Set Sequential Trees

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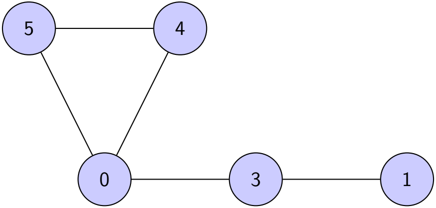
Introduction to Graph Labeling

Basic Problem:

- Vertices are given some sort of labels.
- Edges are labeled with a function of the labels of their vertices.
- Want the labels to have certain properties, such as all being distinct.

Example: Graceful labelings (Rosa, 1967)



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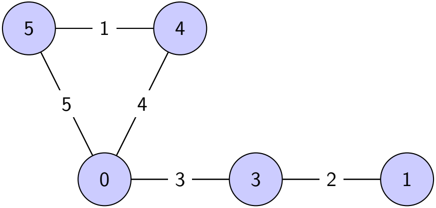
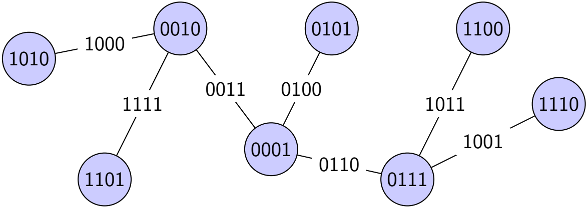


Our Problem: Set-Sequential Trees

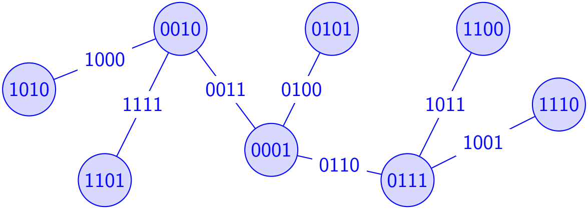
- Label vertices with strings of 0's and 1's of length n (vectors in $\mathbb{F}_2^n$).
- Label edges with binary xor of vertices (sum (mod 2) of the vectors assigned to the vertices).
- Want all vertices and edges together to use each possible nonzero vector exactly once.
- Example: A set-sequential labeling of an 8-vertex tree.



Classifying Set-Sequential Trees

- Goal: Classify all set-sequential trees.
- Set-sequential trees must have 2^{n-1} vertices for some n.
- Conjectured that all trees with 2^{n-1} vertices, all of them of odd degree, are set-sequential.

Adding Pending Edges to a Set-Sequential Tree



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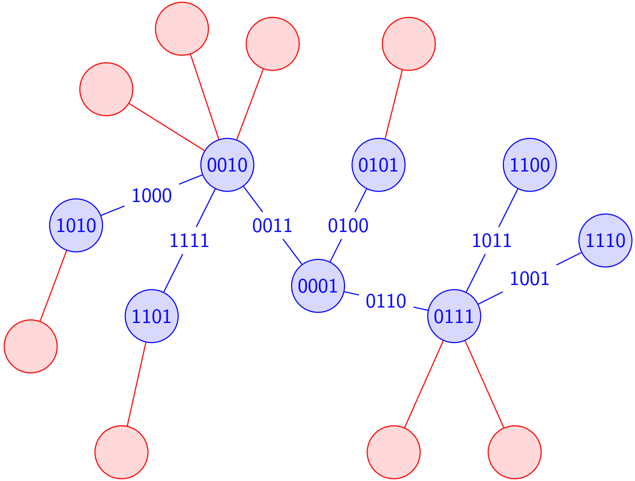

