Folding, Jamming, and Random Walks

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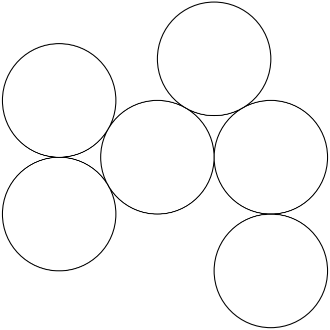
Motivation and Background

- Jammed circular billiard balls in space.
- All same size, all same mass, all externally tangent to adjacent balls.
- Suppose total kinetic energy is constant.

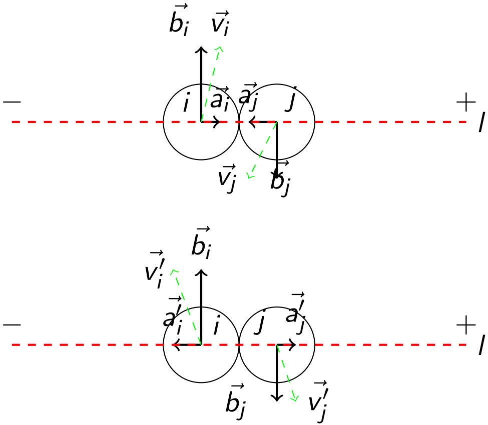


Considering Collisions

- Suppose balls $1, 2, \ldots, n$.
- Pick pair of balls $(i, j), 1 \leq i \leq j \leq n$.
- If balls i, j not touching, don't do anything.
- If balls i, j tangent, consider velocity vectors $\vec{v_i}, \vec{v_j}$.

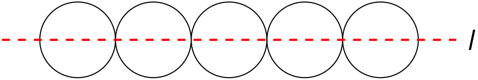
Result of Collisions

- Split $\vec{v_i}, \vec{v_j}$ into two vectors $\vec{b_i}, \vec{b_j}$ perpendicular to line l connecting circle centers and two vectors $\vec{a_i}, \vec{a_j}$ lying on l such that $\vec{a_i} + \vec{b_i} = \vec{v_i}, \vec{a_j} + \vec{b_j} = \vec{v_j}$.
- If $a_i > a_j$, exchange $\vec{a_i}$ and $\vec{a_j}$ between balls i, j.



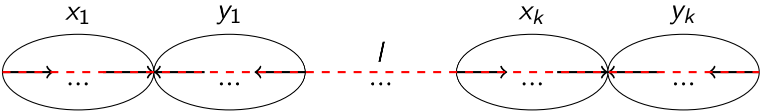
Special Case

- All n balls collinear, with centers lying on line l.



- Consider components of vectors lying on l on the left pointing left or on the right pointing right, won't change end state. Example:

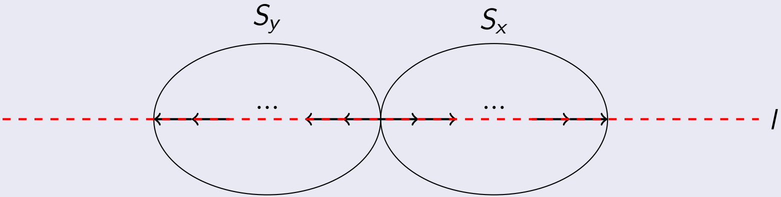
- Configuration of balls' velocity vectors lying on l:



Special Case End State

Theorem

Follow the previous notation using $x_1, \ldots, x_k, y_1, \ldots, y_k$ as length of the k groups of right and left pointing vectors respectively on line l. Let $S_y = y_1 + \cdots + y_k, S_x = x_1 + \cdots + x_k$. No matter the order of moves made, the end state is the following:



The Folding Random Walk

- Jammed billard balls configuration motivates the following random walk, called *Folding*:
- Consider m hyperplanes in $\mathbb{R}^n$ all passing through the origin.
- Assign $+$ and $-$ sign to different sides of all m hyperplanes.
- Start at some point P in $\mathbb{R}^n$. Choose one of the m hyperplanes at random.
- If P on $+$ side of hyperplane, don't do anything.
- If P on $-$ side of hyperplane, reflect P about the hyperplane.

No all pluses

- Cannot have a region on plus side of every hyperplane, else points in this region will always stay fixed under *Folding*.

