Higher Gonalities of Erdős-Rényi Random Graphs

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Abstract

We consider the asymptotic behavior of the second and higher gonalities of an Erdős-Rényi random graph and provide upper bounds for both via the probabilistic method. Our results suggest that for sufficiently large n, the second gonality of an Erdős-Rényi random Graph $G(n,p)$ is strictly less than and asymptotically equal to the number of vertices under a suitable restriction of the probability p. We also prove an asymptotic upper bound for all higher gonalities of large Erdős-Rényi random graphs that adapts and generalizes a similar result on complete graphs. We suggest another approach towards finding both upper and lower bounds for the second and higher gonalities for small $p = \frac{c}{n}$, using a special case of the Riemann-Roch Theorem, and fully determine the asymptotic behavior of arbitrary gonalities when $c \leq 1$.

Contents

1 Introduction

The topic of graph divisors is a fundamental notion within computer science and combinatorics, especially when considered in the context of the chip-firing game. The notion of the chip firing game itself manifests when discussing the complex structures of the sandpile model, in [6]. In fact, the chip-firing game can be related to the Tutte polynomial, self-organized criticality, and even theoretical physics; these connections are documented in [6] and in [7]. One important notion related to the chip-firing game on graph divisors is the gonality of a given graph. The problem of the gonality also arises when considering the Brill-Noether theorem in algebraic geometry, as the gonality for graphs is a discrete analogue of the gonality of algebraic curves. In [2] the problem of the k-th gonality of complete graphs for arbitrary k was discussed and resolved conclusively. The notion of v-reduced divisors, a key concept used to study the rank and gonality of graph divisors in [2] and elsewhere, has also been linked to potential graphs in [9]. The Erdős-Rényi model was developed by Paul Erdős and Alfréd Rényi in [2] to examine the connectivity of graphs and to formalize the notion of a random graph, noting a phase transition in the connectivity of random graph. Meanwhile, [10] examines the treewidth in Erdős-Rényi random graphs, a lower bound for the first and therefore second gonality as noted in [3]. In particular, [10] noted a phase transition in the expected treewidth of a random graph around $p = \frac{c}{n}$, a case that we also considered worthy of attention. The Erdős-Rényi definition has also been used in [11] in conjunction with the SIR model to model the spread of disease, among various other applications, as a formal notion of a randomly selected simple graph.

We considered the question, "What is the expected value of the second gonality of an Erdős-Rényi random graph?" The corresponding problem for the first gonality of an Erdős-Rényi random graph was partially resolved in [3]; it is equal to the number of vertices in the graph under a suitable restriction of the probability p. However, the asymptotic behavior for the second gonality of an Erdős-Rényi random graph had not been established conclusively. Meanwhile, the exact second and higher gonalities of the special case of complete graphs was determined in [2], and in particular, the second gonality of a complete graph was exactly equal to the number of vertices in such a graph. This led us to use the Erdős-Rényi model to prove that, under the same set of restrictions on the probability as considered in [3], the second gonality is also asymptotically equal to the number of vertices, using the results in [3] as a strong lower bound.

Later, we proved the stronger result that for sufficiently large Erdős-Rényi random graphs, the expected second gonality is strictly less than the number of vertices. We also adapted our method for different regimes of the probability p by using a corollary of the Riemann-Roch theorem. Finally, we partially proved a conjecture of ours regarding the higher gonalities of random graphs that was motivated by the results in [2] and generalizes them as well.

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3 Basic definitions and Results

3.1 Nomenclature

Here, we provide some definitions of terms relating to graphs and their divisors.

Definition. For any probability $0 < p < 1$ and a number of vertices n, an **Erdős-Rényi Random Graph** $G(n, p)$ is a simple graph with n vertices and an edge between any two distinct vertices with probability p. We say that $n = |V|$ is the **size** of the random graph.

Definition. We say that a connected component of a graph is a **chain** if it consists of k vertices $v_1, v_2, \ldots, v_k$ so that v_i, v_j are joined by an edge iff $|i - j| = 1$.

Definition. A **divisor** D on a graph G is a formal $\mathbb{Z}$-linear combination of the vertices of G,

$$D = \sum_{v\ \in V(G)} D(v)v.$$

It can be interpreted via the chip-firing game as an arbitrary combination of chips of the vertices in the graph. Figure 1 depicts a possible divisor.

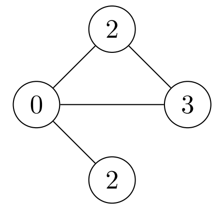

Figure 1: An effective divisor on four vertices with degree 7.

Definition. The **degree** of a graph divisor is defined as the sum of the $D(i)$: it is the number of total chips on the vertices of the divisor. We denote the degree of a graph divisor D as $\deg(D)$. We see that the degree of the divisor in Figure 1 is $0 + 2 + 2 + 3 = 7$.

Definition. To avoid confusion with the typical definition of the degree of a vertex in a graph, we will denote the **chip-degree** of a vertex of a divisor by the number of chips on it. We will denote this by $\text{cdeg}(v)$.

Definition. A divisor is said to be **effective** if it possesses a non-negative number of chips on each of its vertices. Figure 1 depicts an effective divisor.

Definition. A vertex is said to be **fired** in a process known as *chip-firing* when one chip is transferred from that vertex to an adjacent vertex via a connecting edge exactly once for each such edge. Note that the degree is invariant under chip-firing. The result of such a chip-firing is depicted in Figure 2.

