j} f = -x_3^2 + x_4^2 + x_j x_3 - x_j x_4 \in \mathcal{B}$ for all $j \neq 3, 4$. Finally, it remains to compute $-D_{y_3} f = 0$. (Note that $-D_{y_4}$ acts in the same way as $-D_{y_3}$ by virtue of swapping indices.) Hence $D_{y_i-y_j} f \in \ker \mathcal{B} \implies f \in \ker \mathcal{B}$. \square

Proposition 2.22. When $p = 3$, the polynomials $x_i^3 - x_i^2 x_j + x_j^3 \in \ker \mathcal{B}$.

Proof. It suffices to prove that all Dunkl operators $D_{y_i-y_j}$ send $x_1^3 - x_1^2 x_2 + x_2^3$ to a degree 2 singular polynomial. Let $f = x_1^3 - x_1^2 x_2 + x_2^3$. Note that $-D_{y_1} f = -x_1 x_2 - x_2^2 + x_1 x_2 + x_2^2 \in \ker \mathcal{B}$, and the action of $-D_{y_2}$ is exactly the same by symmetry (and by Proposition 2.21 we may replace $x_1^2 x_2$ by $x_1 x_2^2$). Finally, for $j \neq 1, 2$, $-D_{y_j} f = (x_2 - x_j)(x_1 - x_2 - x_j) \in \ker \mathcal{B}$. Hence $D_{y_i-y_j} f \in \ker \mathcal{B} \implies f \in \ker \mathcal{B}$. \square

Remark. This shows that for $p = 3$, the dimension of $\mathcal{L}_{0,c}[3]$ is actually $n - 1$, because we do not need the polynomial $x_i^2 x_j$ to be in \mathcal{L} anymore. So long as all x_i^3 are for $i < n$, we can recover the $x_i^2 x_j$.

From now on, we work in $\mathcal{M}'_{0,c} = \mathcal{M}_{0,c} / (x_i^2 x_j - x_i x_j^2)$ (and also with $\ker \mathcal{B} / (x_i^2 x_j - x_i x_j^2)$). Thus, we can shift exponents around in any monomial so long as all of the exponents remain positive. We will therefore not concern ourselves with specific exponents, but only with the variables that appear in the monomial.

Definition 2.23. Denote $x_{s_1}^{e_1} x_{s_2}^{e_2} \dots x_{s_b}^{e_b}$ by the tuple (s_1, \dots, s_b) . (The degree will be specified each time.)

Denote the singular polynomials from Proposition 2.15 as $(i) - (j) + (j, k) - (i, k)$. Notice that unless a new singular polynomial appears, then the $\ker \mathcal{B}[3]$ polynomials are either the symmetric polynomial $(1) + (2) + \dots + (n)$ or multiples of the singular polynomials, namely $(i) - (i, j) + (i, j, k) - (i, k)$ or $(i, l) - (j, l) + (j, k, l) - (i, k, l)$.

Definition 2.24. Denote $(*)_j$ as the set of polynomials $(i) - (j) + (j, k) - (i, k) \in \mathcal{M}'_{0,c}[j]$. Similarly, define $(\dagger)_j$ as the set of polynomials $(i) - (i, j) + (i, j, k) - (i, k) \in \mathcal{M}'_{0,c}[j]$.

Definition 2.25. Define \mathcal{I} to be the ideal generated by the polynomials from 2.15.

Remark. It's worth pointing out that $\mathcal{I} \subset \mathcal{B}$ and that if no new polynomials (which are sent into $\ker \mathcal{B}$ upon action by any Dunkl operator) appear in some gradation $\mathcal{M}_{0,c}[j]$, then $\mathcal{I}[j] = \ker \mathcal{B}[j]$.

Proposition 2.26. In $\ker \mathcal{B}[3]$, the following polynomials constitute a basis for $(*)_3$ and $(\dagger)_3$:

- $(*)_3$: $(i) - (j) + (j, k) - (i, k)$; choose i, j, k in the same as we did for degree 2 singular polynomials
- $(\dagger)_3$: $(i) - (i, j) + (i, j, k) - (i, k)$; choose $i < j < k$.

Proof. The basis for $(*)_3$ generates all polynomials in $(*)_3$, which is proven in the same fashion as in Proposition 2.16. We can see that all polynomials in the basis of $(\dagger)_3$ are linearly independent with each other and the basis of $(*)_3$ because they contain a unique term (i, j, k) . Furthermore, we can generate all polynomials of the form $(a) - (a, b) + (a, b, c) - (a, c)$ for distinct $a, b, c \in [n]$. If $a < b < c$, then take $a = i, b = j, c = k$. If $b < a < c$, then take the polynomial $(b) - (b, a) + (b, a, c) - (b, c) \in (\dagger)$. We know that from the basis of $(*)$ we can form the polynomial $(a) - (b) + (b, c) - (a, c)$. Adding the two yields the result. Similarly, we can take any polynomial in $(\dagger)_3$ and obtain all polynomials which are permutations of its indices by adding or subtracting polynomials of the form $(i) - (j) + (j, k) - (i, k)$. \square

Proposition 2.27. We can generate all polynomials in $\mathcal{I}[3]$ using $(*)_3$ and $(\dagger)_3$.

Proof. As noted in Proposition 2.26, we simply need to generate all polynomials of the form $(i) - (i, j) + (i, j, k) - (i, k)$ or $(i, l) - (j, l) + (j, k, l) - (i, k, l)$, and the symmetric polynomial. The first kind, $(i) - (i, j) + (i, j, k) - (i, k)$, is exactly produced by the basis of $(\dagger)_3$. The second kind, $(i, l) - (j, l) + (j, k, l) - (i, k, l)$, can be written as $[(l) - (j, l) + (j, k, l) - (k, l)] - [(i) - (i, l) + (i, k, l) - (i, k)] + [(i) - (l) - (i, k) + (k, l)]$, and hence is also generated by $(*)_3$ and $(\dagger)_3$.

