COEFFICIENTS OF GAUSSIAN POLYNOMIALS MODULO N

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Abstract. The q-analogue of the binomial coefficient, known as a q-binomial coefficient, is typically denoted $\begin{bmatrix} n \\ k \end{bmatrix}_q$. These polynomials are important combinatorial objects, often appearing in generating functions related to permutations and in representation theory.

Stanley conjectured that the function $f_{k,R}(n) = \#\left\{i : [q^i] \begin{bmatrix} n \\ k \end{bmatrix}_q \equiv R \pmod N\right\}$ is quasipolynomial for $N = 2$. We generalize, showing that this is in fact true for any integer $N \in \mathbb{N}$ and determine a quasi-period $\pi'_N(k)$ derived from the minimal period $\pi_N(k)$ of partitions with at most k parts modulo N. We also determine the generating function $F_{k,R}(x) = \sum_{n\geq k} f_{k,R}(n)x^n$ and study cases where it simplifies.

1. Introduction

The q-analogue of the binomial coefficient is typically denoted $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and is defined by the rational expression

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[n-k]![k]!},$$

where $[n]! = \prod_{i=1}^{n} (1-q^i)/(1-q)$. These are polynomials with degree $k(n-k)$.

These polynomials appear in combinatorics and have connections to the theory of symmetric polynomials as well as representation theory. In particular, an important characterization is that they enumerate Grassmanian $\mathbf{Gr}(k, \mathbb{F}_q^n)$:

Theorem 1.1 (Stanley)**.** *The number of k-dimensional subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

This fact reveals that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial. To see why, note that if a rational function f, given by $f(q) = P(q)/Q(q)$ where $P, Q \in \mathbb{Z}[q]$, is integral for infinitely many $q \in \mathbb{R}$ (here, at prime powers q where $\mathbb{F}_q$ exists) then it is a polynomial. Recent works such as [1] or [2] have sparked interest in these objects and their coefficients.

In this paper, we investigate the behavior of these coefficients modulo some positive integer $N \in \mathbb{N}$. One motivation for this is the classical Lucas' theorem:

Theorem 1.2 (Lucas' Theorem)**.** *For p prime, let $n, k \in \mathbb{N}$ have base p expansions $n = \sum_{i\geq 0} n_i p^i, k = \sum_{i\geq 0} k_i p^i$. Then*

$$\binom{n}{k} \equiv \prod_{i\geq 0} \binom{n_i}{k_i} \pmod p.$$

By fixing k, the values of $\binom{n}{k} \pmod p$ can be shown to form a repeating sequence related to the base p expansion of k. This extends to modulo N, as seen by the following corollary from [3]:

Corollary 1.3 (Kwong)**.** *Let the prime factorization of N be given by $\prod p_i^{e_i}$ for primes p_i. Then $\binom{n}{k}$ is purely periodic modulo N, with period*

$$P = \prod p_i^{e_i+b-1},$$

where $b \in \mathbb{N}$, $p^{b-1} < k < p^b$.

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The q-binomial coefficients are an example of a "q-analogue", in the sense that $\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$. As a result, it is reasonable to expect similar structured behavior modulo p or even with general composites in the coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$, since this shows $\binom{n}{k} = \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{i\geq 0} [q^i] \begin{bmatrix} n \\ k \end{bmatrix}_q$. Here, $[q^i]f(q)$ denotes the coefficient of q^i in f.

We prove and generalize Conjecture 1.8, that the 'residue counting' function for these coefficients is a *quasipolynomial*. From [4], we have the following definition of a quasipolynomial function:

Definition 1.4 (Stanley)**.** A function $f : \mathbb{N} \to \mathbb{R}$ is *quasipolynomial* with degree d if

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} \ldots + c_0(n)$$

where $c_i(n)$ is a periodic function with integer period Q that is not identically 0. We call Q a *quasi-period* of f. Note that Q is not unique, since kQ is a quasiperiod for $k \in \mathbb{N}$.

Equivalently, we can say $f(n) = P_i(n)$ for $n \equiv i \pmod Q$ where $P_i \in \mathbb{Z}[x]$. In order to state the main result (Theorem 1.9), we make the following definitions.