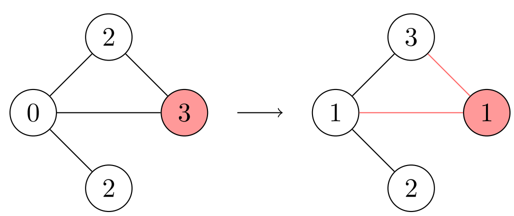

Figure 2: The rightmost vertex of the divisor on the left is fired, resulting in the divisor on the right. These two divisors are equivalent.

Definition. Two divisors on a graph are said to be **equivalent** if one can be obtained from the other through a series of chip-firing moves.

Definition. Two divisors on the same graph are **added** or **subtracted** by adding or subtracting the numbers of chips at each vertex.

Definition. A divisor D has **rank** r if r is the largest integer such that for every effective divisor E with degree r, $D - E$ is equivalent to an effective divisor. Note that if a divisor D has degree less than 0 it is defined to have a rank of -1.

Remark. The degree of a divisor is an upper bound for the rank of that divisor, but the two are not always equal. The following divisor has degree 0, but the bottom two vertices will differ by $1 \pmod 3$ regardless of our choice of chip-firing moves and therefore our divisor cannot be made effective by such moves alone.

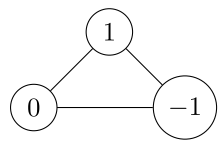

Figure 3: A divisor with degree 0 and rank -1.

Definition. Given a fixed graph G, the k-th **gonality** of G, which we denote as $g_k(G)$, is the minimum degree for a divisor on G to have rank k. The first gonality is often referred to as simply the gonality.

Definition. We say that two vertices are **equivalent** if their corresponding effective divisors of degree one are themselves equivalent. In other words, we may transfer a single chip from either vertex to the other without disturbing the other chips through a suitable sequence of chip-firing moves.

3.2 Preliminary Results on Graph Divisors

We now present several results concerning graph divisors, the gonality, and Erdős-Rényi random graphs.

Theorem 3.1. *Any two vertices on a tree are equivalent.*

Proof. Denote the vertices of the tree by v_t for some $0 \leq i < n$. Suppose that our divisors are $D_i = v_i$, $D_j = v_j$. It suffices to show that $D_i \sim D_j$ when v_i, v_j are connected by an edge, since we can find a unique path between any two distinct vertices on the tree. Thus, assume that v_i, v_j are connected by an edge.

Let G' be the resulting graph when v_j and all of its incident edges are removed. Let S be the set of vertices in the same connected component of G' as v_i. Firing all the vertices in S transfers a chip from v_i to v_j, turning D_i into D_j. Thus, $D_i \sim D_j$ as desired. □

Theorem 3.2. *Any divisor of degree k on a tree has rank k.*

Proof. This is an easy consequence of Theorem 3.1.

Again denote the vertices of the divisor by $v_0, v_1, \cdots, v_{n-1}$. We can clearly choose m, a_i, b_i, c_i, d_i so that $D = \sum_{i=0}^{k-1} v_{a_i} + \sum_{i=0}^{m} (v_{c_i} - v_{d_i})$ and let $E = \sum_{i=0}^{k-1} v_{b_i}$ where a_i, b_i, c_i, d_i are not necessarily distinct integers between 0 and $n-1$ inclusive. Here we are denoting a divisor by a linear combination of vertices. Then by Theorem 3.1, a sequence of chip fires takes $v_{a_i} - v_{b_i}$ as well as $v_{c_i} - v_{d_i}$ to the zero divisor for all i. Concatenating these sequences takes $D - E = \sum_{i=0}^{k-1} (v_{a_i} - v_{b_i}) + \sum_{i=0}^{m} (v_{c_i} - v_{d_i})$ to the zero divisor, as desired. □

Theorem 3.3. *The k-th gonality of a graph is the sum of the k-th gonalities of its connected components.*

We simply note that it is both necessary and sufficient to provide the number of chips corresponding to the k-th gonalities of each of the connected components of the graph.

Theorem 3.4. *For any graph G, $g_k(G) \geq g_m(G)$ if $k \geq m$.*

Proof. Consider a divisor D on G with rank k and degree $g_k(G)$. Then D certainly has rank at least m and thus $g_m(G) \leq \deg(D) = g_k(G)$. □

Theorem 3.5 (Trivial Bound)**.** *Let G have n vertices. Then $g_k(G) \leq kn$.*

Proof. Consider the divisor $D = \sum_{i=0}^{n-1} kv_i$. Then $D - E$ is effective for any divisor E of degree k, so D has rank at least k and degree kn. Then the k-th gonality of the graph is bounded above by the degree of D. □

Now we make note of a pair of results that ultimately prove useful for considering the expected gonality for smaller probabilities p.

Theorem 3.6 (Riemann-Roch)**.** *In a connected graph with genus $g = |E| - |V| + 1$, the k-th gonality $g_k(G)$ satisfies*

$$g_k(G) \leq k + g.$$

Equality holds when G is a tree.

Corollary 3.7. *In a general graph with genus $g = |E| - |V| + 1$ and m connected components, the k-th gonality $g_k(G)$ satisfies*

$$km \leq g_k(G) \leq m + g + mk - 1.$$

Equality holds when G is a forest.

We now present a motivating result from [3] regarding the first gonality that proved useful in our discussions on the second gonality.

