# Verma modules for the Virasoro algebra

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# Lie Algebras

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  - Bilinear: $[x+y,z] = [x,z] + [y,z], \quad \alpha[x,y] = [\alpha x, y]$
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  - Jacobi identity $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$

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## Example

- $\mathfrak{gl}_n(\mathbb{C})$, $n \times n$ complex matrices
- $[a, b] = ab - ba$

## Example

Abelian Lie Algebras: $\mathfrak{g} = V$, a vector space, with $[\cdot, \cdot] = 0$

# Examples of Lie Algebras

**Example**

$\mathfrak{sl}_2(\mathbb{C})$: $2 \times 2$ complex matrices with trace 0.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

# More Examples of Lie Algebras

**Example**

Heisenberg Algebra:

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0$$

- Jacobi: $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$

# Motivation For Representation Theory

- Goal: understand Lie algebras, specifically the Virasoro, through linear algebra, i.e., matrices.
- Main Tool: Representation Theory.

# Representations of Lie Algebras

## Definition

- An $n$-dimensional **representation** of $\mathfrak{g}$ is an $n$-dimensional vector space $V = \mathbb{C}^n$ and a map $\rho : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ such that:

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

- Here:
  1. $[x, y]$ is the Lie bracket of $x$ and $y$ in $\mathfrak{g}$
  2. $\rho(x)$ and $\rho(y)$ are just $n \times n$ matrices and $\rho(x)\rho(y)$ refers to matrix multiplication.
- We think of the matrices $\rho(x)$ and $\rho(y)$ as operators on $V$ since $n \times n$ matrices act on $\mathbb{C}^n$.

# Examples of Representations

**Example**

Trivial representation: Here, $V$ is any vector space and $\rho(x)$ is the 0 matrix for all $x \in \mathfrak{g}$.

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**Example**

One dimensional representations: If $V = \mathbb{C}$, then $\mathfrak{gl}_1(\mathbb{C}) = \mathbb{C}$. So a choice of a representation is just a choice of scalar for each $x \in \mathfrak{g}$ such that $\rho([x, y]) = 0$ for each $x, y \in \mathfrak{g}$.

# SL2

### Example

$\mathfrak{sl}_2(\mathbb{C})$ has a natural representation on $\mathbb{C}^2$ where $e, f, h$ are represented by the defining matrices:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Irreducible Representations

- Irreducible representations are building blocks for other representations and are hence the first things to study.
- A **subrepresentation** is a vector subspace $U \subseteq \mathbb{C}^n$ invariant under all matrices that come from the Lie algebra $\mathfrak{g}$.
- $V$ is **irreducible** if $V$'s only subrepresentations are 0 and $V$.

## The Virasoro Algebra

- As a vector space, the Virasoro algebra has basis

$$\cdots L_{-2}, L_{-1}, L_0, L_1, L_2, \cdots ,$$

  along with a central element $c$
- Lie bracket satisfies $[L_n, c] = 0$ and

$$[L_m, L_n] = (m-n)L_{m+n} + c \cdot \frac{m^3 - m}{12} \cdot \delta_{m+n,0}$$

# Irreducibles for Virasoro and Singular Vectors

- **Singular Vectors**: Given a representation $V$ for Virasoro, a vector $v$ is singular if
  1. There exist some complex numbers $h, c$ such that
  $$L_0(v) = hv,\ c(v) = cv.$$
  2. For every $k > 0$, $L_k(v) = 0$.

  The pair $(h, c)$ is called the weight of the singular vector.
- Irreducibles for Virasoro are labelled by weights $(h, c)$. For each irreducible representation $V$, there is a unique singular vector $v \in V$ and its weight is the same as the label for $V$. This singular vector is called the highest weight vector of $V$.