Definition 1.5. For a natural number N that we call the *modulus*, $R \in \mathbb{Z}/N\mathbb{Z}$, and $k \in \mathbb{N}$, we define

$$f_{k,R}(n) = \#\left\{\alpha : [q^\alpha] \begin{bmatrix} n \\ k \end{bmatrix}_q \equiv R \pmod N\right\}.$$

This function counts the number of coefficients congruent to R modulo N.

Definition 1.6. Define $\pi_N(k)$ as the minimal period of $p_{\leq k}(n)$ modulo N, where $p_{\leq k}(n)$ denotes the number of partitions with at most k parts.

Remark 1. From [5], we see that $p_{\leq k}$ is also an example of a quasipolynomial function.

Definition 1.7. Define $\pi'_N(k)$ as follows:

$$\pi'_N(k) = \begin{cases} \frac{\pi_N(k)}{\pi_N(k-1)}\pi'_N(k-1) & \text{if } N \Big| \frac{\pi_N(k)}{\pi_N(k-1)} \\ N\frac{\pi_N(k)}{\pi_N(k-1)}\pi'_N(k-1) & \text{otherwise} \end{cases}.$$

Stanley originally conjectured the following:

Conjecture 1.8. *The function $f_{k,R}$ is quasipolynomial for the modulus $N = 2$.*

The following theorem, which generalizes Conjecture 1.8, is the main result of this paper. This is shown in Sections 3 and 4.

Theorem 1.9. *For a modulus N, the function $f_{k,R}(n)$ is quasipolynomial, with a quasi-period $\pi'_N(k)$ and degree one.*

The idea will be to formulate an equivalent restatement of Theorem 4.3, which makes a more direct statement about the structure of the coefficients modulo N. In Section 5, we investigate the structure of the generating function

$$F_{k,R}(x) = \sum_{n\geq k} f_{k,R}(n)x^n.$$

2. Coefficients of low degree terms in $\begin{bmatrix} n \\ k \end{bmatrix}_q$

We first try to understand the behavior of the coefficient of q^i in $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for small i.

Lemma 2.1. *Let $n_0, k \in \mathbb{N}$ be arbitrary, and $n \geq n_0 + k$. Then for $0 \leq i < n_0$, we have $[q^i] \begin{bmatrix} n \\ k \end{bmatrix}_q = p_{\leq k}(i)$.*

Proof. Fix k, and consider what happens as n_0 increases. We want to show that the first n_0 coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are constant when $n - k \geq n_0$. Use the well-known identity from ([4] §1.7)

$$\sum_{i \geq 0} p(j,k,i) q^i = \begin{bmatrix} j+k \\ k \end{bmatrix}_q,$$

where $p(j,k,i)$ denotes the number of partitions $\lambda \vdash i$ with at most k parts and maximal part $\leq j$. Now if $i < j$ this last condition can be dropped, leaving $p(j,k,i) = p_{\leq k}(i)$. Then letting $j + k = n$, if $n - k = j \geq n_0$ then the first n_0 coefficients are constant for all such n, and are equal to $p_{\leq k}(i)$ for each i. $\square$

Remark 2. A similar result is true for the last n_0 coefficients. For λ a partition with a Young diagram fitting in a $j \times k$ box, let λ' be its complement as shown in Figure 1. We rotate this to make it a Young diagram. This establishes the symmetry of the q-binomial coefficients, and explains why a similar result holds.

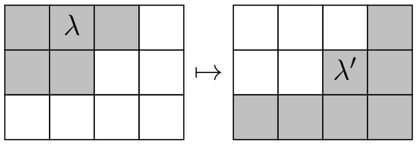

FIGURE 1. The complement map $\lambda \mapsto \lambda'$ on a 4x3 box, where $\lambda = (3,2)$ and $\lambda' = (4,2,1)$.

This warrants an investigation of the function $p_{\leq k}(i)$ modulo N. The following theorem from [6] shows that it is purely periodic, and characterizes the minimal period for prime powers. By the Chinese Remainder Theorem, understanding the behavior of $p_{\leq k}(i)$ modulo prime powers is sufficient to understand its behavior modulo N.