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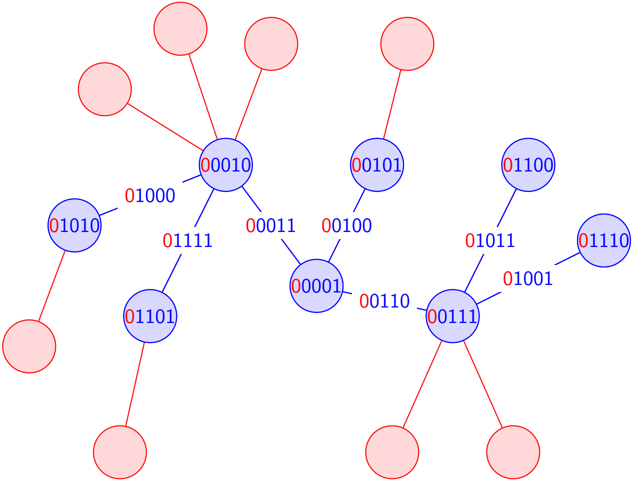

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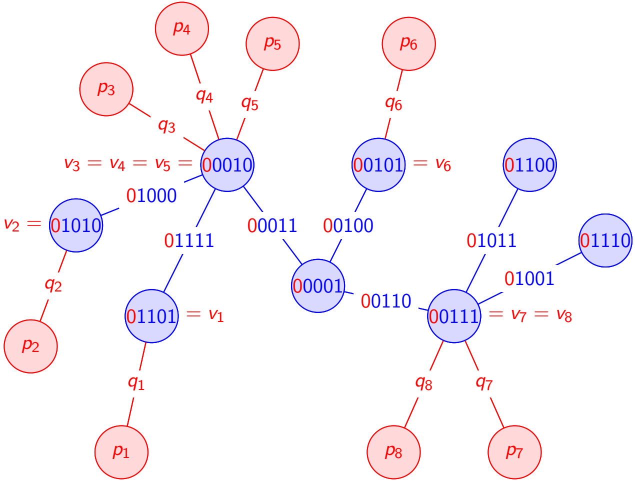

A Conjecture on Producing Set-Sequential Trees

Conjecture (Balister, Győri & Schelp, 2009)

For any 2^{n-1} non-zero vectors $v_1, \dots, v_{2^{n-1}} \in \mathbb{F}_2^n$ with $n \geq 2$ and $\sum_{i=1}^{2^{n-1}} v_i = 0$, there exists a partition of $\mathbb{F}_2^n$ into pairs of vectors (p_i, q_i) for $1 \leq i \leq 2^{n-1}$ such that $v_i = p_i + q_i$ for all i.

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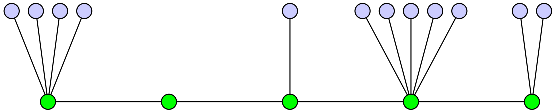
Previously known to hold when:

- $n \leq 5$, or
- Half of all v_i are equal and each v_i occurs an even number of times.

Caterpillars

- Caterpillar: a tree in which all vertices are connected to a center path.
  - Example: A caterpillar with diameter 6 (center path in green)



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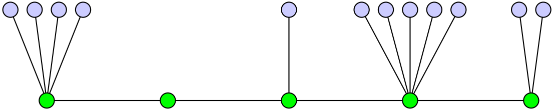

- Our approach: partially resolve the Conjecture to classify set-sequential caterpillars.

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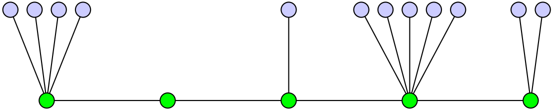

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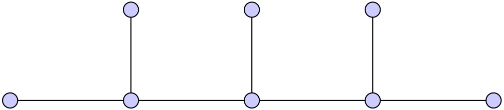

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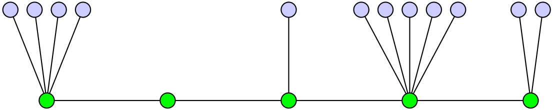

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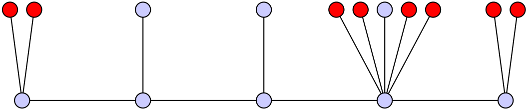

Relevant Previous Work on Set-Sequential Trees

Assuming $[\text{number of vertices}] + [\text{number of edges}] = 2^n - 1$:

- All caterpillars of diameter at most 5 with only odd-degree vertices are set-sequential.
- All paths with at least 16 vertices are set-sequential.
- No graph with exactly 1 or 2 vertices of even degree is set-sequential.
  - No similar known restriction on trees with only odd-degree vertices.