Finally, it remains to show that $(1) + (2) + \dots + (n)$ can be generated by $(*)$ and (\dagger) . But notice that from $(*)$,

$$\begin{aligned} \sum_{k=1}^{n-2} [(k) - (n) + (n-1, n) - (1, n-1)] &= (1) + (2) + \dots + (n-2) + (n) - (n-1, n) + (n-1) + (n, n-1) \\ &= (1) + (2) + \dots + (n). \end{aligned}$$

\square

By Proposition 2.16, the basis for $\mathcal{I}[2]$ is given by $(*)_2$. The basis for $\mathcal{I}[3]$ is given by $(*)_3$ and $(\dagger)_3$. Now heading into higher degrees, the $(*)_j$ will always generate $(*)_{j+1}$ and $(\dagger)_{j+1}$, but the set of polynomials $(\dagger)_3$ will produce both $(\dagger)_4$ and the set of polynomials of the form $(i, l) - (i, j, l) + (i, j, k, l) - (i, k, l)$.

Definition 2.28. Denote the set of polynomials in $\ker \mathcal{B}[j]$ (for $q > 3$) of the form $(i, l_1, l_2, \dots, l_{q-3}) - (i, j, l_1, \dots, l_{q-3}) + (i, j, k, l_1, l_2, \dots, l_{q-3}) - (i, k, l_1, \dots, l_{q-3})$ as $(\dagger^q)_j$.

Remark. Setting $q = 3$ recovers the set (\dagger) .

Proposition 2.29. In $\ker \mathcal{B}[j]$ for $j > 3$, the following polynomials constitute a basis for $(*)_j$, $(\dagger)_j$, and $(\dagger^q)_j$ for $q = 4, 5, \dots, j$:

- $(*)_j$: $(i) - (j) + (j, k) - (i, k)$; choose i, j, k in the same as in Proposition 2.16,
- $(\dagger)_j$: $(i) - (i, j) + (i, j, k) - (i, k)$; choose $i < j < k$,

$(\dagger^q)_j : (i, l_1, l_2, \dots, l_{q-3}) - (i, j, l_1, \dots, l_{q-3}) + (i, j, k, l_1, l_2, \dots, l_{q-3}) - (i, k, l_1, \dots, l_{q-3})$ for $q = 4, 5, \dots, j$; choose $i < j < k < l_1 < l_2 < \dots < l_{q-3}$ for each q .

Proof. We already know that the bases for $(*)_j$ and $(\dagger)_j$ are linearly independent and generate all of $(*)_j$ and $(\dagger)_j$. But we can easily see that for the bases of the $(\dagger^q)_j$'s, they are all linearly independent due to each one containing a unique term of $(i, j, k, l_1, l_2, \dots, l_{q-3})$. Hence we can inductively show that each basis for $(\dagger^q)_j$ is linearly independent with all the basis polynomials for $(*)_j$, $(\dagger)_j$, and $(\dagger^r)_j$ for $r < q$.

It thus remains to show that the basis for $(\dagger^q)_j$ can generate all of $(\dagger^q)_j$. Choose an arbitrary q . Then we have polynomials of the form $(i, l_1, l_2, \dots, l_{q-3}) - (i, j, l_1, \dots, l_{q-3}) + (i, j, k, l_1, l_2, \dots, l_{q-3}) - (i, k, l_1, \dots, l_{q-3})$ for $q = 4, 5, \dots, j$ with $i < j < k < l_1 < l_2 < \dots < l_{q-3}$. But note that we can arbitrarily shuffle the order of $i, j, k, l_1, \dots, l_{q-3}$ by adding and subtracting polynomials in the basis of $(\dagger^{q-1})_j$. Thus, by induction, we have all of $(\dagger^q)_j$ for each q . \square

Proposition 2.30. The polynomials $(*)_j$, $(\dagger)_j$, and $(\dagger^q)_j$ for $4 \leq q \leq j$ generate all of $\mathcal{I}[j]$.

Proof. Fix a $j > 3$ (the case $j = 3$ was already done in Proposition 2.26). Then to obtain $j + 1$, each of the basis polynomials for $\mathcal{I}[j]$ are multiplied by an x_i , and there is a new symmetric polynomial. However this symmetric polynomial is explicitly given as follows:

$$\begin{aligned} \sum_{k=1}^{n-2} [(k) - (n) + (n-1, n) - (1, n-1)] &= (1) + (2) + \dots + (n-2) + (n) - (n-1, n) + (n-1) + (n, n-1), \\ &= (1) + (2) + \dots + (n). \end{aligned}$$

Multiplying $(*)_j$ yields $(*)_{j+1}$ or $(\dagger)_{j+1}$. Multiplying $(\dagger)_j$ yields $(\dagger)_{j+1}$ or $(\dagger^4)_{j+1}$. Multiplying $(\dagger^q)_j$ yields either $(\dagger^q)_{j+1}$ or $(\dagger^{q+1})_{j+1}$, hence the result. \square

Proposition 2.31. The polynomial $(i) - (i, j) + (j)$ is singular in degree p .

Proof. We will use the polynomial $f = x_1^p - x_1 x_2^{p-1} + x_2^p$ and first check the action of the Dunkl operator $-D_{y_1}$. We will freely replace monomials with the notation (i, j) .

We have that

$$\begin{aligned} -D_{y_1} f &= \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} (x_1^p - x_1 x_2^{p-1} + x_2^p), \\ &= x_1 x_2 \frac{x_1^{p-2} - x_2^{p-2}}{x_1 - x_2} + \sum_{k > 2} \frac{x_1^p - x_k^p}{x_1 - x_k} - x_2^{p-1} \sum_{k > 2} \frac{x_1 - x_k}{x_1 - x_k}, \\ &= x_1 x_2 \frac{x_1^{p-2} - x_2^{p-2}}{x_1 - x_2} + x_2^{p-1} - x_1^{p-1} - x_1^{p-1} - x_2^{p-1} + \sum_{k > 2} (x_1^{p-2} x_k + \dots + x_1 x_k^{p-2}), \\ &= (p-2) \cdot (1, 2) + (2) - (1) - (1) - (2) + (p-2) \sum_{k > 2} (1, k), \\ &= -2 \cdot (1, 2) - 2 \cdot (1) - 2 \cdot [-(1) - (1, 2)], \\ &= 0. \end{aligned}$$

Since $x_1^p - x_1 x_2^{p-1} + x_2^p$ is invariant under the action of σ_{12} in $\mathcal{M}'_{0,c}$, we have $-D_{y_2} (x_1^p - x_1 x_2^{p-1} + x_2^p) = 0$.

