A Near-Optimal Spectral Method for Simulating Fluids in a Cylinder

David Darrow

Mentors:
Dr. Alex Townsend and Prof. Grady Wright

MIT Primes

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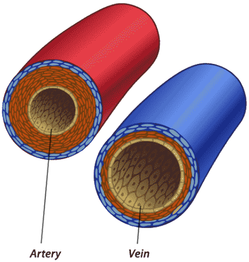
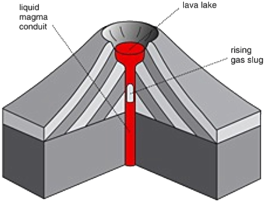
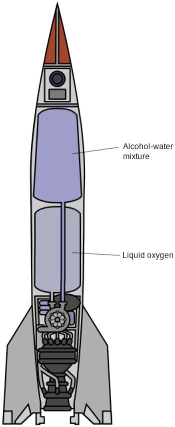






The Navier–Stokes Equations

The *Incompressible Navier–Stokes Equations* model the motion of incompressible fluids (e.g. liquid):

$$\nabla \cdot \vec{v} = 0, \qquad \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) \vec{v} = \frac{1}{Re} \nabla^2 \vec{v} - \nabla p.$$

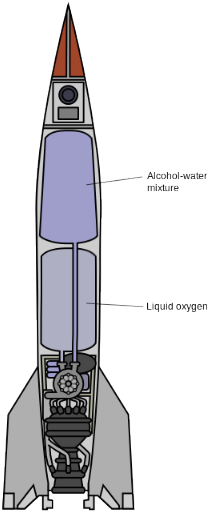
$\vec{v} =$ fluid velocity
$p =$ internal pressure
$Re =$ "Reynolds number"

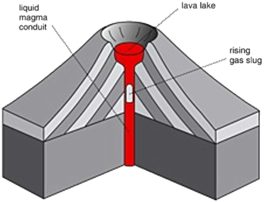
Low Re, high viscosity

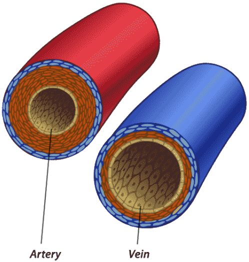
High Re, low viscosity

Solving in a Cylinder

We will focus on solving these equations in a cylinder.







Fluids Move in Mysterious Ways

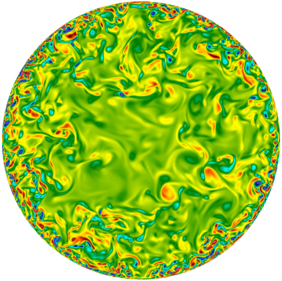
Why are the Navier–Stokes equations a challenge?

- There is no analytical solution.
- Designing accurate/efficient numerical methods is non-trivial.
- They are *highly nonlinear* when $Re \gg 0$.
- There is no explicit equation for the pressure.

Current Methods

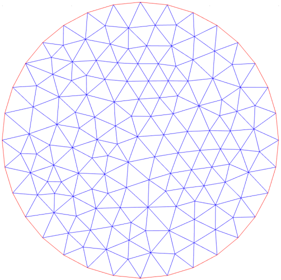
There are three issues with current solution methods:

- They do not appropriately resolve the boundary of the domain.
- They have relatively low accuracy.
- Their computational cost is high, leading to long simulation times.



Turbulent flow is more complicated and important to model near the boundary.

Some methods try to approximate the boundary with a polygon.



Starting Off Simple

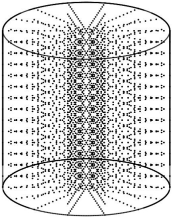
When $Re \approx 0$, the Navier–Stokes equations resemble the *heat equation*:

$$\left(\frac{\partial}{\partial t} - c\nabla^2\right)\vec{v} = f(x, y, z, t), \qquad c > 0.$$

On the cylinder, though, this is still difficult to solve efficiently.

How Do We Solve the Heat Equation?

To approximate a solution to the heat equation, we only enforce it at points on a *discretization grid*:

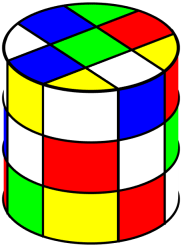


Our solution satisfies the heat equation at each black point on this cylinder.

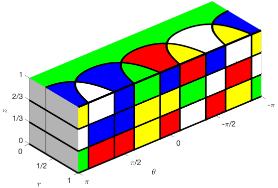
By choosing more points, we get a more accurate solution for the whole cylinder.

Doubling up the Discretization Grid

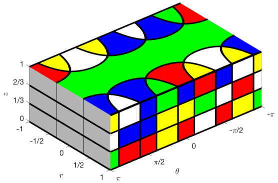
By "doubling up" the grid, we remove a fake boundary at the centerline of the cylinder and spread points out more evenly.



A colored cylinder



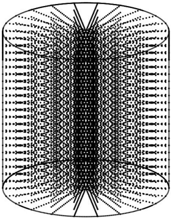
Mapped onto rectangular coordinates



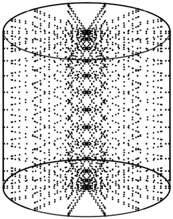
Doubled up

Increasing the Accuracy

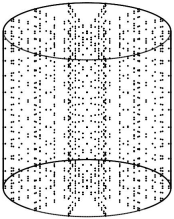
We also employ new, fast, and accurate “spectral” methods for solving differential equations, which means we do not require as many points in the discretization grid to have high accuracy.