The Conjecture with Restrictions on $v_1, \ldots, v_{2^{n-1}}$

Theorem (L.G. & C.K., 2017)

The Conjecture holds for $v_1, \ldots, v_{2^{n-1}} \in \mathbb{F}_2^n$ if $\sum_{i=1}^{2^{n-1}} v_i = 0$ and if any of the following conditions is true:

1. $\dim(\mathrm{span}\{v_1, \ldots, v_{2^{n-1}}\}) \leq 5$.
2. $\dim(\mathrm{span}\{v_1, \ldots, v_{2^{n-1}}\}) = 6$ *and each vector v_i occurs an even number of times.*
3. *There are at most n distinct vectors in $\{v_1, \ldots, v_{2^{n-1}}\}$.*
4. $\dim(\mathrm{span}\{v_1, \ldots, v_{2^{n-1}}\}) \leq n/2$ *and each vector v_i occurs an even number of times.*

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2. $\dim(\text{span}\{v_1, \ldots, v_{2^{n-1}}\}) = 6$ *and each vector* v_i *occurs an even number of times.*
3. *There are at most* n *distinct vectors in* $\{v_1, \ldots, v_{2^{n-1}}\}$.
4. $\dim(\text{span}\{v_1, \ldots, v_{2^{n-1}}\}) \leq n/2$ *and each vector* v_i *occurs an even number of times.*

- Applicable to showing that many trees are set-sequential.

Caterpillars with Small Diameters

Theorem (L.G. & C.K., 2017)

All caterpillars with diameter at most 18 and containing only odd-degree vertices (with 2^n total vertices for some n) are set-sequential.

- Need base cases for induction on caterpillars.
- 10 base cases for diameter ≤ 18.
- Finitely many for larger diameters, but too many to verify through brute force.

Caterpillars with Large Average Degree of Path Vertices

- Idea: Every time pending edges are added, the diameter can increase by 1.
- Large enough caterpillars of a given diameter can be created by repeatedly adding pending edges to a star.

Theorem (L.G. & C.K., 2017)

All caterpillars with diameter $k+1$ that have only odd-degree vertices and have at least 2^k vertices (with 2^n total vertices for some n) are set-sequential.

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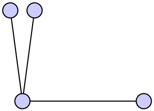

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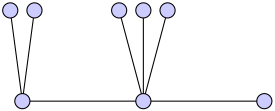

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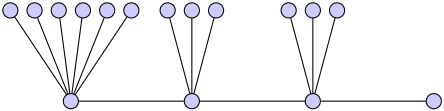

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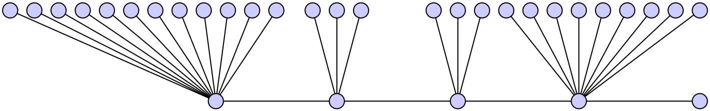

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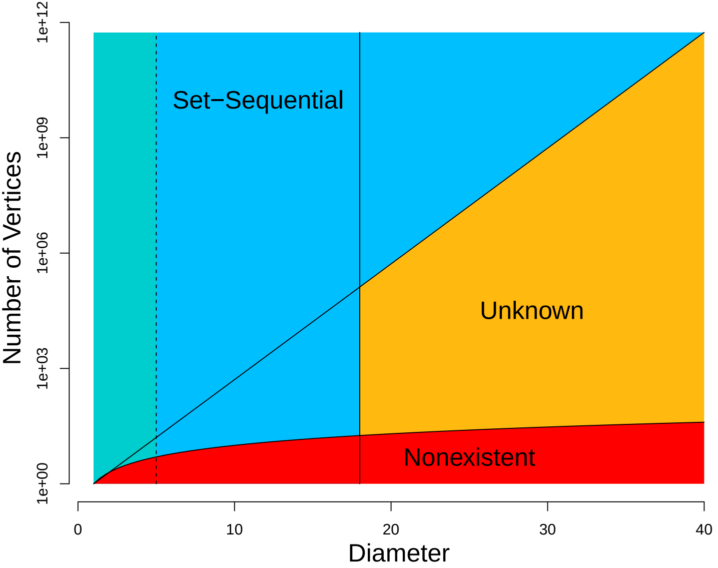
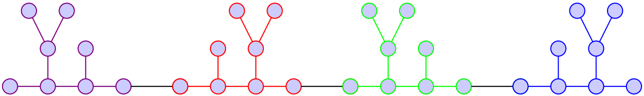
Summary of Set-Sequential Caterpillars



