Hilbert Series of the Representation of Cherednik Algebras

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What is an Algebra?

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- For example, $A = K[x_1, x_2, ..., x_n]$ is the algebra of polynomials in n variables with coefficients in K. One basis for this algebra as a vector space is the set of all monomials.
- $\{1, x_1, x_2, ..., x_n, x_1^2, x_1x_2, ..., x_1x_n, x_2^2, x_2x_3, ...\}$

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- $P = 3 + 2x + 5x^3$
- $P = (3, 2, 0, 5) = 3(1, 0, 0, 0) + 2(0, 1, 0, 0) + 5(0, 0, 0, 1)$

Graded Algebra

- Suppose our algebra A can be written as

$$\bigoplus_{i \geq 0} A_i = A_0 \oplus A_1 \oplus ...$$

where each A_i is a vector space, and $A_i A_j \subset A_{i+j}$ for all i, j. Then our algebra is a **graded algebra**.

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- $K[x_1, x_2, \ldots, x_n]$ is a graded algebra with A_i being the vector space of homogeneous polynomials of degree i.
- $\{1\}$
- $\{x_1, x_2, \ldots, x_n\}$
- $\{x_1^2, x_1 x_2, \ldots, x_1 x_n, x_2^2, x_2 x_3, \ldots\}$
- $\{x_1^3, x_1^2 x_2, \ldots, x_1^2 x_n, \ldots\}$
- ...

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- $\dim(A_0) = 1$
- $\dim(A_1) = |\{x_1, x_2, ..., x_n\}| = n$
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- See a pattern?
- $h(t) = \sum_{i \geq 0} t^i \binom{n+i-1}{i} = \sum_{i \geq 0} (-t)^i \binom{-n}{i} = (1-t)^{-n}$
- rational function

Dunkl Operators

- Let S_{ij} be the operator that switches x_i and x_j in a polynomial
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- e.g. $S_{12}(x_0 + x_1^2 + x_2) = (x_0 + x_1 + x_2^2)$
- For a given constant c, the **Dunkl operators** D_i are such that $D_i(P) = \frac{\partial P}{\partial x_i} - c \sum_{j \neq i} \frac{P - S_{ij}(P)}{x_i - x_j}$
- P has degree $d \Rightarrow D_i(P)$ is 0 or has degree $d - 1$
- They commute

Dunkl Example

$$D_1(x_1^3) = \frac{\partial x_1^3}{\partial x_1} - c\sum_{j\neq 1}\frac{x_1^3 - S_{1j}(x_1^3)}{x_1 - x_j}$$
$$= 3x_1^2 - c\sum_{j\neq 1}\frac{x_1^3 - x_j^3}{x_1 - x_j}$$
$$= 3x_1^2 - c\sum_{j\neq 1}(x_1^2 + x_1 x_j + x_j^2)$$

Cherednik Algebra

- The Cherednik algebra, H, is generated by all of $x_1, x_2, ..., x_n$, all possible $y_i - y_j$, and all S_{ij}, with some relations between them, with the additional fact that $x_1 + x_2 + ... + x_n = 0$.
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- The x_i and y_i don't necessarily commute (but the x_i commute and the y_i commute).
- A **representation** of this Cherednik algebra is just another algebra A such that all elements of the Cherednik algebra act as operators on A.
- For each element k of H, we assign a function $\rho_k : A \to A$, often just notated k.
- Addition and multiplication become addition and composition.

The Polynomial Representation

- Let the x_i act as operators where $x_i(P) = x_i \cdot P$, and the $y_1, y_2, ..., y_n$ are such that $y_i(P) = D_i(P)$
- $I = (x_1 + x_2 + ... + x_n)$ (the ideal)
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- $I = (x_1 + x_2 + ... + x_n)$ (the ideal)
- For $P \in I, Q \in H, Q(P) \in I$
- So, let $A = K[x_1, x_2, ..., x_n]/I$
- The representation of the Cherednik algebra where the elements act as operators on A is called the **polynomial representation**
- $(y_i - y_j)((x_1 + x_2 + ... + x_n) \cdot (P)) = (x_1 + x_2 + ... + x_n) \cdot (y_i - y_j)(P)$

Subrepresentation

- If for some subspace $B \subset A$ we have that all the operators fix B, then B gives a subrepresentation.
- The existence of a nontrivial subrepresentation means our representation is not irreducible.

Shapovalov Form

- Consider the symmetric bilinear form $\beta : AxA^{*} \Rightarrow K$ defined with $\beta(1,1) = 1$, $\beta(P, yQ) = \beta(D_y(P), Q)$, and $\beta(xP, Q) = \beta(P, D_x(Q))$
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- The exact definition of D_x is unimportant
- The kernel of the form is $\{P \in A | \forall Q, \beta(P, Q) = 0\}$.
- $\beta(xP, Q) = \beta(P, D_x(Q)) = 0, \beta(D_y(P), Q) = \beta(P, yQ) = 0$, symmetric
- kernel is a subrepresentation

Irreducible Representation

We consider H acting on $A/\ker(\beta) = B$, which turns out to be an irreducible representation. The problem is then to describe $A/\ker(\beta)$. K is taken to be a field of positive characteristic $p > 0$.

Past Results

Remember that the Hilbert series is $h(t) = \sum_{i \geq 0} t^i \dim(A_i)$.

For $p|n$, Devadas and Sun have shown that for all but finitely many c (in the set $\{-1, 1, ..., \frac{p-1}{2}\}$), the Hilbert series for the irreducible representation is $(1 + t + t^2 + ... + t^{p-1})^{n-1}$. Note that the total dimension, which is $(n-1)p$, is finite.

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- $P = x_1^{d_1} x_2^{d_2} ... x_n^{d_n}$
- $(x_1^{d_1} x_2^{d_2} ... x_n^{d_n}, y_1^{d_1} y_2^{d_2} ... y_n^{d_n}) = d_1! d_2! ... d_n!$

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- If $y_1^{e_1} y_2^{e_2} ... y_n^{e_n} \neq y_1^{d_1} y_2^{d_2} ... y_n^{d_n}, (x_1^{d_1} x_2^{d_2} ... x_n^{d_n}, y_1^{e_1} y_2^{e_2} ... y_n^{e_n}) = 0$
- If $\exists d_i \geq p,\ d_1! d_2! ... d_n! = 0 \Rightarrow P \in ker$

Current Work: The c=0 Case

- $B = A/\ker(\beta)$
- $\dim(B_d) = |\{x_1^{d_1}x_2^{d_2}...x_n^{d_n} | d_i < p \forall i, \sum d_i = d\}|$

Current Work: The c=0 Case

- $B = A / \ker(\beta)$
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Theorem

When $c = 0$, the Hilbert series is just the generating function $(1 + t + t^2 + ... + t^{p-1})^{n-1}$.

Current and Future Work - p Dividing n

- As mentioned before, Devadas and Sun proved that the Hilbert Series is $(1 + t + ... + t^{p-1})^{n-1}$, unless $c \in \{-1, 1, ..., \frac{p-1}{2}\}$.
- We have written some code to compute Hilbert series.

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For $K = \mathbb{F}_3$ and $p = n = 3$, $c = 0, 1, 2$ all give $(1 + t + t^2)^2$.

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For $K = \mathbb{F}_5$ and $p = n = 5$, $c = 4 = -1$ gives $(1 + 3t + t^2)(1 + t + t^2 + t^3 + t^4)$ (instead of $((1 + t + t^2 + t^3 + t^4)^4)$.

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