Generalizations of Hall-Littlewood Polynomials

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Symmetric Polynomials

Definition (symmetric polynomial)

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Examples:

- $x_1 x_2 \dots x_n$
- $x_1 x_2^2 + x_1^2 x_2 + x_2 x_3^2 + x_2^2 x_3 + x_3 x_1^2 + x_3^2 x_1$
- Not symmetric: $x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2$

Monomial Symmetric Polynomials

Definition (partition)

A *partition* is a finite nonincreasing sequence of nonnegative integers. If λ is a partition, its elements are denoted $\lambda_1 \geq \lambda_2 \geq \dots$.

Example: $(4, 3, 3, 1, 0)$

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Definition (monomial symmetric polynomial)

If λ is a partition of length n, then $m_\lambda(x_1, x_2, \dots, x_n)$ is the symmetric polynomial that is the sum of all **distinct** terms that result from permuting the variables in $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$.

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Examples:

- $m_{(4,2,1)}(x_1, x_2, x_3) = x_1^4 x_2^2 x_3 + x_1^4 x_2 x_3^2 + x_1^2 x_2^4 x_3 + x_1^2 x_2 x_3^4 + x_1 x_2^4 x_3^2 + x_1 x_2^2 x_3^4$
- $m_{(2,2,0)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2$
- $m_{(1,1,1)}(x_1, x_2, x_3) = x_1 x_2 x_3$

These polynomials form a basis of all symmetric polynomials.

Hall-Littlewood Polynomials

Definition (Hall-Littlewood polynomial)

For a constant t and a partition λ, the *Hall-Littlewood polynomial* is given by:

$$P_\lambda(x_1, \ldots, x_n; t) = \text{const}_\lambda(t) \sum_{\sigma \in S_n} \sigma \left(\prod_{1 \le i \le n} x_i^{\lambda_i} \prod_{1 \le i < j \le n} \frac{x_i - t x_j}{x_i - x_j} \right)$$

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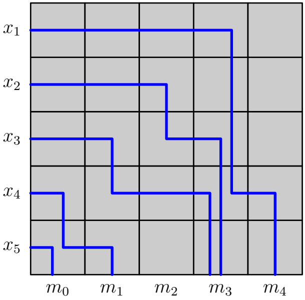
- $t = 1$ gives the monomial symmetric polynomials.
- $t = 0$ gives the Schur polynomials, another important basis of symmetric polynomials.

Path Ensembles



$$\lambda = (4, 3, 3, 1, 0)$$

This is a *type-A path ensemble with end state* λ with n paths, where n is the length of λ.

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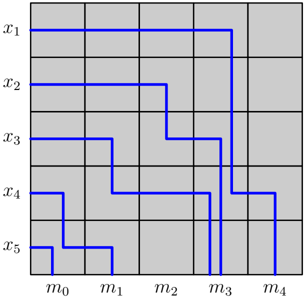

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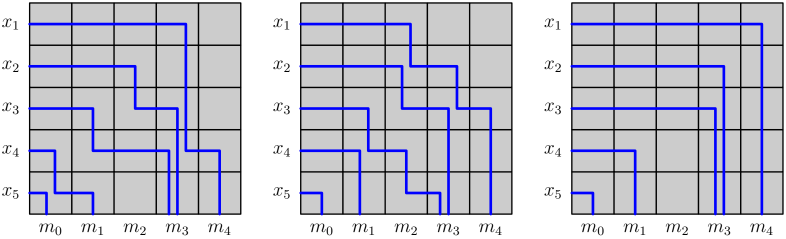
This is a *type-A path ensemble with end state* λ with n paths, where n is the length of λ.

- Paths may only go right and down and cannot intersect.
- Paths enter at the left and exit at the bottom.
- Only one horizontal path may enter or exit each tile.
- The number of paths exiting the bottom in column i is the number of instances of i in λ.
- Each tile is associated with a variable x_i, and each tile in a row is associated with the same variable.

Path Ensembles



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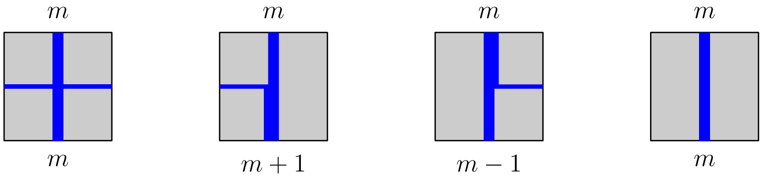
Path Ensembles

Each tile is assigned a *weight*, depending on the paths that enter and exit it and the variable x associated with it (t is, again, a constant):



$$x \qquad x \qquad 1 - t^m \qquad 1$$

The weight of a path ensemble is the product of the weights of all its tiles.