Caterpillars with Small Average Degree of Path Vertices

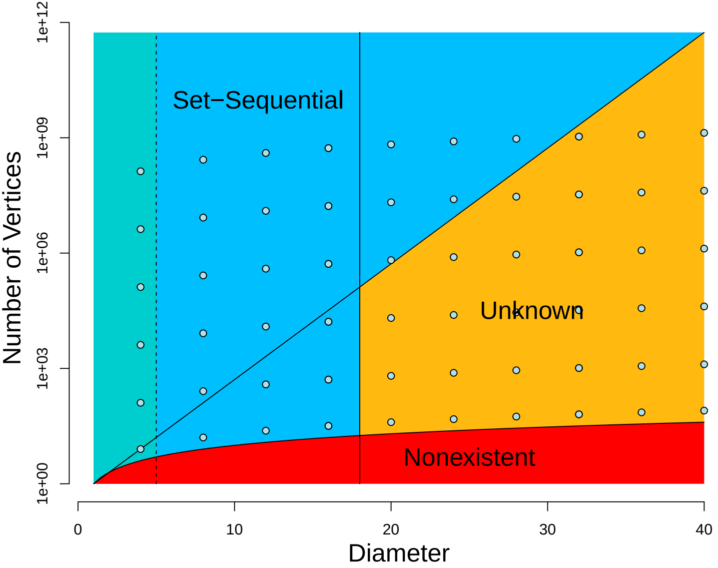
Idea: String together 4 copies of a set-sequential tree (extension of proof for set-sequentialness of paths by Mehta and Vijayakumar).



Theorem (L.G. & C.K., 2017)

For any set-sequential tree T with at least 3 vertices, let u and v be any distinct vertices in T with degree 1. Let $u_1, \ldots, u_4$ and $v_1, \ldots, v_4$ be the vertices corresponding to u and v respectively in 4 distinct copies of T. Then the tree obtained by adding (u_1, u_2), (v_2, v_3), and (u_3, u_4) as edges is set-sequential.

Caterpillars with Small Average Degree of Path Vertices



Future Work

- Make more progress on the Conjecture.
- Completely classify set-sequential caterpillars (and more generally, set-sequential trees).
  - Specifically focusing on caterpillars (and other trees) with few leaves.
- Find more general methods for proving set-sequentialness of trees.

Acknowledgements

I would like to thank:

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- MIT PRIMES
- My family

References

1. K. Abhishek and G. K. Agustine, "Set-Valued Graphs," Journal of Fuzzy Set Valued Analysis, vol. 2012, pp. 1-17, 2012.
2. K. Abhishek, "Set-Valued Graphs II," Journal of Fuzzy Set Valued Analysis, vol. 2013, pp. 1-16, 2013.
3. A. R. Mehta and G. R. Vijayakumar, "A note on ternary sequences of strings of 0 and 1," arXiv preprint arXiv:0803.4079, 2008.
4. S. M. Hegde, "Set colorings of graphs," European Journal of Combinatorics, vol. 30, pp. 986-995, May 2009.
5. P. N. Balister, E. Győri, and R. H. Schelp, "Coloring vertices and edges of a graph by nonempty subsets of a set," European Journal of Combinatorics, vol. 32, pp. 533-537, May 2011.