Ho-Zimmerman

Exists $2[\binom{m-1}{0} + \binom{m-1}{1} + \cdots + \binom{m-1}{n-1}]$ regions formed by m distinct hyperplanes in $\mathbb{R}^n$ all passing through origin.

Corollary

If no region on plus side of every hyperplane, then $m \geq n+1$. Also, if $m \geq n+1$, we can assign the $+$ and $-$ signs of the hyperplanes so that there does not exist a region that is on the plus side of every hyperplane.

Graph under *Folding*

- Consider the following directed graph: for all vertices v in the orbit of starting point P under *Folding*, draw directed edge from v to all vertices w that can be reached from v under one step of *Folding*.
- Exists self-loops.

Theorem

In two dimensions, the graph is bipartite.

Conjecture

In any dimension, the graph is bipartite.

Characterizing 2d Orbit

- We work in two dimensions.
- Without loss of generality, suppose one of the lines is the x-axis.
- Suppose that the starting point is unit distance from the origin.

Lemma

If two lines make an angle that is an irrational multiple of π with each other, then the orbit under *Folding* is dense in the unit circle.

Lemma

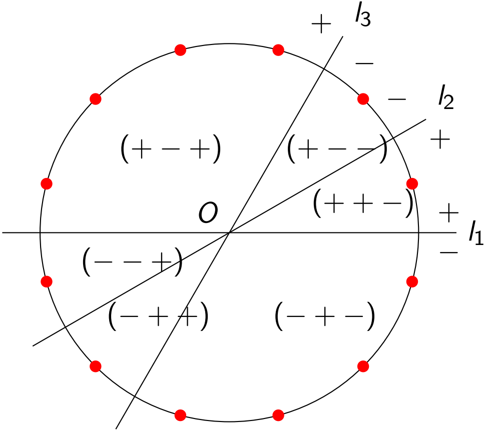
Suppose m lines all with angles that are rational multiples of π, say $\frac{p_1}{q_1}\pi, \ldots, \frac{p_m}{q_m}\pi$. There exists $\operatorname{lcm}(q_1, q_2, \cdots, q_m)$ points in orbit under *Folding* if the starting point can be written as a linear combination of $\frac{1}{q_1}\pi, \ldots, \frac{1}{q_m}\pi$ with integer coefficients and $2\operatorname{lcm}(q_1, q_2, \cdots, q_m)$ otherwise.

Simplifying the random walk

- Consider the transition matrix of *Folding* in two dimensions, with the following conditions.
- Three lines l_1, l_2, l_3 with angles $0 < \frac{p}{q}\pi < \frac{r}{s}\pi \leq \frac{\pi}{2}$ respectively.
- Start at $\frac{1}{2\operatorname{lcm}(q,s)}\pi$ on unit circle. Orbit is a regular $2\operatorname{lcm}(q,s)$-gon.

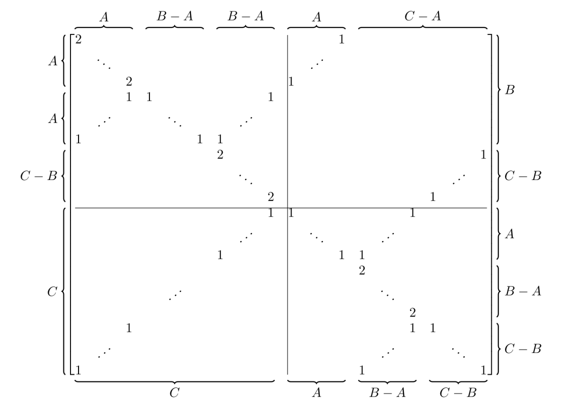
Assignment of Pluses and Minuses

- Assign the plus and minus signs to the three lines as seen in the following example with three lines, the x axis, $\frac{1}{6}\pi$, and $\frac{1}{3}\pi$.



Adjacency Matrix

For convenience set $A = \frac{\text{lcm}(q,s)}{q} \cdot p$, $B = \frac{\text{lcm}(q,s)}{s} \cdot r$, $C = \text{lcm}(q,s)$. Then given the previous conditions imposed, the adjacency matrix always takes the following form:



Example Adjacency Matrix

Examples

Consider angles $0, \frac{1}{4}\pi, \frac{1}{2}\pi$. Suppose we start *Folding* at point with angle $\frac{1}{8}\pi$ on unit circle.