Now take $j \neq 1, 2$. Then we have

$$\begin{aligned}
D_{y_j} f &= \sum_{k \neq j} \frac{1 - \sigma_{jk}}{x_k - x_j} \left(x_1^p - x_1 x_2^{p-1} + x_2^p \right), \\
&= \frac{1 - \sigma_{1j}}{x_1 - x_j} \left(x_1^p - x_1 x_2^{p-1} + x_2^p \right) + \frac{1 - \sigma_{2j}}{x_2 - x_j} \left(x_1^p - x_1 x_2^{p-1} + x_2^p \right), \\
&= (1) + (j) + (p-2) \cdot (1, 3) - (2) + (2) + (j) + (p-2) \cdot (2, j) - (1, 2) - (1, j) - (p-3) \cdot (1, 2, j), \\
&= (1) + 2 \cdot (j) - 3 \cdot (1, j) - 2 \cdot (2, j) - (1, 2) + 3 \cdot (1, 2, j), \\
&= [(1) - (j) + (2, j) - (1, 2)] - 3[(j) - (1, j) + (1, 2, j) - (2, j)] \in \ker \mathcal{B}[p-1].
\end{aligned}$$

Hence all Dunkl operators $D_{y_k - y_l}$ send all degree p polynomials of the form $(i) - (i, j) + (j)$ into the kernel, so they are in $\ker \mathcal{B}[p]$. \square

Proposition 2.32. For any j , the dimension of $\mathcal{L}_{0,c}[j]$ is at most n .

Proof. We will show that the n polynomials $(1), (2), \dots, (n-1)$, and $(1, 2)$, combined with $(*)_j, (\dagger)_j$, and $(\dagger^q)_j$ (for $4 \leq q \leq j$) linearly generate the entire subspace of homogeneous degree j polynomials.

First, we easily obtain all polynomials of the form (i) , since the only missing one is (n) but the symmetric polynomial fills that in.

Next, we obtain all polynomials of the form (i, j) from $(*)_j$, since each is of the form $(i) - (j) + (j, k) - (i, k)$. We can remove $(i) - (j)$ and we are left with $(j, k) - (i, k)$. Setting $k = 1$ and $i = 2$ allows us to obtain all of the form $(1, j)$ and next setting $i = 1$ and j, k to be anything allows us to obtain all (j, k) .

We can then obtain all (i, j, k) from $(\dagger)_j$, since they are of the form $(i) - (i, j) + (i, j, k) - (i, k)$. Removing the necessary terms leaves us with (i, j, k) .

For $q > 3$, any polynomial in $(\dagger^q)_j$ contains one term with q distinct variables and other terms with less than q distinct variables. Inductively we can remove all other terms to obtain all terms of the form (i_1, i_2, \dots, i_q) . When we reach $q = j$, we are done. \square

Proposition 2.33. For $2 \leq j \leq p-1$, the dimension of $\mathcal{L}_{0,c}[j]$ is n .

Proof. We showed that $\dim \mathcal{L}_{0,c}[2] = n$ in Proposition 2.19, so assume that $j > 2$. It suffices to show that no linear combination of $(1), (2), \dots, (n-1)$, and $(3, 4)$ is in $\ker \mathcal{B}$.

Let us examine the operator $D_{y_1 - y_2}$. We perform computations strictly in $\mathcal{L}_{0,c}[j-1]$ (which has been found by the prior inductive step), adding and subtracting polynomials from $\ker \mathcal{B}$ freely. We have that

$$\begin{aligned}
D_{y_1 - y_2} x_1^j &= - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} x_1^j + \sum_{k \neq 2} \frac{1 - \sigma_{2k}}{x_2 - x_k} x_1^j, \\
&= (j-2) \cdot (1) - (2) + (j-2) \cdot (1, 2).
\end{aligned}$$

We also have that

$$\begin{aligned}
D_{y_1 - y_2} x_2^j &= - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} x_2^j + \sum_{k \neq 2} \frac{1 - \sigma_{2k}}{x_2 - x_k} x_2^j, \\
&= -(j-2) \cdot (2) + (1) - (j-2) \cdot (1, 2).
\end{aligned}$$

For $r \neq 1, 2$, then

$$\begin{aligned}
D_{y_1 - y_2} x_r^j &= - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} x_r^j + \sum_{k \neq 2} \frac{1 - \sigma_{2k}}{x_2 - x_k} x_r^j, \\
&= (1) - (2) + (j-2) \cdot (1, r) - (j-2) \cdot (2, r) + (j-2) [(1) - (2) + (2, t) - (1, r)], \\
&= (j-1) \cdot [(1) - (2)].
\end{aligned}$$

Finally,

$$\begin{aligned}
D_{y_1-y_2} x_3 x_4^{j-1} &= - \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} x_3 x_4^{j-1} + \sum_{k \neq 2} \frac{1 - \sigma_{2k}}{x_2 - x_k} x_3 x_4^{j-1}, \\
&= (1, 3) - (2, 3) + (j-3) \cdot (1, 3, 4) - (j-3) \cdot (2, 3, 4) - (j-3) [(1) - (1, 3) + (1, 3, 4) - (1, 4)], \\
&= (j-2) \cdot (1, 3) - (2, 3) - (j-3) \cdot (1) + (j-3) \cdot (1, 4) - (j-3) \cdot (2, 3, 4) \\
&\quad + (j-3) [(2) - (2, 3) + (2, 3, 4) - (2, 4)], \\
&= (j-2) \cdot (1, 3) - (j-2) \cdot (2, 3) - (j-3) [(1) - (2) + (2, 4) - (1, 4)] \\
&\quad + (j-3) [(1) - (2) + (2, 4) - (1, 4)], \\
&= (j-2) [(1, 3) - (2, 3) + (1) - (2) + (2, 3) - (1, 3)], \\
&= (j-2) [(1) - (2)].
\end{aligned}$$

Suppose that we have such a polynomial, $f = a_1 x_1^j + a_2 x_2^j + \cdots + a_{n-1} x_{n-1}^j + b x_3 x_4^{j-1}$. Then we note that $a_1 = a_2$ to remove the $(1, 2)$ terms. By symmetry (using Dunkl operators), we have $a_1 = a_2 = a_3 = \cdots = a_{n-1}$. Obviously $a_i \neq 0$, since $x_3 x_4^{j-1}$ is not singular. Now we assume without loss of generality that $a_i = \frac{1}{j-1}$, to obtain that

$$D_{y_1-y_2} \left(\sum a_i x_i^j \right) = (n-2) [(1) - (2)] = (2) - (1).$$

We thus conclude that $b = -\frac{1}{j-2}$. But a quick check using the operator $D_{y_1-y_3}$ shows that f is not singular after all, and hence no linear combination exists. \square

Proposition 2.34. The dimension of $\mathcal{L}_{0,c}[p]$ is $n-1$.

