Pattern Avoidance on Binary Matrices

William Zhang

Under the mentorship of Jesse Geneson

May 20, 2017
Seventh Annual PRIMES Conference

Motivation

- Resolves the Stanley-Wilf conjecture

Motivation

- Resolves the Stanley-Wilf conjecture
- Bounds the complexity of an algorithm applied in motion planning in robotics

Motivation

- Resolves the Stanley-Wilf conjecture
- Bounds the complexity of an algorithm applied in motion planning in robotics
- Bounds the number of unit distances between vertices in a convex n-gon

Definitions

Binary Matrix

A *binary matrix* is a matrix that has only 0's or 1's for entries. They are also known as *0-1* matrices.

Definitions

Binary Matrix

A *binary matrix* is a matrix that has only 0's or 1's for entries. They are also known as *0-1* matrices.

For convenience, we represent them with dots.

$$\begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Definitions (continued)

Representation

Given two binary matrices A and B, we say that A *represents* B if they have the same dimensions and corresponding coefficient satisfies $B_{ij} \leq A_{ij}$.

Definitions (continued)

Representation

Given two binary matrices A and B, we say that A *represents* B if they have the same dimensions and corresponding coefficient satisfies $B_{ij} \leq A_{ij}$.

$$L_1 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & & \end{pmatrix} \qquad L_2 = \begin{pmatrix} & \bullet & \bullet & \\ \bullet & & & \bullet \\ & \bullet & \bullet & \bullet \end{pmatrix}$$

L_2 represents L_1.

Definitions (continued)

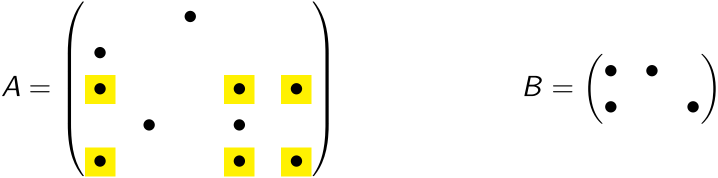
Containment and Avoidance

Let A and B be two binary matrices. We say that A *contains* B if some submatrix of A represents B. Otherwise, A *avoids* B.

Definitions (continued)

Containment and Avoidance

Let A and B be two binary matrices. We say that A *contains* B if some submatrix of A represents B. Otherwise, A *avoids* B.



Definitions (continued)

Weight

The *weight* of a binary matrix A is the number of one entries in A. We denote it $w(A)$.

Definitions (continued)

Weight

The *weight* of a binary matrix A is the number of one entries in A. We denote it $w(A)$.

The extremal function

Given a binary matrix A, we define $\text{ex}\,(A, n)$ to be the largest possible weight of an $n \times n$ binary matrix that avoids A. This function is only defined if A is a nonzero matrix.

Examples and Facts

Let B be a $k \times 1$ binary matrix of all ones. Then $\operatorname{ex}(B, n) = n(k-1)$.

Examples and Facts

Let B be a $k \times 1$ binary matrix of all ones. Then $\operatorname{ex}(B, n) = n(k-1)$.

$$A = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \qquad B = \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix}$$

Examples and Facts (continued)

Let A and B be binary matrices such that A contains B, Then $\operatorname{ex}(A, n) \geq \operatorname{ex}(B, n)$.

Examples and Facts (continued)

If B is an 0-1 matrix, then $\mathrm{ex}\,(B, m+n) \geq \mathrm{ex}\,(B, m) + \mathrm{ex}\,(B, n)$ for all m, n.

Examples and Facts (continued)

If B is an 0-1 matrix, then $\text{ex}(B, m+n) \geq \text{ex}(B, m) + \text{ex}(B, n)$ for all m, n.
Let M be an $m \times m$ matrix that avoids B, and let N be an $n \times n$ matrix that avoids B.

Examples and Facts (continued)

If B is an 0-1 matrix, then $\text{ex}(B, m+n) \geq \text{ex}(B, m) + \text{ex}(B, n)$ for all m, n.

Let M be an $m \times m$ matrix that avoids B, and let N be an $n \times n$ matrix that avoids B.

$$\begin{pmatrix} & M & & 0 & \cdots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & N & \\ 0 & \cdots & 0 & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & N & \\ 0 & \cdots & 0 & & & \\ & & & 0 & \cdots & 0 \\ & M & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \end{pmatrix}$$

avoids B.

Examples and Facts (continued)

Let B be any nonzero binary matrix except for the 1×1 matrix of a single 1 entry. Then $\mathrm{ex}(B, n) = \Omega(n)$.

Examples and Facts (continued)

Let B be any nonzero binary matrix except for the 1×1 matrix of a single 1 entry. Then $\mathrm{ex}\,(B, n) = \Omega(n)$.

$$A = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & & & & & & \bullet & \bullet \\ \bullet & & & & & & \bullet & \bullet \\ \bullet & & & & & & \bullet & \bullet \\ \bullet & & & & & & \bullet & \bullet \\ \bullet & & & & & & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \qquad B = \begin{pmatrix} \bullet \end{pmatrix}$$

Results

Lemma

Let A be an $r \times c$ binary matrix. Then $\operatorname{ex}(A, n) = \Omega\left(n^{2-\frac{r+c-2}{w(A)-1}}\right)$.