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Stationary Measure, 0 mod 4

- Consider angles $0, \frac{1}{n}\pi, \frac{1}{2}\pi$ where n is $0 \pmod 4$, starting at angle $\frac{1}{2n}\pi$ on the unit circle.

0 mod 4

Suppose $n = 2k$. Stationary measure vector under *Folding* is the following, going in counterclockwise order starting from point making angle $\frac{1}{2n}\pi$ with x-axis.

$$\frac{1}{4(2^k + \frac{2^{k+1}-5}{3})}(3, 1, 2^2, 2^2, \ldots, 2^{k-2}, 2^{k-2}, 2^k, 2^k, \ldots, 2^4, 2^4, 2^2, 2^2,$$

$$1, 3, 3\cdot 2^2, 3\cdot 2^2, \ldots, 3\cdot 2^{k-2}, 3\cdot 2^{k-2}, 2^{k-1}, 2^{k-1}, \ldots, 2^3, 2^3, 2, 2)$$

Stationary Measure, 2 mod 4

- Consider angles $0, \frac{1}{n}\pi, \frac{1}{2}\pi$ where n is $2 \pmod 4$, starting at angle $\frac{1}{2n}\pi$ on the unit circle.

2 mod 4

Suppose $n = 4j + 2, j \geq 1$. Stationary measure vector under *Folding* is the following, going in counterclockwise order starting from point making angle $\frac{1}{2n}\pi$ with x-axis.

$$v = (2^{2j+1} - 1, \frac{(2^{2j+2}-1)\cdot 2^{2i}}{3}, \frac{(2^{2j+2}-4)\cdot 2^{2i}}{3},$$

for $i = 0, 1, \ldots, j-1$.

$$\frac{2^{2j+2i+4}+3\cdot 2^{2j}-2^{2i+4}}{3}, \frac{2^{2j+2i+4}-3\cdot 2^{2j}-2^{2i+2}}{3}, \frac{2^{2j+3}-2^{2j}-4}{3},$$

for $i = j-1, j-2, \ldots, 0$.

$$\frac{2^{2j}-1}{3}, 2^{2j+2i+2} - 2^{2j} - 2^{2i}, 2^{2j+2i+2} + 2^{2j} - 2^{2i+2},$$

for $i = 0, 1, \ldots, j-1$.

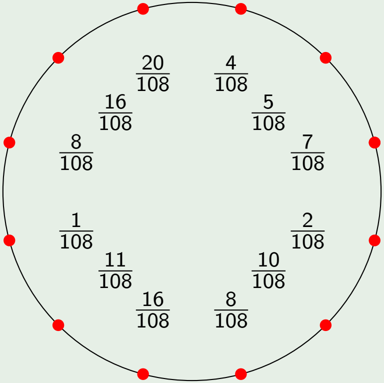
$$\frac{(2^{2j+2}-4)\cdot 2^{2i+1}}{3}, \frac{(2^{2j+2}-1)\cdot 2^{2i+1}}{3}, \frac{2^{2j+1}-2}{3})$$

for $i = 0, 1, \ldots, j-1$.

Example Stationary Measure

Examples

Consider angles $0, \frac{1}{6}\pi, \frac{1}{2}\pi$. Suppose we start *Folding* at point with angle $\frac{1}{12}\pi$ on unit circle. Vector is $\frac{1}{108}(7, 5, 4, 20, 16, 8, 1, 11, 16, 8, 10, 2)$.



Future Research

- Find stationary measure for n odd, with angles $0, \frac{1}{n}\pi, \frac{1}{2}\pi$.
- First generalize stationary measure to $0, \frac{a}{n}\pi, \frac{1}{2}\pi$ where $0 < a < n$. We have conjectures on the stationary measure in the case when $n \equiv 0 \pmod 4$, which depend on $n \pmod{16}$.
- Use Matrix Tree Theorem to help generalize in general 2d case.
- Generalize to multiple dimensions, using hyperplanes that result from the jammed billiard ball configurations.
- Use ratio of two largest magnitude eigenvalues of adjacency matrix to find the amount of time it takes to reach stationary measure.
- Work on jammed ball configurations problem directly.

Acknowledgements

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