Proof. From Proposition 2.32, $(1), (2), \dots, (n-1)$, and $(1, 2)$ generate all of $\mathcal{L}_{0,c}[p]$. But by Proposition 2.31, $(1) - (1, 2) + (2) \in \ker \mathcal{B}$. Thus $(1, 2)$ is not needed and $\dim \mathcal{L}_{0,c}[p] \leq n-1$.

To show that the dimension is exactly $n-1$, we consider a linear combination $f = a_1 x_1^p + a_2 x_2^p + \cdots + a_{n-1} x_{n-1}^p$. Using the Dunkl operator $D_{y_1-y_2}$, we compute that (for $r \neq i, j$)

$$\begin{aligned}
D_{y_i-y_j} x_i^p &= -(j), \\
D_{y_i-y_j} x_2^p &= (i), \\
D_{y_i-y_j} x_r^p &= (i) - (j).
\end{aligned}$$

This implies that $a_i = a_j$ for all i, j . But then

$$D_{y_1-y_2} (x_1^p + x_2^p + \cdots + x_{n-1}^p) = (n-2) [(1) - (2)] \neq 0.$$

Thus there does not exist such a linear combination and the dimension is exactly $n-1$. \square

Proposition 2.35. The dimension of $\mathcal{L}_{0,c}[p+1]$ is 1.

Proof. From Proposition 2.32, the set $\{x_1^{p+1}, x_2^{p+1}, \dots, x_{n-1}^{p+1}, x_1 x_2^p\}$ generates $\mathcal{L}_{0,c}[p+1]$. However, note that $(i) - (i, j) + (j) \in \ker \mathcal{B}[p]$. Multiplying by (i) yields $(i) - (i, j) + (i, j) = (i) \in \ker \mathcal{B}[p+1]$. Hence $\dim \mathcal{L}_{0,c}[p+1] \leq 1$. To prove equality, it suffices to check that in $\mathcal{L}_{0,c}$, $D_{y_1-y_2} x_1 x_2^p = (1, 2) + (2) = (1, 2) \neq 0$. \square

Proposition 2.36. The dimension of $\mathcal{L}_{0,c}[p+2]$ is 0.

Proof. From Proposition 2.35, $x_i^{p+2} = (i) \in \ker \mathcal{B}[p+2]$, and $x_i x_j^{p+1} = (i, j) \in \ker \mathcal{B}[p+2]$. This covers all polynomials described in Proposition 2.32, implying that $\ker \mathcal{B}[p+2]$ contains all of $\mathcal{M}_{0,c}[p+2]$. \square

Proposition 2.37. For all $m > p+2$, the dimension of $\mathcal{L}_{0,c}[v]$ is 0.

Proof. This follows from the fact that $\ker \mathcal{B}$ is an ideal. \square

Theorem 2.38. *The Hilbert series for $\mathcal{L}_{0,c}$ over a field with prime characteristic p is*

$$h_{\mathcal{L}_{0,c}}(z) = \left(\frac{1 - z^p}{1 - z} \right) (1 + (n - 2)z + z^2).$$

Proof. We simply expand and match coefficients from the previous Propositions. Note that this case also covers $p = 2$ from Theorem 2.12. \square

3 The case $t = 1$

In this case, the Dunkl operator is

$$D_{y_i - y_j} = \partial_{x_i} - \partial_{x_j} - c \sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} + c \sum_{\ell \neq j} \frac{1 - \sigma_{j\ell}}{x_j - x_\ell}.$$

We will again study each gradation (by degree) explicitly from $\deg = 0$ upwards and find a collection of polynomials which belong in $\ker \mathcal{B}$. We will prove that any polynomial not in the span of that collection is not in $\ker \mathcal{B}$ using explicit Dunkl operator actions. We will also utilize a result from [BC13] which constricts the form of the Hilbert series of $\mathcal{L}_{1,c}$.

3.1 Characteristic $p = 2$

From Proposition 1.15 ([BC13]), we know that the Hilbert series is of the form $h_{\mathcal{L}_{1,c}}(z) = (1 + z)^{n-1} Q(z^2)$ for some integer polynomial Q . Assume c is transcendental over \mathbb{F}_2 . We will let $Q(z^2) = Q_0 + Q_2 z^2 + Q_4 z^4 + \dots$ and compute term by term: each time we compute $\dim \mathcal{L}_{1,c}[d]$ for some even d , we can expand $h_{\mathcal{L}_{1,c}}(z) [z^d]$ to find Q_d .

There is a basis of $\mathcal{M}_{t,c}$ consisting of elements of $\mathbb{F}_2[c, x_1, x_2, \dots, x_n]$, so we assume that all coefficients are from $\mathbb{F}_2[c]$.

Any polynomial f can be graded by powers of c . We divide out by a power of c so that $f \not\equiv 0 \pmod{c}$. Since the Dunkl operator acts with two separate gradations, we may consider them separately.

Definition 3.1. For a given polynomial $f \in \mathbb{F}_2[c, x_1, \dots, x_{n-1}]$, let $f = \sum_{k \geq 0} c^k f^{(k)}$ where each $f^{(k)} \in \mathbb{F}_2[x_1, \dots, x_{n-1}]$ and $f^{(0)} \neq 0$.

Definition 3.2. Denote $\alpha_{ij} = \partial_{x_i} - \partial_{x_j}$ and $\beta_{ij} = -\sum_{k \neq i} \frac{1 - \sigma_{ik}}{x_i - x_k} + \sum_{\ell \neq j} \frac{1 - \sigma_{j\ell}}{x_j - x_\ell}$, so that $D_{y_i - y_j} = \alpha_{ij} + c\beta_{ij}$.

We will use this notation throughout the rest of the paper. We will again use the substitution $x_n = x_1 + x_2 + \dots + x_{n-1}$.

Proposition 3.3. The dimension of $\mathcal{L}_{1,c}[0]$ is 1.

Proof. All elements are constants. \square

Corollary 3.3.1. We have that $Q_0 = 1$.

Proposition 3.4. The dimension of $\mathcal{L}_{1,c}[1]$ is $n - 1$.