Theorem 3.8 (Deveau et al.)**.** *Consider an Erdős-Rényi Random Graph $G(n, p)$ with $p = \frac{c(n)}{n}$ with $c(n) \ll n$ and $c(n) \to \infty$. Then*

$$\mathbb{E}(g_1(G(n, p))) > n - o(n).$$

Theorem 3.9 (Cools, Panizzut)**.** *Suppose that t is the smallest integer with $\frac{t(t+3)}{2} \geq k$. Let $h = \frac{t(t+3)}{2} - k$ and suppose that $\frac{(d-1)(d-2)}{2} > k$. Then the k-th gonality of the complete graph K_d satisfies*

$$g_k(K_d) = td - h.$$

Remark. To simplify notation, we will denote $\mathbb{E}(g_k(G(n,p)))$ by $F(k,n,p)$ in the future.

4 Probabilistic Bounds for Higher Gonalities

We now discuss our results on the asymptotic behavior of the second and higher gonalities of Erdős-Rényi random graphs.

4.1 Bounds on the second gonality

We first prove the simplest version of our probabilistic bounds.

Theorem 4.1. *Let $p = \frac{c(n)}{n}$ where $1 \ll c(n) \ll n$ and $c(n) \to \infty$. Then*

$$F(2,n,p) \leq n(1+e^{-c(n)}).$$

In particular, this means that

Corollary 4.2.

$$\frac{F(2,n,p)}{n} \sim 1$$

for $c(n) \to \infty$.

Proof. Consider the divisor D with two chips on each vertex with degree zero and one chip on all other vertex. We claim D has rank at least two. Any divisor E with two chips on different vertices trivially satisfies $D - E$ effective, whereas if both chips of E are on a vertex v, then firing all other vertices in the graph of divisor $D - E$ leaves an effective divisor: if both chips of E are on an isolated vertex, $D - E$ is effective and left unchanged by the firings, whereas if both chips of E are on a vertex v with nonzero degree, the chip-firings transfer a chip from each of the neighbors of v to v and thus $D - E$ is now effective.

Thus, the expected gonality is bounded above by $n + k$ where k is the expected number of vertices with degree zero. The probability that any given vertex has degree zero is $(1-p)^{n-1} = \left(1 - \frac{c(n)}{n}\right)^{n-1}$ and thus the expected number of isolated vertices is $k = n(1-p)^{n-1} = n(1-\frac{c(n)}{n})^{n-1} \leq ne^{-c(n)}$ because $\frac{n}{c(n)} \to \infty$ as $n \to \infty$. Hence our upper bound is $F(2,n,p) \leq n(1+e^{-c(n)})$.

By Theorem 3.4, the second gonality is bounded below by the first gonality. Combining these previous results with Theorem 3.8 yields

$$n - o(n) < F(1,n,p) \leq F(2,n,p) \leq n(1+e^{-c(n)}) < n + o(n)$$

so that

$$1 \leq \lim_{n\to\infty} \frac{F(2,n,p)}{n} \leq \lim_{n\to\infty} 1 + e^{-c(n)} = 1$$

as desired. ☐

We use similar reasoning to strengthen the previous result in order to show that $F(2,n,p) < n$ for sufficiently large n. This time, we consider vertices of degree 1.

Theorem 4.3. *Let $p = \frac{c(n)}{n}$ where $1 \ll c(n) \ll n$ and $c(n) \to \infty$. For all sufficiently large n and any $\epsilon > 0$ we have*

$$F(2,n,p) < n(1 - e^{-c(n)}(c(n) - 1 - \epsilon)).$$

We provide an example of a divisor with rank at least two and sufficiently small degree; we compute and bound this average degree with a simple probabilistic argument. We delay our proof of the theorem until our desired divisor is constructed. Construct a divisor D in several steps:

1. Place a chip on each vertex.

2. Remove the chips from all vertices of degree 1.

3. Add a single chip to all vertices within a connected component of size 1 or 2.

4. Add a single chip to all vertices that are connected only to at least two vertices of degree 1 and no other vertices.

5. Add a single chip to the end vertices of a chain of length at least five and remove all the other chips.

Lemma 4.4. *The divisor D has rank at least two.*

Proof. It suffices to prove that the divisors formed by the connected components of the graph have rank at least two. We split into cases.

Case 1. *The component is a tree.*

Suppose the tree has more than one vertex with degree greater than 1. Then the tree has size at least four. It will also be unaffected by step 4. If the tree is not a chain, then all vertices with degree at least 2 have exactly 1 chip and all other vertices have 0 chips, so the tree has degree at least two. If the tree is a chain, the tree will have degree exactly 2 by virtue of step 5.

Otherwise, suppose the tree has no vertex with degree 2 or more. Then it must consist of either a single vertex or a connected component of degree 2, so it will have degree 2. If the tree has exactly one vertex with degree 2 or more, step 4 ensures that the degree of this tree is exactly 2. Thus, by construction, the degree of the component is at least two and by Theorem 3.2, the rank of the component is at least two as well.

Case 2. *The component is not a tree.*

Then steps $3, 4, 5$ cannot affect the component in any way since they only affect components without any cycles. We also see that any vertex v with degree one is equivalent to its only neighbor; firing v transfers a chip to its sole neighbor, and firing all vertices other than v transfers a chip from its sole neighbor back to v.

Thus, we may merge any vertices with degree one with their neighbors, since by step two, all such vertices within this component have had their single chip removed. All vertices in this connected component have chip-degree one, and the connected component must still contain a cycle of at least three vertices (since removing vertices of degree one cannot break any cycles).

