PRIMES 2017 General Math Problems

Dear PRIMES applicant:

These are General Math Problems in the PRIMES 2017 Math Problem Set. Please send us your solutions as part of your PRIMES application by the application deadline (December 1, 2016). For complete rules, see `http://math.mit.edu/research/highschool/primes/apply.php`

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`smith-solutions`". Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions / results / ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help.* This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden! Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General Math Problems

Problem G1. A positive multiple of 11 is *good* if it does not contain any even digits in its decimal representation.

(a) Find the number of good integers less than 1000.

(b) Determine the largest such good integer.

(c) Fix $b \geq 2$ an even integer. Find the number of positive integers less than b^3 which are divisible by $b+1$ and do not contain any even digits in their base b representation. (This is the natural generalization of part (a) with 10 replaced by b.)

Problem G2. A fair six-sided die whose sides are labelled 1, 4, 9, 16, 25, 36 is rolled repeatedly until the sum of the rolled numbers is nonzero and either even or a multiple of 3.

(a) Compute the probability that when we stop, the sum is odd.

(b) Find the expected value of the number of rolls it takes until stopping.

Partial marks may be awarded for approximate answers obtained by computer simulation. (This is also a good way to check your answer!)

Problem G3. Let $\mathcal{H}$ be a hyperbola with center Z. Points A and B are selected on $\mathcal{H}$. Suppose that the tangents to $\mathcal{H}$ at points A and B intersect at a point C distinct from A, B, Z. Prove that line ZC passes through a point X in the interior of segment AB and determine the ratio AX/AB.

Problem G4. Suppose $P(n)$ is a monic polynomial with integer coefficients for which $P(0) = 17$, and suppose distinct integers $a_1, \dots, a_k$ satisfy $P(a_1) = \cdots = P(a_k) = 20$.

(a) Find the maximum possible value of k (over all P).

(b) Determine all P for which this maximum is achieved.

Problem G5. A positive integer N is *nice* if all its decimal digits are 4 or 7.

(a) Find all nine-digit nice numbers which are divisible by 512.

(b) How many d-digit nice numbers are divisible by 512 for each d?

Problem G6. A sequence x_1, x_2, ... is defined by $x_1 = 10$ and

$$x_n = 3n \cdot x_{n-1} + n! - 3^n \cdot (n^2 - 2)$$

for integers $n \geq 2$. Derive a closed form for x_n (not involving Σ summation).

Problem G7. Let ABC be a triangle, let a, b, c be the lengths of its sides opposite to A, B, C respectively, and let h_A, h_B, and h_C be the lengths of the altitudes from A, B, and C. Suppose that

$$\sqrt{a + h_B} + \sqrt{b + h_C} + \sqrt{c + h_A} = \sqrt{a + h_C} + \sqrt{b + h_A} + \sqrt{c + h_B}. \quad (1)$$

(a) Show that $(a + h_B)(b + h_C)(c + h_A) = (a + h_C)(b + h_A)(c + h_B)$.

(b) Prove that the three terms on the left-hand side of (1) are obtained by a permutation of the three terms on the right-hand side of (1). (Possible hint: consider polynomials with the three terms as roots.)

(c) Show that triangle ABC is isosceles.