Proof. Under the substitution for x_n , we have $\dim \mathcal{L}_{1,c}[1] \leq n - 1$. To show equality, assume that $f = a_1 x_1 + a_2 x_2 + \dots + a_{n-1} x_{n-1} \in \ker \mathcal{B}$ where $a_i \in \mathbb{F}_2[c]$. Then $\alpha_{in} f^{(0)} = (\partial_{x_i} - \partial_{x_n}) f^{(0)} = a_i = 0$ ensures that $f^{(0)} = 0$, which contradicts our assumption of a nonzero constant term. \square

Remark. We actually get this for free by the form of the Hilbert series given in [BC13].

Proposition 3.5. The dimension of $\mathcal{L}_{1,c}[2]$ is $\binom{n}{2}$.

Proof. Under the substitution for x_n , we can concern ourselves only with polynomials in $\{x_1, x_2, \dots, x_{n-1}\}$. We show that no singular polynomials exist, which implies that $\dim \mathcal{L}_{1,c}[2] = \binom{n}{2}$. In these $n-1$ variables, $\ker \mathcal{B}[1] = \{0\}$. Consider some singular polynomial f . Then $\alpha_{in}f^{(0)} = (\partial_{x_i} - \partial_{x_n})f^{(0)} = \partial_{x_i}f^{(0)} = 0$ implies that it cannot contain any term of the form $x_i x_j$. Let the remaining terms be $f^{(0)} = x_1^2 + x_2^2 + \dots + x_C^2$ for some $C < n$. Obviously $\alpha_{ij}f^{(0)} = 0$, but $\beta_{1n}f^{(0)} = Cx_n + (1-C)x_1$ which is x_1 or $x_1 + x_2 + \dots + x_{n-1}$. Since $\alpha_{1n}f^{(1)} + \beta_{1n}f^{(0)} = 0$, this implies that $\alpha_{1n}f^{(1)} = x_1$ or $x_1 + \dots + x_{n-1}$, which is impossible. \square

Corollary 3.5.1. *We have $Q_2 = n-1$.*

Proposition 3.6. The dimension of $\mathcal{L}_{1,c}[3]$ is $\binom{n+1}{3}$.

Proof. Expand and look at the coefficient of z^3 in $h_{\mathcal{L}_{1,c}}(z)$ via the form from [BC13]. \square

Remark. This implies that under the substitution $x_n = x_1 + x_2 + \dots + x_{n-1}$, there are no singular polynomials in the space of homogeneous degree 3 polynomials in $\{x_1, x_2, \dots, x_{n-1}\}$.

Proposition 3.7. The polynomials $R_{ij} = \frac{c+1}{c} (x_i^4 + x_i^2 x_j^2 + x_j^4) + (x_i^3 + x_i^2 x_j + x_i x_j^2 + x_j^3) \left(\sum_{k \neq i,j,n} x_k \right) + (x_i^2 + x_j^2) \left(\sum_{l \neq i,j,n} x_l^2 \right) + (x_i + x_j) \left(\sum_{a,b \neq i,j,n; a \neq b} x_a^2 x_b \right)$ are singular for all $i, j \in [n-1]$.

Remark. Note that R_{ij} can be rewritten as

$$R_{ij} = \frac{1}{c} (x_i^4 + x_i^2 x_j^2 + x_j^4) + x_i^2 x_j^2 + (x_i + x_j) \sum_{k \neq i,j} x_k^3.$$

Proof. We will prove that $f = cR_{12}$ is singular (the rest are the same by symmetry). Let $f^{(0)} = x_1^4 + x_1^2 x_2^2 + x_2^4$ and $f^{(1)} = x_1^2 x_2^2 + (x_1 + x_2) \sum_{k>2} x_k^3$. (We use the form of $f^{(1)}$ as given in the remark, not the proposition statement.)

We compute the action of $D_{y_j} = \partial_{x_j} - cD_j$ where $D_j = \sum_{k \neq j} \frac{1-\sigma_{1j}}{x_j - x_k}$. We have that

$$D_{y_j} cR_{12} = \left(\partial_{x_j} f^{(0)} \right) + c \left(D_j f^{(0)} + \partial_{x_j} f^{(1)} \right) + c^2 \left(D_j f^{(1)} \right),$$

so we are interested in the action upon each degree (when it is viewed as a polynomial in c with coefficients in the ring $\mathbb{k}[x_1, x_2, \dots, x_n]$). Computing the action of D_{y_1} , we have

$$\begin{aligned} \partial_{x_1} f^{(0)} &= 0, \\ \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} f^{(0)} &= x_1^2 x_2 + x_1 x_2^2 + \sum_{k=1}^n x_k^3, \\ \partial_{x_1} f^{(1)} &= \sum_{k>2} x_k^3, \\ \sum_{k \neq 1} \frac{1 - \sigma_{1k}}{x_1 - x_k} f^{(1)} &= \sum_{k>2} \left[x_2^2 (x_1 + x_k) + x_1^3 + x_1^2 x_k + x_1 x_k^2 + x_k^3 + x_2^3 + x_2 (x_1^2 + x_1 x_k + x_k^2) + \sum_{j \in [n]} x_j^3 \right]. \end{aligned}$$

From this, we note that (still viewing $D_{y_1} cR_{12}$ as a polynomial in c with coefficients in $\mathbb{k}[x_1, \dots, x_n]$) the constant term is 0. The coefficient of c is then $x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$. As for the coefficient of c^2 , we sum and using the fact that $x_n = x_1 + \dots + x_{n-1}$, we ultimately obtain 0. Hence $D_{y_1} R_{12} = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$.

Note that the action of D_{y_2} is identical to the action of D_{y_1} by symmetry, so it suffices to compute the

action of D_{y_3} (since all other $j > 2$ are analogous). Then

$$\begin{aligned}
\partial_{x_3} f^{(0)} &= 0, \\
\sum_{j \neq 3} \frac{1 - \sigma_{3j}}{x_3 - x_j} f^{(0)} &= x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + (x_1 + x_2) x_3^2, \\
\partial_{x_3} f^{(1)} &= (x_1 + x_2) x_3^2, \\
\sum_{j \neq 3} \frac{1 - \sigma_{3j}}{x_3 - x_j} f^{(1)} &= \sum_{j \neq 3} \frac{1 - \sigma_{3j}}{x_3 - x_j} \left(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 + (x_1 + x_2) \sum_{j \in [n]} x_j^3 \right), \\
&= \frac{1 - \sigma_{13}}{x_3 - x_1} (x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4) + \frac{1 - \sigma_{23}}{x_3 - x_2} (x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4) = 0.
\end{aligned}$$

Via the discussion above, we again see that the constant term in $D_{y_3} c R_{12}$ is 0. The coefficient of c is $x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$, and the coefficient of c^2 is 0. Thus $D_{y_k} = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$ for all $k \in [n]$, and hence all Dunkl operators $D_{y_i - y_j}$ send all R_{kl} to 0. \square

Proposition 3.8. The singular polynomials R_{ij} in degree 4 are linearly independent for $i, j < n$.