Now, consider any effective divisor E with rank 2. If E is the sum of two distinct vertices, we are done.

Otherwise, E consists of twice a single vertex v and $D - E$ is effective with the exception of -1 chips on vertex v. Fire all vertices other than v. Then the neighbors of v, of which there are at least one, will all donate their sole chip to v and the resulting divisor is effective, as desired. □

We are ready to prove our main result regarding the second gonality.

Proof of Theorem 4.3. We notice that $F(2, n, p) \leq \mathbb{E}(\deg{(D)})$ because Lemma 4.3 ensures that D has rank at least two. So it suffices to prove that

$$\mathbb{E}(\deg{(D)}) < n(1 - e^{-c(n)}(c(n) - 1 - \epsilon)).$$

We use the principle of the linearity of expectation; let d_i be the expected number of chips added by step i.

The first step contributes $d_1 = n$ to the expected degree, without exception.

The second step contributes $d_2 = -np_2$, where p_2 is the probability that any given vertex has degree one.

This probability is given by

$$p_2 = n\frac{c(n)}{n}\left(1 - \frac{c(n)}{n}\right)^{n-2} = c(n)\left(1 - \frac{c(n)}{n}\right)^{n-2}$$

and thus the contribution to the expected degree is

$$d_2 = -nc(n)\left(1 - \frac{c(n)}{n}\right)^{n-2}.$$

We apply the same reasoning to the third step. The probability that any given vertex is its own connected component is $\left(1-\frac{c(n)}{n}\right)^{n-1}$, and the probability that any two given vertices form their own connected component is $\frac{c(n)}{n}\left(1-\frac{c(n)}{n}\right)^{2n-4}$. For each connected component of size one, Step 2 adds one chip, and for each connected component of size two, Step 2 adds two chips, for a net contribution of

$$d_2 = n\left(1-\frac{c(n)}{n}\right)^{n-1} + 2\binom{n}{2}\frac{c(n)}{n}\left(1-\frac{c(n)}{n}\right)^{2n-4} < n\left(1-\frac{c(n)}{n}\right)^{n-1} + nc(n)\left(1-\frac{c(n)}{n}\right)^{2n-4}.$$

Now factor in the fourth and fifth steps. In the fourth step, the connected component in question must have size $k \geq 3$. The probability that any given vertex is the unique "center" of such a connected component of size k is

$$p_4^k = \binom{n-1}{k-1}\frac{c(n)^{k-1}}{n^{k-1}}\left(1-\frac{c(n)}{n}\right)^{1+kn-\frac{k^2+3k}{2}}$$

and the net increase in degree for any given $k \geq 3$ is

$$d_4^k = n\binom{n-1}{k-1}\frac{c(n)^{k-1}}{n^{k-1}}\left(1-\frac{c(n)}{n}\right)^{1+kn-\frac{k^2+3k}{2}} = k\binom{n}{k}\frac{c(n)^{k-1}}{n^{k-1}}\left(1-\frac{c(n)}{n}\right)^{1+kn-\frac{k^2+3k}{2}}$$

because there are $1+kn-\frac{k^2+3k}{2}$ specific edges that must not exist and $k-1$ specific edges that must exist for any possible tree-component on k vertices.

However, note that for every chain of length $k \geq 5$, step five decreases the degree additionally by at least one as well: it decreases the gonality by a net $k-4$ chips. The number of possible chains of length k is equal to $\frac{k!}{2}\binom{n}{k}$, all of which occur with equal probability, and therefore the expected contribution from the chains of any given length $k \geq 5$ is equal to

$$d_5^k = (k-4)\frac{k!}{2}\binom{n}{k}\frac{c(n)^k}{n^k}\left(1-\frac{c(n)}{n}\right)^{1+kn-\frac{k^2+3k}{2}} \geq k\binom{n}{k}\frac{c(n)^k}{n^k}\left(1-\frac{c(n)}{n}\right)^{1+kn-\frac{k^2+3k}{2}}$$

so the net contributions from $k \geq 5$ from steps 4 and 5 do not increase the second gonality. We thus only include the Step 4 connected components with size three or four. Thus, steps 4 and 5 combined increase the gonality by at most

$$d_4 + d_5 \leq \frac{n}{2}c(n)^2\left(1-\frac{c(n)}{n}\right)^{3n-8} + \frac{n}{6}c(n)^3\left(1-\frac{c(n)}{n}\right)^{4n-13} \leq nc(n)^2\left(1-\frac{c(n)}{n}\right)^{3n}$$

as $\left(1-\frac{c(n)}{n}\right)^{4n-13} \approx e^{-4c(n)}, \left(1-\frac{c(n)}{n}\right)^{3n-8} \approx e^{-3c(n)}$, and $e^{-4c(n)} \ll e^{-3c(n)}$ as $n \to \infty$.