Theorem 2.2 (Nijenhuis and Wilf)**.** *For a prime power p^e, fix a set $S = \{s_0, s_1, \ldots s_l\}$ with entries in $\mathbb{N}$. Let $p(n;S)$ be given by the generating function*

$$P(x;S) := \prod_{s \in S} \frac{1}{1 - x^s} = \sum_{n \geq 0} p(n;S) x^n,$$

where $p(n;S)$ is the number of partitions λ with parts in S and $|\lambda| = n$. Then $p(*;S)$ is purely periodic modulo p^e, with minimal period

$$\pi_{p^e}(S) = p^{b_p(S)+e-1} L_p(S)$$

where $b_p(S)$ is the smallest integer such that

$$p^{b_p(S)} \geq \sum_{s \in S} p^{\nu_p(s)}$$

where $\nu_p(s)$ is p-adic valuation of s and $L_p(S)$ is the p-free part of $\mathrm{lcm}(S)$.

For a more detailed discussion, see [7]. Lemma 2.1 then shows that for n_0 sufficiently large, the first n_0 coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for $n - k \geq n_0$ will follow a repeating pattern of period $\pi_{p^e}(k) = p^{b_p([k])+e-1} L_p([k])$ modulo p^e. Here, $[k] := \{1, 2, \ldots k\}$.

Definition 2.3. Fix k, a modulus $N \in \mathbb{N}$, and $n > \pi_N(k)$. Let $\mathcal{S}$ be the sequence of the first $\pi_N(k)$ coefficients of $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ reduced mod N. It is given by $\mathcal{S} = (s_0, s_1, \ldots s_{\pi_N - 1})$, where $s_\alpha \equiv [q^\alpha] \begin{bmatrix} n+k \\ k \end{bmatrix}_q \pmod{N}$.

Theorems 2.2 and Lemma 2.1 show that $\mathcal{S}$ characterizes the periodic sequence $p_{\leq k}(i)$ modulo N.

Example 1. One example of this sequence for $N = 2$ and $k = 3$ is shown in Figure 2.

s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}
1	1	0	1	0	1	1	0	0	0	0	0

FIGURE 2. Values of $\mathcal{S}$ modulo 2.

Next, we study $\mathcal{S}$ for prime powers p^e as the modulus. The generating function of $p_{\leq k}(n)$ is given by

$$P_{\leq k}(q) := \sum_{n\geq 0} p_{\leq k}(n)q^n = \frac{1}{\prod_{i\in[k]} 1-q^i}.$$

Lemma 2.4. $\operatorname{lcm}([k]) \mid \pi_{p^e}(k)$.

Proof. By Theorem 2.2, it suffices to show that $b_p([k]) \geq \nu_p(\operatorname{lcm}([k])$. We can then see

$$p^{b_p([k])} \geq \sum_{i\in[k]} p^{\nu_p(i)} \geq p^{\max_{i\in[k]}\nu_p(i)} = p^{\nu_p(\operatorname{lcm}([k])}.$$

Taking logs, the result follows. $\square$

Definition 2.5. Define the operator $\Delta_Q : \mathbb{Z}[[q]] \to \mathbb{Z}[[q]]$ for $Q \in \mathbb{N}$ via the formula

$$\begin{aligned}(\Delta_Q F)(q) &= F(q) - q^Q F(q)\\ &= \sum_{n\geq 0} f(n)q^n - \sum_{n\geq Q} f(n-Q)q^n,\end{aligned}$$

This can be viewed as an analogue of the finite difference operator (Δ, as in [4] §1.9) acting on formal power series.

Example 2. Consider the generating function $F(q) = \frac{q}{(1-q)^2} = \sum_{i\geq 0} iq^i$. Suppose we want to calculate $\Delta_5 F(q)$: then we have

$$\begin{aligned}\Delta_5 F(q) &= \frac{q}{(1-q)^2} - \frac{q^6}{(1-q)^2}\\ &= \frac{5q^5}{1-q} + (4q^4 + 3q^3 + 2q^2 + 1q).\end{aligned}$$

This demonstrates a key aspect of the Δ_Q operator: the lowest q monomial terms remain unchanged, while the rest of the sequence can be viewed as a union of Q subsequences with the traditional finite difference operator applied.