"Finite difference" methods

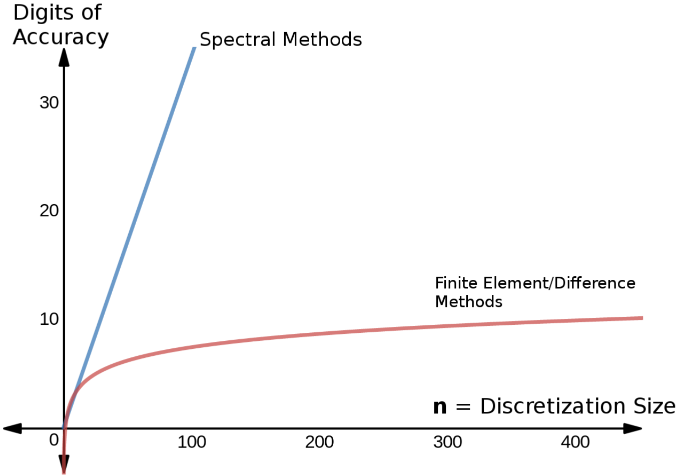


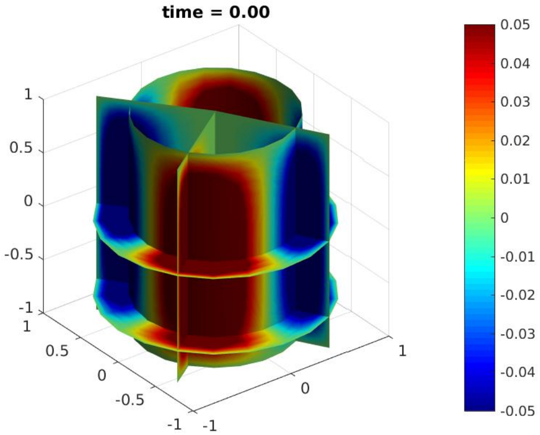
Spectral methods



Spectral methods, doubled

Accuracy vs. Computation Time

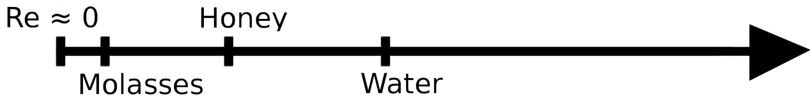




We can now solve the heat equation with $\mathcal{O}(n^3 \log n)$ steps.

Animation

Ramping Up the Reynolds Number



The higher the Reynolds number, the harder a system is to model.

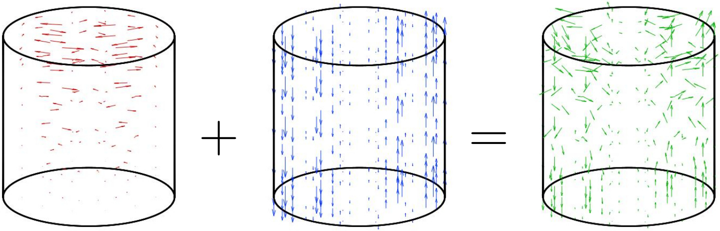
How does heat diffusion relate to the Navier–Stokes equations?

With some more machinery, they can be *turned into heat equations*.

The PT Decomposition

For any vector field $\vec{A}$ and any unit vector $\hat{z}$,

$$\nabla \times [\lambda_a \hat{z}] + \nabla \times \nabla \times [\gamma_a \hat{z}] = \nabla \times \vec{A}.$$



Toroidal Field *Poloidal Field* *Velocity Field*

We can compute the PT decomposition of $\vec{A}$ in $\mathcal{O}(n^3 \log n)$ steps.

Rewriting Navier–Stokes

We can decompose the Navier–Stokes equations into Poloidal and Toroidal components:

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) \vec{v} = \frac{1}{Re} \nabla^2 \vec{v} - \nabla p \qquad (\textit{Velocity Field})$$

becomes

$$\left(\frac{\partial}{\partial t} - \frac{1}{Re} \nabla^2\right) \lambda_\omega = f_1(\vec{\omega}) \qquad (\textit{Toroidal Field})$$

and

$$\left(\frac{\partial}{\partial t} - \frac{1}{Re} \nabla^2\right) \gamma_\omega = f_2(\vec{\omega}), \qquad (\textit{Poloidal Field})$$

which can be numerically solved as two scalar heat equations.

Recap

Current Methods

- have relatively low accuracy.
- either over-resolve the origin or under-resolve the boundary.
- have a high computational cost.

Our method

- has digits of accuracy proportional to n.
- resolves the boundary and selects discretization points more evenly.
- only requires $\mathcal{O}(n^3 \log n)$ operations.

Future Developments

Future Goals:

- Combining the pieces into a single Navier–Stokes code.

Acknowledgments

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Figures:

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- Rocket image retrieved from wikimedia.org
- Honey image retrieved from styletips101.com
- Water image retrieved from nuocuongvihawa.com
- Turbulent flow modeled by the Argonne National Laboratory