Proof. Consider some linear combination of the R_{ij} and consider the lowest gradation by powers of c . This is comprised of the $x_i^4 + x_i^2 x_j^2 + x_j^4$ terms. But the $x_i^2 x_j^2$ part is unique to each R_{ij} , hence they are linearly independent. \square

Proposition 3.9. The dimension of $\mathcal{L}_{1,c}[4]$ is $\binom{n+2}{4} - \binom{n-1}{2}$.

Proof. This is equivalent to the claim that no more singular polynomials exist in $\mathcal{M}_{1,c}[4]$. After the substitution for x_n , consider such a polynomial f . Then $\alpha_{ij} f^{(0)} = 0$, so no terms of the form $x_i^3 x_j$ exist in $f^{(0)}$. Now subtract copies of $c R_{ij}$ to remove the terms of the form $x_i^2 x_j^2$, so that without loss of generality $f^{(0)} = x_1^4 + x_2^4 + \dots + x_C^4$. Then $\alpha_{1n} f^{(1)} + \beta_{1n} f^{(0)} = 0$, or

$$0 = \partial_{x_1} f^{(1)} + (1 + C) \left(x_2^3 + \dots + x_{n-1}^3 + \sum_{i,j < n} x_i^2 x_j^2 \right) + (x_1^3 + \dots + x_{n-1}^3)$$

$$+ x_1^2 (x_1 + \dots + x_C) + x_1 (x_1 + \dots + x_C)^2 + (x_1 + \dots + x_{n-1})^2 (x_1 + \dots + x_C) + (x_1 + \dots + x_{n-1}) (x_1 + \dots + x_C)^2.$$

However, regardless of the parity of C , there must remain an x_1^3 term, which is impossible to produce in $\partial_{x_1} f^{(1)}$. Since by assumption $f^{(0)} \neq 0$, we have the result. \square

Corollary 3.9.1. We have $Q_4 = n - 1$.

Proposition 3.10. The dimension of $\mathcal{L}_{1,c}[5]$ is $\binom{n+3}{5} - (n-1)\binom{n-1}{2}$.

Proof. Expand and look at the coefficient of z^5 in $h_{\mathcal{L}_{1,c}}(z)$ using the form from [BC13]. It turns out to be $\binom{n-1}{5} + (n-1)\binom{n-1}{3} + (n-1)^2$, which is equivalent to that expression. \square

Remark. This means that the only polynomials in degree $\ker \mathcal{B}[5]$ are linear combinations of $x_\ell R_{ij}$ for $i, j, \ell < n$ and $i \neq j$.

The key theorem we will now introduce allows us to answer the question of whether a specific polynomial is in $\ker \mathcal{B}$ for all odd n .

Theorem 3.11. Let f be a (homogeneous) polynomial in k variables (for simplicity, say x_1, x_2, \dots, x_k). Define $G = \deg f$ and S to be the maximal exponent of any of the variables. Then $f \in \ker \mathcal{B}$ in any $\mathcal{M}_{1,c}(S_n, \mathfrak{h})$ iff $f \in \ker \mathcal{B}$ in any $\mathcal{M}_{1,c}(S_n, \mathfrak{h})$ for all $n \leq S + k + G - 2$.

Proof. We work in the ring of polynomials in infinite number of variables $\mathbb{k}[x_1, x_2, \dots]$ and consider the subring $\mathbb{k}[x_1, x_2, \dots, x_n]$. Denote by $D_{y_i}^{(n)}$ the Dunkl operator associated with $\mathcal{H}_{1,c}(\mathfrak{h}, S_n)$; i.e., $D_{y_i}^{(n)} = \partial_{x_i} - c \sum_{j \neq i, j \leq n} \frac{1 - \sigma_{ij}}{x_i - x_j}$. We also denote by $e_s^{(n)} = \sum_{j=1}^n x_j^s$ and $e_0^{(n)} = 1$.

It is easy to see that if $i > k$ then $D_{y_i}^{(n)} f = \sigma_{i,k+1} D_{y_{k+1}}^{(k+1)} f$ hence for $i > k$, the value of $D_{y_i} f$ does not depend on n . But the result starts to depend on a new variable, namely x_i .

Now suppose $i \leq k$ Computing,

$$D_{y_i}^{(n)} f = \partial_{x_i} f - c \sum_{j \neq i, j \leq k} \frac{1 - \sigma_{ij}}{x_i - x_j} f - c \sum_{k < j \leq n} \frac{1 - \sigma_{ij}}{x_i - x_j} f = F(x_1, \dots, x_k) - c \sum_{k < j \leq n} \frac{1 - \sigma_{ij}}{x_i - x_j} f,$$

where $F(x_1, \dots, x_k)$ is a polynomial which depends only on x_1, x_2, \dots, x_k . Now write $f = \sum_{\ell} f_{\ell} x_i^{\ell}$, grading f by degree in x_i . Crucially, each $f_{\ell} \in \mathbb{k}[x_1, x_2, \dots, x_k]$. Then

$$\begin{aligned} D_{y_i}^{(n)} f &= F - c \sum_{k < j \leq n} \sum_{\ell} f_{\ell} (x_i^{\ell-1} + x_i^{\ell-2} x_j + \dots + x_j^{\ell-1}) \\ &= F - c \sum_{\ell} f_{\ell} \left(x_i^{\ell-1} (n - k) + x_i^{\ell-2} (e_1^{(n)} - e_1^{(k)}) + x_i^{\ell-3} (e_2^{(n)} - e_2^{(k)}) + \dots + (e_{\ell-1}^{(n)} - e_{\ell-1}^{(k)}) \right). \end{aligned}$$

Since $n - k = 1 - k$ in \mathbb{k} , that term does not depend on n , and thus we can write

$$D_{y_i}^{(n)} f = \sum_{s=0}^{\deg_{x_i}(f)-1} F_s(x_1, \dots, x_k) e_s^{(n)},$$

where $\deg_{x_i}(f)$ is the degree of f as a polynomial in x_i and each $F_s(x_1, \dots, x_k) \in \mathbb{k}[x_1, x_2, \dots, x_k]$ (recall that $c \in \mathbb{k}$).