Theorem 2.6. *Fix n, k. Consider the sequence $\mathcal{S}$ where we take coefficients modulo $N = p^e$. Then $s_{|\mathcal{S}|-1}, \ldots s_{|\mathcal{S}|-\binom{k+1}{2}+1}$ are all 0 modulo $N = p^e$, and $s_i = (-1)^{k+1} s_{\pi_N(k)-\binom{k+1}{2}-i}$.*

Proof. The main idea behind this result is to exploit the form of the generating function $P_{\leq k}(q)$. We can re-write it as follows, letting $Q := \pi_{p^e}(k)$:

$$P_{\leq k}(q) = \frac{1}{\prod_{i\in[k]} 1-q^i} = \frac{\gamma(q)}{(1-q^Q)^k}, \tag{1}$$

where we can obtain

$$\gamma(q) = \frac{(1-q^Q)^k}{\prod_{i\in[k]} 1-q^i} = \prod_{d\mid Q} \Phi_d(q)^{f(d)} \tag{2}$$

where $f(d) = k - \#\{i \in [k] : d \mid i\} \geq 0$. To show (2), note that $f(d)$ accounts for every factor in the denominator, and is bounded below by 0. This follows from Lemma 2.4 and the fact that each cyclotomic factor of the denominator can appear at most k times (at most once for each factor $(1-q^i)$). Thus we conclude that $\gamma(q) \in \mathbb{Z}[q]$ and that $\deg \gamma(q) = kQ - \binom{k+1}{2}$. Using Δ_Q as in Definition 2.5, we obtain

$$\Delta_Q^k P_{\leq k}(q) = \gamma(q).$$

Using the fact that $P_{\leq k}(q)$ can be written as $P_{\leq k}(q) \equiv \frac{\gamma_0(q)}{1-q^Q} \pmod{p^e}$ for some unique $\gamma_0(q)$ with $\deg \gamma_0 < Q$ by Theorem 2.2, we can see that $\Delta_Q P_{\leq k}(q) = \gamma_0(q)$. It follows from this that

$$\Delta_Q^k P_{\leq k}(q) \equiv \sum_{i \geq 0} (-1)^i \binom{k-1}{i} \gamma_0(q) q^{Qi} \pmod{p^e},$$

using the formula for $\Delta^k f(n)$ from [4] in §1.9. Thus, for $r \in \mathbb{Z}/Q\mathbb{Z}$ we have

$$[q^{r+Q(k-1)}]\gamma(q) \equiv (-1)^{k-1}[q^r]\gamma(q) \pmod{p^e}.$$

Knowing that $\deg \gamma = kQ - \binom{k+1}{2}$, there must be $\binom{k+1}{2} - 1$ zeroes at the end of $\mathcal{S}$. Furthermore, the polynomial $\gamma(q)$ can be shown to be symmetric using (2) and the symmetry of the cyclotomic polynomials (this is only true for Φ_d when $d > 1$, but $d = 1$ is not an issue as $f(1) = 0$). Referring to Figure 3, this shows the symmetry of $\mathcal{S}$ when the trailing zeroes are ignored: by the symmetry of $\gamma(q)$, the elements with label i in Figure 3 are equal. These are also identical instances of $\mathcal{S}$ without the trailing zeroes up to sign, so $\left\{s_0, \ldots s_{|\mathcal{S}| - \binom{k+1}{2}}\right\}$ is symmetric or "anti-symmetric" about its center. Precisely, this says that $s_i = (-1)^{k+1} s_{\pi_N(k) - \binom{k+1}{2} - i}$.