Now, we add these contributions and make use of the fact that $e^{c(n)} \gg c(n)$ when $c(n)$ tends to infinity:

$$\begin{aligned} F(2,n,p) &\leq d_1 + d_2 + d_3 + d_4 + d_5 \\ &\leq n\left(1 - c(n)\left(1-\frac{c(n)}{n}\right)^{n-2} + \left(1-\frac{c(n)}{n}\right)^{n-1} + c(n)\left(1-\frac{c(n)}{n}\right)^{2n-4} + c(n)^2\left(1-\frac{c(n)}{n}\right)^{3n}\right) \\ &\leq n\left(1 - e^{-c(n)}\left(c(n) - 1 - 2c(n)e^{-c(n)} - c(n)^2 e^{-2c(n)}\right)\right) \\ &< n(1 - e^{-c(n)}(c(n) - 1 - \epsilon)) \end{aligned}$$

for any $\epsilon > 0$ and sufficiently large n, as desired. □

4.2 Bounds on Higher Gonalities

We first use Corollary 3.7 to the Riemann-Roch theorem (Theorem 3.6) to prove bounds where $c \leq 1$ and $p = \frac{c}{n}$ for arbitrary gonalities.

Theorem 4.5. *For $c < 1$,*

$$k\left(n - \frac{c(n-1)}{2}\right) \leq F\left(k, n, \frac{c}{n}\right) < kn\left(1 - \frac{c}{2}\right) + \frac{k+1}{2}(-\ln(1-c) - \frac{c^2}{2}) - \frac{c}{2}.$$

For $c = 1$, we also have

$$k\frac{n+1}{2} \leq F\left(k, n, \frac{1}{n}\right) < \frac{kn}{2} + (k+1)\frac{\ln n}{2} + \frac{k-1}{4}.$$

Remark. This shows that $\frac{1}{kn}F(k,n,\frac{c}{n}) \sim 1 - \frac{c}{2}$ and in particular $\frac{1}{n}F(2,n,\frac{c}{n}) \sim 2 - c$ when $c \leq 1$ and $n \to \infty$.

Before proving Theorem 4.5, we first bound the number of connected components m.

Lemma 4.6. *In any graph G with $|V|$ vertices,$|E|$ edges, $|C|$ cycles, and m connected components, we have $|V| - |E| \leq m \leq |V| - |E| + |C|$.*

Proof. We add in the edges one at a time. We begin with $|V|$ connected components. Each time we add an edge that does not create a cycle, we reduce the number of connected components by 1. Otherwise, we add at least one cycle. Thus we decrease the number of connected components at least $|E| - |C|$ times and at most $|E|$ times and thus $|V| - |E| \leq m \leq |V| - |E| + |C|$ as desired. □

We now seek to bound the quantity $\mathbb{E}(|C|)$.

Lemma 4.7. *In an Erdős-Rényi Random graph $G(n, \frac{c}{n})$ with $0 < c < 1$ and $n \geq 3$, the expected number of* *cycles $E(|C|)$ satisfies*

$$\mathbb{E}(|C|) < \frac{1}{2}(-\ln(1-c) - \frac{c^2}{2} - c).$$

When $c = 1$, we have

$$\mathbb{E}(|C|) < \frac{\ln n}{2} - \frac{1}{4}$$

Proof. We consider the expected number of cycles of any given size, and then sum the contributions. Given a set of k vertices, there are $\frac{(k-1)!}{2}$ possible cycles. Then the expected number of cycles of length k is

$$\mathbb{E}(|C_k|) = \frac{(k-1)!}{2}\binom{n}{k}\left(\frac{c}{n}\right)^k < \frac{c^k}{2k}.$$

Note that $\ln(1-x) = -\sum_{i=1}^{\infty}\frac{x^i}{i}$ for any $|x| < 1$. Then $\sum_{i=3}^{\infty}\frac{c^i}{2i} = \frac{1}{2}(-\ln(1-c) - \frac{c^2}{2} - c)$. Adding up these expressions for all $k \geq 3$ gives

$$\mathbb{E}(|C|) = \sum_{i=3}^{n}\mathbb{E}(|C_i|) < \sum_{i=3}^{\infty}\frac{c^i}{2i} = \frac{1}{2}(-\ln(1-c) - \frac{c^2}{2} - c)$$

for $c > 1$, and

$$\mathbb{E}(|C|) = \sum_{i=3}^{n}\mathbb{E}(|C_i|) < \sum_{i=3}^{\infty}\frac{1}{2k} < \frac{\ln n}{2} - \frac{1}{4}$$

as desired. □

We now use Corollary 3.7 and Lemmas 4.6 and 4.7 to conclude the proof of Theorem 4.5 for $c < 1$.

Proof of Theorem 4.5. When $c < 1$, we have

$$\begin{aligned}
k\left(n - \frac{c(n-1)}{2}\right) \leq k\mathbb{E}(m) \leq F(k, n, \frac{c}{n})& \\
&\leq \mathbb{E}(|E|) - \mathbb{E}(|V|) + (k+1)\mathbb{E}(m) \\
&< (\frac{c(n-1)}{2} - n) + (k+1)(n - \frac{c(n-1)}{2} + \frac{1}{2}(-\ln(1-c) - \frac{c^2}{2} - c)) \\
&= kn\left(1 - \frac{c}{2}\right) + \frac{k+1}{2}(-\ln(1-c) - \frac{c^2}{2}) - \frac{c}{2}.
\end{aligned}$$

For $c = 1$, we also see from these previous results that

$$k\left(\frac{n+1}{2}\right) \leq k\mathbb{E}(m) \leq F(k, n, \frac{1}{n}) < \mathbb{E}(|E|) - \mathbb{E}(|V|) + (k+1)\mathbb{E}(m) < \frac{kn}{2} + (k+1)\frac{\ln n}{2} + \frac{k-1}{4}$$

as desired. $\square$

In [2], it is suggested that the $\frac{k^2+3k}{2}$th gonality of a graph is asymptotically proportional to kn, where n is the size of the graph. We have shown this to be true for random graphs when $k = 1$. Motivated by [2], We now generalize our approach to Theorem 4.1 to find an upper bound for arbitrarily high gonalities.