Now let us examine the action of Dunkl operators D_{y_j} on each term $F_s(x_1, \dots, x_k) e_s^{(n)}$. Note that

$$D_{y_j}^{(n)} (F_s e_s^{(n)}) = D_{y_j}^{(n)} (F) e_s^{(n)} + F \partial_{x_i} (e_s^{(n)}).$$

If $j \leq k$, then $D_{y_j}^{(n)} (F_s e_s^{(n)}) = \sum_{s,t} F_{s,t}(x_1, \dots, x_k) e_s^{(n)} e_t^{(n)}$. If $j > k$, then it depends on x_j in an asymmetric way, and thus $D_{y_j}^{(n)} (F_s e_s^{(n)}) = \sum_{s,t} F_{s,t}(x_1, x_2, \dots, x_k, x_j) e_s^{(n)} e_t^{(n)}$. So we can prove the following lemma by induction:

Lemma 3.12. *If $f \in \mathbb{k}[x_1, \dots, x_k]$, then up to an action of $w \in S_{n-k}$ (i.e. up to permuting the rest of the variables) $D_{y_{j_r}} \dots D_{y_{j_1}} f$ can be expressed as:*

$$w \circ D_{y_{j_r}} \dots D_{y_{j_1}} f = \sum_{s_j < S} F_{s_1, \dots, s_r}(x_1, \dots, x_{k+r}) e_{s_1}^{(n)} \dots e_{s_r}^{(n)},$$

where $S = \max(\deg_{x_i}(f))$ and $\max(\deg_{x_i} F_{s_1, \dots, s_r}) \leq \max(\deg_{x_i}(f))$.

Proof. Note that the polynomial in the lemma is not homogeneous in e_s , since $e_0 = 1$.

We will prove by induction on r . For $r = 1$ this follows from the previous discussion. Suppose we know this for $r - 1$. Consider $D_{y_{j_r}} \dots D_{y_{j_1}} f$. We know that

$$w \circ D_{y_{j_r-1}} \dots D_{y_{j_1}} f = \sum_{s_j < S} F_{s_1, \dots, s_{r-1}}(x_1, \dots, x_{k+r-1}) e_{s_1}^{(n)} \dots e_{s_{r-1}}^{(n)}.$$

We can write:

$$\begin{aligned} w \circ D_{y_{j_r}} D_{y_{j_r-1}} \dots D_{y_{j_1}} f &= D_{y_{w(j_r)}} \circ w \circ D_{y_{j_r-1}} \dots D_{y_{j_1}} f, \\ &= \sum_{s_j < S} D_{y_{w(j_r)}} [F_{s_1, \dots, s_{r-1}}(x_1, \dots, x_{k+r-1})] e_{s_1}^{(n)} \dots e_{s_{r-1}}^{(n)} \\ &\quad + \sum_{s_j < S} [F_{s_1, \dots, s_{r-1}}(x_1, \dots, x_{k+r-1})] \partial_{w(j_r)} [e_{s_1}^{(n)} \dots e_{s_{r-1}}^{(n)}]. \end{aligned}$$

We have two cases. First $w(j_r) > k + r - 1$. Then by the discussion before the lemma, the first sum consists of polynomials in variables $x_1, \dots, x_{k+r-1}, x_{w(j_r)}$ and the number of symmetric polynomials does not grow, the second sum consists of polynomials in the same number of variables, but the number of symmetric polynomials drops by one. So after acting by $\sigma_{k+r, w(j_r)}$ we obtain the formula we need. Since no new e_s arise it follows that the bound by S still holds in this case. Also since action of Dunkl operators does not increase the maximal degree in the single variable the second assertion also works.

The second case is $w(j_r) \leq k + r - 1$. In this case the first part of the sum does not depend on any new variables, but we get one new symmetric polynomial in the product. Its index is bounded by maximal degree of $F_{s_1, \dots, s_{r-1}}$ in single variable minus 1, so bounded by S . The second sum consists of polynomials depending on the same set of variables, but with one symmetric polynomial erased. So we again obtain the polynomial of the same form. Hence the Lemma holds. \square

Now the statement that $f \in \ker \mathcal{B}$ will follow from the fact that by acting by any number of Dunkl operators $D_{y_a - y_b}^{(n)}$ on f , we obtain 0. In particular, when \mathbb{k} has characteristic 2, $\ker \mathcal{B}[3]$ consists of only the 0 polynomial after the substitution $x_n = x_1 + \dots + x_{n-1}$. We know that the Dunkl operators have a basis $D_{y_1 - y_u}^{(n)}$ for $u = 2, 3, \dots, n$. Thus it suffices to check that all sequences $u_1, u_2, \dots, u_{G-3} \in \{2, 3, \dots, n\}$ satisfy $D_{y_1 - y_{u_{G-3}}}^{(n)} \dots D_{y_1 - y_{u_1}}^{(n)} f = 0$ (after performing the substitution $x_n = x_1 + \dots + x_{n-1}$). Using the lemma it follows that

$$w \circ D_{y_1 - y_{u_{G-3}}}^{(n)} \dots D_{y_1 - y_{u_{G-4}}}^{(n)} \dots D_{y_1 - y_{u_1}}^{(n)} f = \sum_{s_i \leq S} F_{s_1, s_2, \dots, s_{G-3}}(x_1, \dots, x_k, x_{k+1}, \dots, x_{k+G-3}) e_{s_1}^{(n)} \dots e_{s_{G-3}}^{(n)}.$$

Note that since we work over characteristic 2 and we factored out e_1 it follows that we have only e_{s_i} with s_i - odd and not 1. We can rewrite this as

$$\sum_{s_i \leq S} \tilde{F}_{s_1, \dots, s_{G-3}}(x_1, \dots, x_{k+G-3}) \tilde{e}_{s_1}^{(n)} \dots \tilde{e}_{s_{G-3}}^{(n)},$$

where $\tilde{e}_s^{(n)} = \sum_{t=k+G-2}^n x_t^n$ (remember that each $s_t < S$). Note that the $\tilde{e}_{s_t}^{(n)}$ are algebraically independent when $n - k - G + 3 \geq S - 1$, and thus for $n \geq k + G + S - 2$, if the value is 0, then by algebraic independence all the \tilde{F} are zero, and thus the value is 0 for all n satisfying $n \geq k + G + S - 3$. Hence it suffices to check all combinations of Dunkl operators for all n satisfying $n < k + G + S - 3$ and at least one of the values of n for $n \geq k + g + S - 3$; if it is zero on all of those cases, then $f \in \ker \mathcal{B}$ for all n . \square

This theorem thus easily shows the following polynomials are in $\ker \mathcal{B}$.