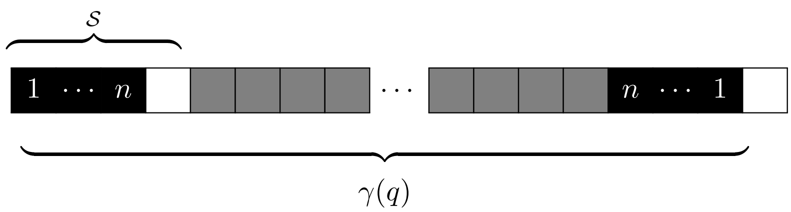

FIGURE 3. Symmetry in $\mathcal{S}$. White boxes represent sections of 0 values modulo p^e, gray sections represent the other values of coefficients of $\gamma(q)$ modulo p^e. The coefficients are ordered from left to right by increasing associated powers of q. We define $n = Q - \binom{k+1}{2}$, so that the white numbers $1, 2, \ldots, n$ enumerate coefficients in the black sections.

□

These ideas can be generalized using the Chinese remainder theorem.

Lemma 2.7. *Partitions with at most k parts are purely periodic modulo N for all $N \in \mathbb{N}$, with period*

$$\pi_N(k) = \operatorname*{lcm}_{p|N}\left(\pi_{p^{\nu_p(N)}}(k)\right).$$

Corollary 2.8. *Theorem 2.6 also holds for $\mathcal{S}$ for the general modulus N.*

Proof. Use Chinese remainder theorem, and note that Theorem 2.6 is preserved when combining congruences modulo different prime powers. □

Theorem 2.9. *Let $k \geq 0$ and N be odd. If k is odd and $\gcd\left(\frac{\pi_N(k+1)}{\pi_N(k)}, N\right) > 1$, then we have*

$$\frac{\pi_N(k+1)}{\pi_N(k)} \sum_{i \in \mathbb{Z}/\pi_N(k)\mathbb{Z}} p_{\leq k}(i) \equiv 0 \pmod{N}.$$

Otherwise we have the stronger result $\sum_{i \in \mathbb{Z}/\pi_N(k)\mathbb{Z}} p_{\leq k}(i) \equiv 0 \pmod{N}$.

Proof. First, we prove this for when k is even. We have two cases. First, suppose $\frac{\pi_N(k) - \binom{k+1}{2} - 1}{2} \in \mathbb{Z}$. This means there exists a 'central' element that is self-inverse[2] in $\mathcal{S}$ by Corollary 2.8. Since N is odd it is 0 mod N. Using Corollary 2.8 we pair all other terms in $\sum_{i \in \mathbb{Z}/\pi_N(k)\mathbb{Z}} p_{\leq k}(i)$ in zero-sum pairs.

[2] $x = -x \bmod N$

Otherwise, $\frac{\pi_N(k)-\binom{k+1}{2}-1}{2} \notin \mathbb{Z}$. There is no central entry, and pairing via Corollary 2.8 suffices to show $\sum_{i\in\mathbb{Z}/\pi_N(k)\mathbb{Z}} p_{\le k}(i) \equiv 0 \pmod N$.

For k odd, we use a different method since $(-1)^{k+1} = 1$. We have

$$\begin{aligned}\sum_{i\in\mathbb{Z}/\pi_N(k+1)\mathbb{Z}} p_{\le k+1}(i) - p_{\le k}(i) &\equiv \sum_{i\in\mathbb{Z}/\pi_N(k+1)\mathbb{Z}} p_{=(k+1)}(i) \pmod N\\ &\equiv \sum_{i\in\mathbb{Z}/\pi_N(k+1)\mathbb{Z}} p_{\le k+1}(i-(k+1)) \pmod N\\ &\equiv 0 \pmod N. \quad \text{(by even case, shift invariance)}\end{aligned}$$

Thus, we obtain

$$\frac{\pi_N(k+1)}{\pi_N(k)} \sum_{i\in\mathbb{Z}/\pi_N(k)\mathbb{Z}} p_{\le k}(i) \equiv 0 \pmod N$$

Unless $\gcd\left(\frac{\pi_N(k+1)}{\pi_N(k)}, N\right) > 1$, the stronger statement $\sum_{i\in\mathbb{Z}/\pi_N(k)\mathbb{Z}} p_{\le k}(i) \equiv 0 \pmod N$ holds since we would have $\frac{\pi_N(k+1)}{\pi_N(k)} \in (\mathbb{Z}/N\mathbb{Z})^\times$. $\square$