# Verma Modules

- How to construct irreducible representations of given highest weight? Verma modules.
- Fix a highest weight $(h, c)$. The Verma module of this highest weight is an infinite dimensional representation of Virasoro.
- Basis: Ordered monomials in the $L_k$ for $k < 0$ of the form $L_{i_1}^{a_i} \cdots L_{i_m}^{a_m}$, where the $i_j$ are nondecreasing negative integers and $a_i$ are non-negative integers.
- Example: $L_{-2}L_{-1}$ is a basis element but $L_{-1}L_{-2}$ is not.
- Vacuum Vector: 1 is a basis element as well, since we allow the exponents to be 0. We call this the vacuum vector.

# Verma modules contd

- The action of the Virasoro is described as follows:
  1. On the vacuum vector 1: $L_0$ acts by $h$ and $c$ by $c$. $L_k$ kills 1 for $k > 0$. $L_{-k}(1) = L_{-k}$.
  2. On other basis vectors, we use the commutation relations of Virasoro to put
  $$L_k L_{i_1}^{a_1} \cdots L_{i_m}^{a_m}$$
  in increasing order.
  3. We may have some terms left over with $L_0, c$ of $L_k > 0$ on the right. In the first two cases, we just multiply by $h, c$. In the last case, we set it equal to 0.
- Key Fact: The irreducible assoc. to $(h, c)$ is the quotient of the Verma module assoc. to $(h, c)$ by all proper subrepresentations.

# Example of Computation

$$\textbf{Recall}: \ [L_m, L_n] = (m-n)L_{m+n} + c \cdot \frac{m^3 - m}{12} \cdot \delta_{m+n,0}$$

$$\begin{aligned} L_1 \cdot (L_{-3}L_{-1}) &= L_{-3}L_1L_{-1} + [L_1, L_{-3}]L_{-1} \\ &= L_{-3}L_1L_{-1} + 4L_{-2}L_{-1} \\ &= L_{-3}L_{-1}L_1 + L_{-3}[L_1, L_{-1}] + 4L_{-2}L_{-1} \\ &= L_{-3}L_{-1}L_1 + 2L_{-3}L_0 + 4L_{-2}L_{-1} \\ &= 0 + 2hL_{-3} + 4L_{-2}L_{-1}. \end{aligned}$$

# Irreducible Factors of Vermas

- Every Verma module $V$ has a composition series of submodules
$$V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_n = 0$$
such that each subquotient $V_i / V_{i+1}$ is irreducible.
- These irreducible subquotients are unique up to reordering and are called the composition factors or irreducible factors of $V$.
- Irreducible factors are in bijection with singular vectors inside $V$.
- The number of times the irreducible assoc. to $(h, c)$ appears as a composition factor in $V$ is the number of independent singular vectors in $V$ of weight $(h, c)$.

# Computing singular vectors

- Goal: Understand composition series of Vermas by computing all singular vectors.
- Method:
  1. We can write each raising operator $L_k$ as a matrix in the given basis for the Verma module. The Verma modules are graded by degree of the basis monomials and $L_k$ raises degree by $k$. Focusing degree by degree gives finite matrices.
  2. Compute the null space of each matrix via computer algebra software.
  3. Compute the intersection of the nullspaces. This intersection is the space of all singular vectors.
  4. Compute the weights of the singular vectors.

## Current Progress

- Have code for computing matrices for $L_{k>0}$ for degree of at most 15
- For example, when the degree is 2, the matrix for the action of $L_1$ is

$$\begin{pmatrix} 0 & 2h & 0 & 0 \\ 0 & 0 & 4h+2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{which encodes} \begin{aligned} L_1 L_0 &= 0 \\ L_1 L_{-1} &= 2h \\ L_1 L_{-1} L_{-1} &= (2+4h) L_{-1} \\ L_1 L_{-2} &= 3L_{-1}. \end{aligned}$$

- Can compute null spaces
- Difficulty: intersecting these null spaces to find singular vectors

# Future Research

- Ultimate goal: do the above for more complicated $q$-deformed Heisenberg-Virasoro algebra.
- Main problem: commutation relations are much more involved than in the Virasoro algebra. Hence, the algorithm is much slower. The only commutation relation that fits on the slide is

$$[T_a, U_b] = \sum_{k=1}^{\infty} c_k \left( q^k U_{b-k} T_{a+k} - T_{a-k} U_{b+k} \right).$$

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