Theorem 4.8. *Let $p = \frac{c(n)}{n}$ where $c(n) \ll n$, $c(n) \to \infty$. Suppose that $k \leq \frac{t(t+3)}{2}$. Then the expected k-th gonality of an Erdős-Rényi Random Graph $G(n,p)$ satisfies*

$$F\left(k,n,p\right) < (t+\epsilon)n$$

for any $\epsilon > 0$ as $n \to \infty$.

Remark. This result is a generalization of Theorem 4.1 when $t = 1$.

We note the following lemma.

Lemma 4.9. *Denote the expected number of vertices with degree less than m, of an Erdős-Rényi random graph $G(n,p)$ by $q(m,n,p)$. Then if $p = \frac{c(n)}{n}$, $1 \ll c(n) \ll n$, and $c(n) \to \infty$,*

$$\lim_{n\to\infty} \frac{q(m,n,p)}{n} = 0$$

for any m.

Proof. Since we are considering $n \to \infty$, we may suppose that $n > 2m$. The probability that any given vertex has degree $0 \leq d < m < \frac{n}{2}$ is

$$p = \binom{n}{d}\left(1 - \frac{c(n)}{n}\right)^{n-d-1}\left(\frac{c(n)}{n}\right)^{d} < \frac{c(n)^d}{d!e^{\frac{c(n)}{2}}} < \epsilon$$

for any $\epsilon > 0$ and sufficiently large n, since $n > 2m \geq 2d+2$. Thus, the total number of vertices with degree less than m, $q(m,n)$, is equal to

$$q(m,n,p) = n\sum_{d=0}^{m-1}\binom{n}{d}\left(1 - \frac{c(n)}{n}\right)^{n-d}\left(\frac{c(n)}{n}\right)^{d} < n\sum_{d=0}^{m-1}\frac{c(n)^d}{d!e^{\frac{c(n)}{2}}} < n\epsilon$$

for any $\epsilon > 0$ and sufficiently large n, and thus $\lim\limits_{n\to\infty} \frac{q(m,n,p)}{n} = 0$ as desired. $\square$

We prove another pertinent lemma.

Lemma 4.10. *Let $n > t$ be positive integers. Suppose a sequence $d_1, d_2, \cdots, d_n$ of non-negative integers satisfies the following property: For any $0 \leq i \leq t+1$, at least i of the d_j are greater than or equal to $t+2-i$.*

Then

$$\sum_{j=0}^{n} d_j \geq \frac{t(t+3)}{2} + 1.$$

Proof. We claim equality holds iff the d_j are $0, 0, \cdots, 0, 1, 2, 3, \cdots, t, t+1$ in some order. The "if" direction is trivial as $\sum_{i=0}^{t+1} i = \frac{t(t+3)}{2} + 1$ and for any i exactly i of the d_j are greater than or equal to $t + 2 - i$.

Without loss of generality, order the d_i so that $d_1 \geq d_2, \cdots \geq d_n$. Then we must have $d_i \geq t + 2 - i$ for all i. Suppose to the contrary that a suitable sequence of d_j exists with $\sum_{j=0}^{n} d_j < \frac{t(t+3)}{2} + 1$. Then we must have some d_i with $d_i < t + 2 - i$, a contradiction. ☐

Now we prove our main result.

Proof of Theorem 4.8. Construct a divisor D as follows: Add t chips to any vertex of degree greater than or equal to $\frac{t^2+5t}{2}$, and add $\frac{t^2+5t}{2}$ chips to any vertex with smaller degree.

Consider any effective divisor E of degree $k \leq \frac{t(t+3)}{2}$, and express it as $E = \sum_{i=1}^{n} d_i v_i$, so that $\sum_{i=1}^{n} d_i = k \leq \frac{k(k+3)}{2}$. By the contrapositive of the lemma, there exists some non-negative integer i, so that at most i of the d_j are greater than $t - j$; in other words, there are at most i vertices of E with chip-degree greater than $t - i$, and thus, at most i vertices of $D - E$ with chip-degree less than i. Firing all the vertices of $D - E$ with chip-degree at least i yields an effective divisor as a result: each vertex with chip-degree at least i loses at most i chips (one chip to each vertex of degree less than i), whereas each vertex v with chip-degree at most $i - 1$ now possesses at least $\frac{t^2+5t}{2} - i + 1 + \mathrm{cdeg}(v) \geq \frac{t^2+5t}{2} - t - \frac{t^2+3t}{2} \geq 0$ chips if its degree is greater than or equal to $\frac{t^2+5t}{2}$, or at least $\frac{t^2+5t}{2} - \frac{t^2+3t}{2} \geq k > 0$ chips if its degree is less than $\frac{t^2+5t}{2}$.