Results

Lemma

Let A be an $r \times c$ binary matrix. Then $\operatorname{ex}(A, n) = \Omega\left(n^{2-\frac{r+c-2}{w(A)-1}}\right)$.

Corollary

If A is an $r \times c$ 0-1 matrix with $w(A) > r + c - 1$, then $\operatorname{ex}(A, n) = \Omega(n^p)$ for some $p > 1$.

Results

Lemma

Let A be an $r \times c$ binary matrix. Then $\text{ex}(A, n) = \Omega\left(n^{2-\frac{r+c-2}{w(A)-1}}\right)$.

Corollary

If A is an $r \times c$ 0-1 matrix with $w(A) > r + c - 1$, then $\text{ex}(A, n) = \Omega(n^p)$ for some $p > 1$.

Lemma (CrowdMath, 2016)

If A is an $r \times c$ 0-1 matrix with $w(A) > r + c - 1$, then $\text{ex}(A, n) = \Omega(n \log n)$.

Furedi-Hajnal limit of permutations

Theorem (Marcus and Tardos, 2004)

Every $k \times k$ permutation matrix P satisfies $\mathrm{ex}(P, n) \leq 2k^4\binom{k^2}{k}n$. More importantly, $\mathrm{ex}(P, n) = \Theta(n)$

Furedi-Hajnal limit of permutations

Theorem (Marcus and Tardos, 2004)

Every $k \times k$ permutation matrix P satisfies $\mathrm{ex}(P, n) \leq 2k^4 \binom{k^2}{k} n$. More importantly, $\mathrm{ex}(P, n) = \Theta(n)$

Furedi-Hajnal limit

If P is a binary matrix such that $\mathrm{ex}(P, n) = \Theta(n)$, then $\lim_{n \to \infty} \frac{\mathrm{ex}(P,n)}{n}$ exists and is called the *Furedi-Hajnal limit*. We denote it with $c(P)$.

More definitions

Distance Vector

In matrix P, the *distance vector* between entries P_{i_1,j_1} and P_{i_2,j_2} is $(i_2 - i_1, j_2 - j_1)$.

More definitions

Distance Vector

In matrix P, the *distance vector* between entries P_{i_1,j_1} and P_{i_2,j_2} is $(i_2 - i_1, j_2 - j_1)$.

r-repetition

A vector (x, y) is *r-repeated* in a permutation matrix P if (x, y) occurs as the distance vector of at least r pairs of 1 entries.

More definitions

Distance Vector

In matrix P, the *distance vector* between entries P_{i_1,j_1} and P_{i_2,j_2} is $(i_2 - i_1, j_2 - j_1)$.

r-repetition

A vector (x, y) is *r-repeated* in a permutation matrix P if (x, y) occurs as the distance vector of at least r pairs of 1 entries.

These definitions also extend to d-dimensional 0-1 matrices.

Recent bounds

Theorem (Cibulka and Kyncl, 2016)

For almost all permutations matrices P, we have $c(P) = 2^{O(k^{2/3}(\log k)^{7/3}/(\log \log k)^{1/3})}$.

Recent bounds

Theorem (Cibulka and Kyncl, 2016)

For almost all permutations matrices P, we have
$c(P) = 2^{O(k^{2/3}(\log k)^{7/3}/(\log\log k)^{1/3})}$.

Lemma (Cibulka and Kyncl, 2016)

Almost all $k \times k$ permutation matrices are $\frac{4\log k}{\log\log k}$-repetition free.

Main result

Multidimensional permutation matrices

A *d-dimensional k-permutation matrix* is a d-dimensional matrix such that every $(d-1)$-dimensional cross section of it has exactly a single 1 entry.

Theorem

Almost all d-dimensional k-permutation matrices are $\left(\frac{2d \log k}{\log \log k}\right)$-repetition free for $d > 2$.

Further Directions

- Prove that $c(P) = 2^{o\left(k^{2/3}\right)}$ for all $k \times k$ permutation matrices P.

Further Directions

- Prove that $c(P) = 2^{o\left(k^{2/3}\right)}$ for all $k \times k$ permutation matrices P.
- Extend the known bounds for $c(P)$ to d-dimensional permutations.

Further Directions

- Prove that $c(P) = 2^{o\left(k^{2/3}\right)}$ for all $k \times k$ permutation matrices P.
- Extend the known bounds for $c(P)$ to d-dimensional permutations.
- Find stronger upper and lower bounds on the extremal function of a binary matrix based on its size and weight.

Acknowledgments

- MIT-PRIMES program
- MIT Math Department
- My mentor, Dr. Jesse Geneson
- My parents