Maps between Critical Groups of Group Representations

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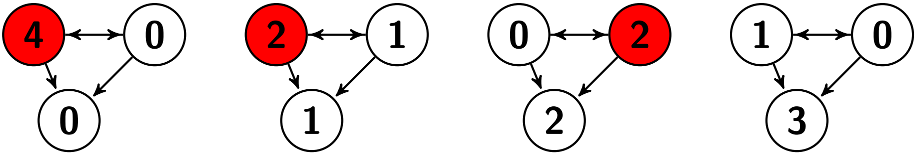
Introduction to Chip Firing



$[0, 0, 4]^t$ $[1, 1, 2]^t$ $[2, 2, 0]^t$ $[3, 0, 1]^t$

Definitions for the Graph Case

- Let there be v_i chips on node i of our graph G. Define a chip configuration, $v = [v_0, v_1, .., v_l]^t \in \mathbb{N}^{l+1}$.

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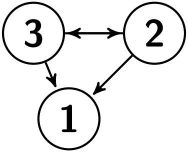
- *Stable Configurations* are chip configurations $v < d^C$ which do not permit additional firings.
- *Recurrent Configurations* are stable configurations v such that for all chip-configurations w, selectively adding chips to w and stabilizing yields v.

Laplacian Matrix

A Laplacian Matrix, $L(G) = D - A$, where D is the degree matrix such that $D_{ij} = deg(\text{node i})$ if $i = j$ and $D_{ij} = 0$ if $i \neq j$. A is an adjacency matrix such that A_{ij} is the number of edges from node i to node j.

Define a dynamical firing on node i that sends v to $v - r_i$, where r_i corresponds to the ith row of $L(G)$, the Laplacian Matrix of G. Let d^C be the diagonal of $L(G)$.

Example of Graph Laplacian



$$L(G) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

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Critical Groups

Definition

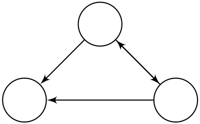
Let G be a digraph on n vertices with global sink s. The critical group of G is the group quotient:

$$K(G) = \mathbb{Z}^n / \mathrm{im}(L^t(G))$$

Theorem

Let G be a digraph with a global sink. The set of all recurrent chip on G is an abelian group under the operation $(v, w) \to stab(v + w)$, and it is isomorphic via the inclusion map to the critical group $L(G)$.

Revisiting the Graph Example



$$L(G) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$K(G) = \mathbb{Z}/3\mathbb{Z}$$

Recurrent Configurations: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Each recurrent must have order 3:
$[1,1]^t$ is the zero recurrent of order 1
$[0,1]^t$ has order 3 since $stab([0,3]^t) = [1,1]$
$[1,0]^t$ has order 3 since $stab([3,0]^t) = [1,1]$

Introduction to Representation Theory

- A group G is a set of elements that are closed under a certain binary operator (the group operation), associativity, identity, and invertibility. For example, the 3rd roots of unity form a group under the operation of normal multiplication.

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- A representation of a group G on a vector space V, is a homomorphism or map $p : G \to GL(V)$ such that:

$$p(g_1)p(g_2) = p(g_1 g_2)$$

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  for all $g_1, g_2 \in G$
- An explicit example for the cyclic group C_3 with elements $1, g, g^2$ is:

  $1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad g \to \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ $\quad g^2 \to \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

Analogy to the Group Case

For a representation V of the group G: Define the McKay-Cartan Matrix to be $\tilde{C} = nI - M$, where n is the dimension of V and:

$$\chi_V \chi_i = \sum m_{ij} \chi_j$$

The chip-firing game applies to any $\mathbb{Z}$-matrices and we introduce the McKay-Cartan marix here instead of the graph Laplacian Matrix.
The reduced McKay-Cartan Matrix can be defined as the submatrix by removing the first row and column. In this way, the critical group is defined analagously as:

$$K(V) = \mathbb{Z}^n / \mathrm{im}(C^t(V))$$

The Abelian Group Case

In the case of abelian groups, we have a complete classification of the map p and found an exact correspondance of p to regular covering maps on graphs. From Reiner-Tseng, we also have a combinatorial interpretation of the kernel of our map. In fact, we have discovered the following theorem:

Definition

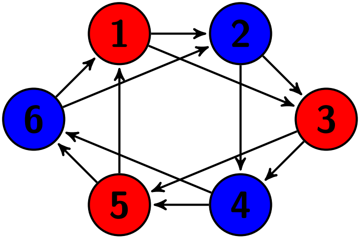
The Cayley Graph of a group G with generating set S has elements of the group as its node and edges between g and gs for elements $s \in S$. The nodes of our Cayley Graph are the irreducible representations of G and the edges correspond to the choice of our faithful representation V.

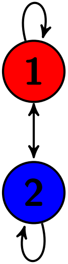
Theorem

There is a surjection of critical groups from $K(V)$ to $K(Res_H^G V)$ corresponding to the map p, a graph covering map on the Cayley Graphs of each group.

Cyclic Group Example

Let $G = C_6 =< g \mid g^6 = e >$ and consider the representation $V = V_{w^2} \oplus V_w$, where V_{w^k} sends $g \to w^k$ where $w^6 = 1$. Consider the subgroup $H = C_2$ and the regular covering can be depicted by:





General Maps Between Critical Groups

Define the map $p : \mathbb{Z}^{l+1} \to \mathbb{Z}^{l'+1}$, with standard matrix A, such that:

$$ResV_i = \oplus W_j^{\oplus A_{ij}}$$

Theorem

The following diagrams commute:

$$\begin{array}{ccc} \mathbb{Z}^{l+1} & \xrightarrow[C^t(V)]{} & \mathbb{Z}^{l+1} \\ p\downarrow & & \downarrow p \\ \mathbb{Z}^{l'+1} & \xrightarrow[C^t(ResV)]{} & \mathbb{Z}^{l'+1} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}^{l'+1} & \xrightarrow[C^t(ResV)]{} & \mathbb{Z}^{l'+1} \\ p^t\downarrow & & \downarrow p^t \\ \mathbb{Z}^{l+1} & \xrightarrow[C^t(V)]{} & \mathbb{Z}^{l+1} \end{array}$$

Hence, we have a map, $\pi : K(V) \to K(ResV)$ on cosets:

$$\pi : u + im(C^t(V)) \to p(u) + im(C^t(ResV))$$

Proof Outline

- From characters, p corresponds to restriction of virtual representations, considered in $\mathbb{Z}^{l+1}$

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- Check that $ResV_1 \otimes ResV \cong Res(V_1 \otimes V)$ (Commutativity with M)
- Check that $IndW_1 \otimes V \cong Ind(W_1 \otimes ResV)$

The Symmetric Group

- It is known that p is surjective as a linear map and also as a map of cosets: $p : K(S_n) \twoheadrightarrow K(S_{n-1})$.

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- Induction and restriction of irreducible representations is well-understood with the concept of Young Diagrams (Binary Matrix for p).

Symmetric Case Continued

Theorem from Gaetz

Let γ be the reflection representation of S_n and let $p(j)$ denote the number of partitions of the integer j. Then:

$$K(\gamma) \cong \bigoplus_{i=2}^{p(n)-p(n-1)} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{1 \leq j \leq n, p(j)-p(j-1) \geq i} j$$

Lemma

The kernel of our map, p, for γ is as follows:

$$ker(p) = (\mathbb{Z}/n\mathbb{Z})^{p(n)-p(n-1)-1}$$

Future Work

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- Give explicit formulas and/or bounds on critical groups for specific representations of the symmetric group (Using repeated character values and injectivity of p^t)

Theorem from Gaetz

If χ_γ is real-valued, as in the symmetric group case, and $\chi_\gamma(c)$ is an integer character value achieved by m different conjugacy classes, then $K(\gamma)$ contains a subgroup isomorphic to $(\mathbb{Z}/(n-\chi_\gamma(c))\mathbb{Z})^{m-1}$

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- Investigate other potential maps such as dualization (commutes with the same diagram)
- Identify connections to chip firing

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