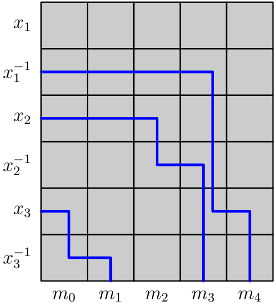
Path Ensembles and Hall-Littlewood Polynomials

Theorem (Tsilevich)

For a partition λ, the sum of the weights of all type-A path ensembles with end state λ, divided by $\prod_{1\leq i\leq n} x_i$, is the Hall-Littlewood polynomial:

$$\mathrm{const}_\lambda(t) \sum_{\sigma\in S_n} \sigma\left(\prod_{1\leq i\leq n} x_i^{\lambda_i} \prod_{1\leq i<j\leq n} \frac{x_i - tx_j}{x_i - x_j}\right)$$

Type-BC Path Ensembles



$$\lambda = (4, 3, 1)$$

This is a *type-BC path ensemble with end state λ.*

- The construction is similar to type-A path ensembles, but there are twice as many rows, with tiles associated with the variable x_i or x_i^{-1}.
- At the left, exactly one path enters in each pair of rows with variables x_i and x_i^{-1}.
- Weights are the same, but with an additional weight of $-t$ each time a path enters in the row associated with x_i^{-1}.
- (The model is simplified slightly here for brevity.)

Type-BC Hall-Littlewood Polynomials

Theorem (Wheeler, Zinn-Justin)

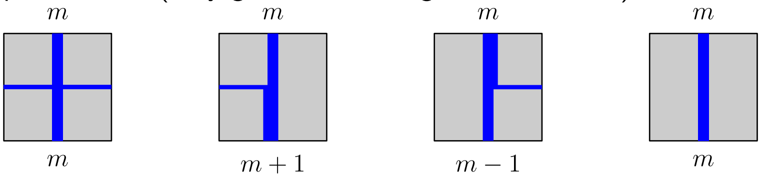
The sum of the weights of all type-BC path ensembles with end state λ, divided by $\prod_{1\le i\le n}(x_i - tx_i^{-1})$, is the type-$BC$ Hall-Littlewood polynomial:

$$\mathrm{const}_\lambda(t) \sum_{\omega\in S_n\rtimes\{\pm1\}^n} \omega\left(\prod_{1\le i\le n} \frac{x_i^{\lambda_i}}{1-x_i^{-2}} \prod_{1\le i<j\le n} \frac{(x_i - tx_j)(x_ix_j - t)}{(x_i - x_j)(x_ix_j - 1)}\right)$$

where $S_n \rtimes \{\pm1\}^n$ is similar to the set of permutations of the variables, except that it can take x_i to either x_j or x_j^{-1}.

Generalized Weights

Generalized weights are defined with an additional constant parameter s (they give normal weights when $s = 0$):



$\frac{x-st^m}{1-sx}$	$\frac{(1-s^2t^m)x}{1-sx}$	$\frac{1-t^m}{1-sx}$	$\frac{1-st^m x}{1-sx}$

Problem

Theorem (Borodin)

The sum of the generalized weights of all type-A path ensembles with end state λ, divided by $\prod_{1\leq i\leq n} x_i$, is:

$$\mathrm{const}_\lambda(t,s)\sum_{\sigma\in S_n}\sigma\left(\prod_{1\leq i\leq n}\frac{((x_i-s)/(1-sx_i))^{\lambda_i}}{1-sx_i}\prod_{1\leq i<j\leq n}\frac{x_i-tx_j}{x_i-x_j}\right).$$

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Question (Borodin)

How do we similarly generalize type-BC Hall-Littlewood polynomials?

Our Result

Theorem

The sum of the generalized weights of all type-BC path ensembles with end state λ, divided by $\prod_{1\le i\le n}(x_i - tx_i^{-1})$, is:

$$\mathrm{const}_\lambda(t,s) \sum_{\omega\in S_n \rtimes \{\pm 1\}^n} \omega\left(\prod_{1\le i\le n} \frac{((x_i - s)/(1-sx_i))^{\lambda_i}}{(1-x_i^{-2})(1-sx_i)} \prod_{1\le i<j\le n} \frac{(x_i - tx_j)(x_ix_j - t)}{(x_i - x_j)(x_ix_j - 1)} \right).$$

Further Research

- Type-BC polynomials have a *dual* formed by slightly modifying the weights. What is the corresponding dual of generalized type-BC polynomials?

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- Type-BC polynomials have a *dual* formed by slightly modifying the weights. What is the corresponding dual of generalized type-BC polynomials?
- Borodin defined *skew* generalized type-A Hall-Littlewood polynomials by allowing paths to enter at the top. Is there an analog for type-BC Hall-Littlewood polynomials?
- What previously proven identities can be generalized to apply to generalized type-BC Hall-Littlewood polynomials?

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