It remains to determine the expected degree of our divisor D. But Lemma 4.8 ensures that

$$\mathbb{E}(g(D)) = tn + \frac{t^2+3t}{2} q\left(\frac{t^2+5t}{2}, n, p\right) < tn + \epsilon n = (t+\epsilon)n$$

as desired. ☐

5 Empirical Observations on the Second Gonality

5.1 Exact Expressions for small n

We computed the exact expressions for $\frac{1}{n}F(2, n, p)$ for $n \leq 5$, as a function of p.

n	$\frac{1}{n}F(2,n,p)$
1	2
2	$2-p$
3	$2-2p+p^3$
4	$2-3p+3p^3+2.25p^4-4.5p^5+1.25p^6$
5	$2-4p+6p^3+9p^4-10.8p^5-37p^6+58p^7-6p^8-30p^9+13.8p^{10}$

Table 1: $\frac{1}{n}F(2,n,p)$ precisely evaluated for small n.

Here, we notice a distinct pattern in the constant, p^1, and p^2 terms. In particular, where $p \approx 0$, we see that

$$\frac{1}{n}F(2,n,p) \approx 2-(n-1)p = 2-c\left(1-\frac{1}{n}\right) = \frac{1}{n}\cdot 2\left(n-\frac{c(n-1)}{2}\right),$$

which is exactly the bound predicted by Theorem 4.5.

5.2 Graphical Depiction of Bounds

We use this subsection to illustrate graphically the various bounds we proved in Section 4 regarding the second gonality over the various regimes of probability.

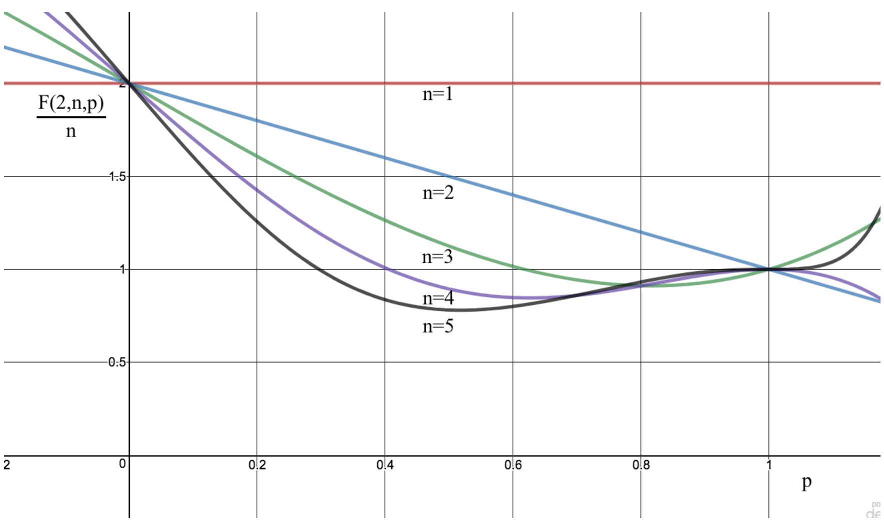

Figure 4: The expected second gonality for small n as a function of p.

Figure 4 depicts the behavior of $\frac{F(2,n,p)}{n}$. Notice that the function is less than 1 for sufficiently large probabilities p: this represents $F(2,n,p) < n$ as proven in Theorem 4.3.

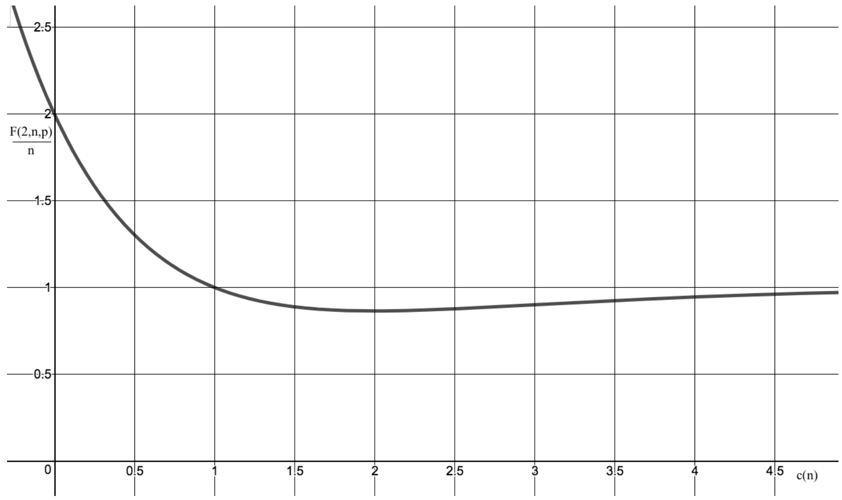

Figure 5: The behavior of the function $f_1(c) = (1 - e^{-c}(c-1))$.

Figure 5 shows the behavior of the function $f_1(c(n)) = (1 - e^{-c(n)}(c(n)-1))$, which represents an upper bound for $\frac{F(2,n,p)}{n}$ when $p = \frac{c(n)}{n}$, $c(n) \to \infty$.