Proposition 3.13. • The polynomials $x_i^6 \in \ker \mathcal{B}$.

- The polynomials $x_i^5 x_j^2 x_k^2 \in \ker \mathcal{B}$.
- The polynomials $x_i^4 x_j^4 \in \ker \mathcal{B}$.
- The polynomials $x_i^3 x_j^3 x_k^3 \in \ker \mathcal{B}$.
- The polynomials $x_i^2 x_j^2 x_k^2 x_\ell^2 \in \ker \mathcal{B}$.

Proof. The proof is exhausting all cases using a computer, as outlined in Theorem 3.11. \square

However, its use is not limited to showing that a polynomial is in $\ker \mathcal{B}$. The method of proof of the theorem can also show that a polynomial is not in $\ker \mathcal{B}$.

Proposition 3.14. The polynomial $x_1^5 x_2 \notin \ker \mathcal{B}$.

Proof. We show that $D_{y_1 - y_2} D_{y_1 - y_2} D_{y_1 - y_2} x_1^5 x_2 = c(x_1 x_2^2 + x_2^3)$. Adopting the proof of Theorem 3.11, for all n , $D_{y_1 - y_2} D_{y_1 - y_2} D_{y_1 - y_2} x_1^5 x_2$ will be a polynomial in $\tilde{e}_0^{(n)}, \tilde{e}_1^{(n)}, \tilde{e}_3^{(n)}$, where $\tilde{e}_s^{(n)} = \sum_{j=3}^n x_j^s$, with coefficients from $\mathbb{k}[x_1, x_2]$. By algebraic independence and by the fact that the Dunkl operators do not depend on n , if the result is the same up to $n = 5 + 2 + 6 - 2 = 11$, then the coefficients are always the same for any n . Checking the action of $(D_{y_1 - y_2})^3 (x_1^5 x_2)$ for $n = 3, 5, 7, 9, 11$ (with a computer, for example) shows that it always holds. \square

Corollary 3.14.1. *We have $Q_6 \geq 1$.*

Proof. Check the coefficient of z^6 in the Hilbert series for $\mathcal{L}_{1,c}$ using the form from [BC13], noting that $\dim \mathcal{L}_{0,c}[6] \geq 1$. \square

Corollary 3.14.2. *We have $\dim \mathcal{L}_{1,c}[n+5] \geq 1$.*

Proof. Expand the Hilbert series for $\mathcal{L}_{1,c}$ and check the coefficient of z^6 . \square

Proposition 3.15. We have the equality $\dim \mathcal{L}_{1,c}[n+5] = 1$.

Proof. From Theorem 3.11, it is easy to check that $x_1^3 x_2^3 x_3^2 + c(x_2^3 x_3^5 + x_1 x_2^2 x_3^5) \in \ker \mathcal{B}$. Multiplying this polynomial by x_1 yields $x_1^4 x_2^3 x_3^2 + c(x_1 x_2^3 x_3^5 + x_1^2 x_2^2 x_3^5) \in \ker \mathcal{B}$. Noting that $x_1^2 x_2^2 x_3^5 \in \ker \mathcal{B}$, we obtain that $x_1^4 x_2^3 x_3^2 + c x_1 x_2^3 x_3^5 \in \ker \mathcal{B}$. If either monomial were in $\ker \mathcal{B}$, then combined with Proposition 3.13, every monomial of degree $n+5$ would be contained in $\ker \mathcal{B}$, hence $\dim \mathcal{L}_{1,c}[n+5] \leq 1$. But since $\dim \mathcal{L}_{1,c}[n+5] \geq 1$ by Corollary 3.14.2, equality is achieved. \square

Corollary 3.15.1. *We have that $Q_6 = 1$.*

We will denote by (s_1, s_2, \dots) , a monomial whose (nonzero) exponents (of its distinct variables) are s_1, s_2, \dots for $s_1 \geq s_2 \geq \dots$ and all variables are x_i for $i < n$ (we may simply substitute $x_n = x_1 + \dots + x_{n-1}$).

Proposition 3.16. For $n \geq 5$, $\dim \mathcal{L}_{1,c}[n+7] = 0$.

Proof. We proceed by Pigeonhole principle on the exponents of any monomial and use Proposition 3.13 to show that such a monomial is contained in $\ker \mathcal{B}$. If the highest degree in a single variable is at least 6, then by $x_i^6 \in \ker \mathcal{B}$, it is in $\ker \mathcal{B}$. If its highest degree in a single variable is 5, then by the Pigeonhole principle it is either $(5, 5, \dots)$ or $(5, 4, \dots)$ or $(5, 3, 3, \dots)$ or $(5, 3, 2, \dots)$ or $(5, 2, 2, 2, \dots)$, all of which can be formed via Proposition 3.13. If its highest degree in a single variable is 4, then it is either $(4, 4, \dots)$ or $(4, 3, 3, \dots)$ or $(4, 3, 2, 2, \dots)$ or $(4, 2, 2, 2, \dots)$. If its highest degree in a single variable is 3, then it must have at least three other variables with exponent at least 2. If its highest degree in a single variable is 2, then there must be at least 4 distinct variables with exponent 2. Hence every possible monomial is contained in $\ker \mathcal{B}[n+7]$. \square

Corollary 3.16.1. *We have that $Q_8 = 0$.*

Putting these together, we find the Hilbert series for $\mathcal{L}_{1,c}$ for $n \geq 5$. For $n = 1$, there is not much to say (and the formula does not apply), and for $n = 3$, a quick Sage computation shows that the Hilbert series matches the same form.

Theorem 3.17. *The Hilbert series for $\mathcal{L}_{1,c}$ over a field with characteristic 2 is*

$$h_{\mathcal{L}_{1,c}}(z) = (1+z^2)(1+z)^{n-1}(1+(n-2)z^2+z^4),$$

or alternatively,

$$h_{\mathcal{L}_{1,c}}(z) = (1+z)^{n-1}(1+(n-1)z^2+(n-1)z^4+z^6).$$

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