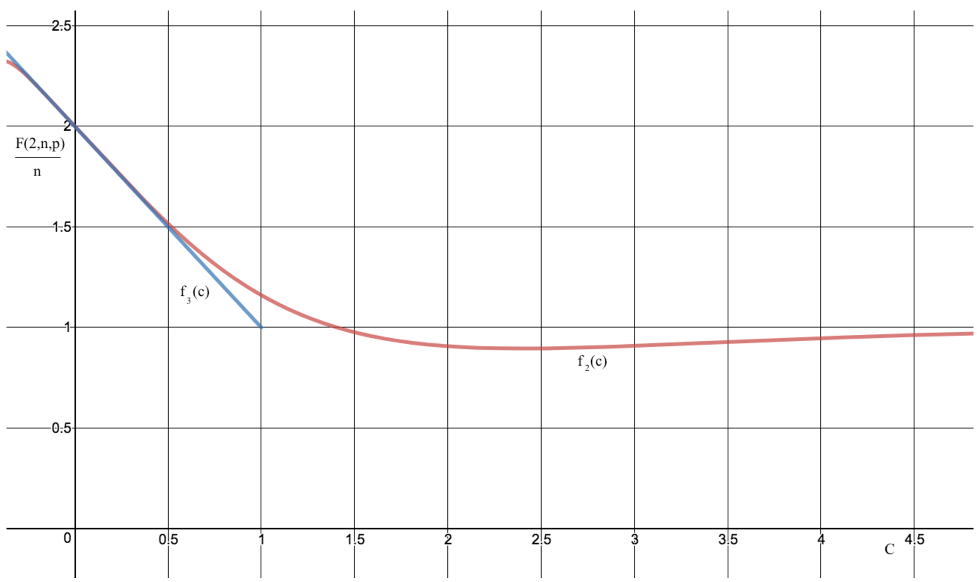

Figure 6: The behavior of the function $f_2(c) = (1 - e^{-c}(c-1) + ce^{-2c} + \frac{c^2e^{-3c}}{2} + \frac{c^3e^{-4c}}{6} - \frac{11c^4e^{-5c}}{24})$ juxtaposed with that of $f_3(c) = 2 - c$.

Figure 6 depicts the behavior of $f_2(c)$, which from the proof of Theorem 4.3 is an upper bound for $\frac{F(2,n,p)}{n}$ for $p = \frac{c}{n}$, as well as $f_3(c)$, the exact asymptotic behavior of $\frac{F(2,n,p)}{n}$ for $c \leq 1$. We see that $|f_1(c) - f_2(c)| \to 0$ as $c \to \infty$, as expected from our mathematical results. In addition, we note that $f_3(c) < f_2(c)$ when $c \leq 1$, derived from the proof of Theorem 4.5 when $k = 2$.

We also note that $\frac{F(2,n,p)}{n} \leq f_2(c) < 1$ for $c > 1.405$ and n sufficiently large. By explicitly considering the excess contributions of chains of length 6, 7, or 8, we can refine our result and show that $\frac{F(2,n,p)}{n} < 1$ for any $c > 1.395$ and sufficiently large n.

6 Conclusion

6.1 Summary of Results

In Section 4.1, we proved the inequality chain

$$n - o(n) = \mathbb{E}(\mathrm{tw}(G(n,p))) \leq F(1,n,p) \leq F(2,n,p) < n$$

for sufficiently large n under for $p = \frac{c(n)}{n}, c(n) \to \infty, c(n) \ll n$. We use the notation in [3] of $\mathrm{tw}(G)$ to denote the treewidth of a graph. Although we have shown that asymptotically

$$n - o(n) < F(2,n,p) < n.$$

for $p = \frac{c(n)}{n}$, $1 \ll c(n) \ll n$, and $c(n) \to \infty$ we are certainly open to improved bounds on $F(2,n,p)$. We also proved that $F(2,n,\frac{1}{n}) < n + o(n)$ as a corollary to Theorem 4.5 and also that $F(2,n,\frac{c}{n}) < n$ for all $c > 1.395$.

Our work in section 4.2 extended and generalized known bounds on the gonalities of complete graphs to the case of Erdős-Rényi random graphs, proving an analogous upper bound. It generalizes our results in section 4.1 on the second gonality. We proved an upper bound for the $\frac{k(k+3)}{2}$-th gonality:

$$F\left(\frac{k(k+3)}{2}, n, \frac{c(n)}{n}\right) < (k+\epsilon)n$$

and by extension, all lower gonalities as well. If we use the same notation as in Theorem 3.9, we have

$$F(k,n,p) < (t+\epsilon)n$$

where t is the minimal positive integer with $t(t+3) \geq 2k$.

6.2 Conjectures on Higher Gonalities

We have already proven that $F(2,n,\frac{c}{n}) < n + o(n)$ for any $c = 1, c > 1.395$ and also that $F(2,n,\frac{c}{n}) \sim (2-c)n$ for $c < 1$. This naturally leads us to the following conjecture:

Conjecture 6.1. For any $c \geq 1$,

$$F\left(2, n, \frac{c}{n}\right) < n + o(n).$$

To prove this conjecture, however, we need better bounds on the number of connected components in an Erdős-Rényi random graph. We therefore seek improved bounds on m, the expected number of connected components of an Erdős-Rényi Random Graph, when $1 < c < 1.395$. One might apply Corollary 3.7 in order to find improved bounds on the higher gonalities for these cases.

We have already proved that the $\frac{k(k+3)}{2}$-th gonality is bounded above by $(k+\epsilon)n$. We conjecture that kn is in fact a lower bound for this gonality as well: in other words,

Conjecture 6.2. Let k be a positive integer. Set t to be the minimal positive integer so that $\frac{t(t+3)}{2} \geq k$. Then for any $\epsilon > 0$, the k-th gonality of an Erdős-Rényi random graph $G(n,p)$ satisfies

$$(t-\epsilon)n < F(k,n,p) < (t+\epsilon)n$$

for all sufficiently large n.

This result parallels the main result of [2] on complete graphs; we believe that their bound is also accurate for Erdős-Rényi random graphs. New lower bounds on the gonality would certainly be intriguing.

7 References

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