Corollary 2.10. *Let p be an odd prime. Then for all k except odd k where $\gcd\left(\frac{\pi_p(k+1)}{\pi_p(k)}, p\right) > 1$ and $k \nmid \pi_p(k-1)$, we have*

$$\sum_{i\in\mathbb{Z}/\pi_p(k)\mathbb{Z}} p_{\le k}(i) \equiv 0 \pmod p.$$

3. Decomposition of $\begin{bmatrix} n \\ k \end{bmatrix}_q$

In this section and the next, we exploit the results from Section 2 regarding the periodicity of $\mathcal{S}$ and the structure of $\mathcal{S}$ (as described by Theorem 2.6) in order to prove Theorem 1.7.

Definition 3.1. For a modulus N, we define the function $\mathcal{L}_{k,R}^{(i)}$ by

$$f_{k,R}(n) = \mathcal{L}_{k,R}^{(i)}\left(\frac{n-i}{\pi'_N(k)}\right),$$

if $n \equiv i \pmod{\pi'_N(k)}$.

Remark 3. The change of variables $n \mapsto \frac{n-i}{\pi'_N(k)}$ is used to simplify proofs.

The aim is now to show that the functions $\mathcal{L}_{k,R}^{(i)}$ are linear, from which it follows by definition that $f_{k,R}$ is quasipolynomial. To do this, we will use the following general strategy:

- Divide the coefficients of $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ into different sections with periodic behavior.
- Using the periodicity of the first n coefficients (by Lemma 2.7), inductively show that these sections are also periodic using a friendly partition decomposition (Lemma 4.2).
- Use this last fact to show that $n \mapsto n + \pi'_N(k)$ changes $f_{k,R}(n+k)$ a constant amount depending only on r.
- Conclude $f_{k,R}$ is quasipolynomial, since the previous point shows $\mathcal{L}_{k,R}^{(i)}$ are linear.

We begin with the division of coefficients in $\begin{bmatrix} n \\ k \end{bmatrix}_q$ into different sections.

Definition 3.2. The ith *section* of the q-binomial coefficient $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ is the sequence of coefficients denoted by S_i with jth term given by

$$p_{\le k}^{(i)}(j) = [q^{in+j}] \begin{bmatrix} n+k \\ k \end{bmatrix}_q,$$

where $j \in \mathbb{Z}/n\mathbb{Z}$. As a special case, S_0 is just a concatenation of copies of $\mathcal{S}$.

Recall the aforementioned identity

$$\sum_{i\geq 0} p(n,k,i)q^i = \begin{bmatrix} n+k \\ k \end{bmatrix}_q,$$

where $p(n,k,i)$ denotes the number of partitions $\lambda \vdash i$, with at most k parts and maximal part $\leq n$.

This definition allows us to loosely characterize a section by saying terms in the sequence contain the number of partitions which fit in a $n \times k$ box of size $|\lambda| = l$ for l such that there exists a partition of $\lambda \vdash l$ covering i complete rows but no partition covering $i+1$ rows.

Definition 3.3. Let $X = (x_0, \ldots, x_{|X|-1}$ and $Y = (y_0, \ldots, y_{|Y|-1})$ be finite sequences. The concatenation operator $\oplus$ is defined as $X \oplus Y = (x_0, x_1, \ldots x_{|X|-1}, y_0, y_1, \ldots y_{|Y|-1})$.

We then make the following decomposition of S_i that proves useful:

$$\mathsf{S}_i = \left(\bigoplus_{j\in[l]} \mathsf{B}_i^j \right) \oplus \mathsf{R}_i,$$

where the B_i^j are $\pi'_N(k)$-length subsequences and R_i is the remainder after these $l = \lfloor \frac{n}{\pi'_N(k)} \rfloor$ consecutive subsequences are removed from S_i. Informally, if we regard $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ as a sequence ordered by the associated exponents of q, we can relate $X = \bigoplus_{i\in[k]} \mathsf{S}_{i-1} \oplus (1)$ to its corresponding q-binomial coefficient. Here, (1) is just a sequence only containing 1. We can index X starting at 0, obtaining

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{x_i\in X} x_i q^i \pmod N.$$

The net result of this decomposition is illustrated in Figure 4.

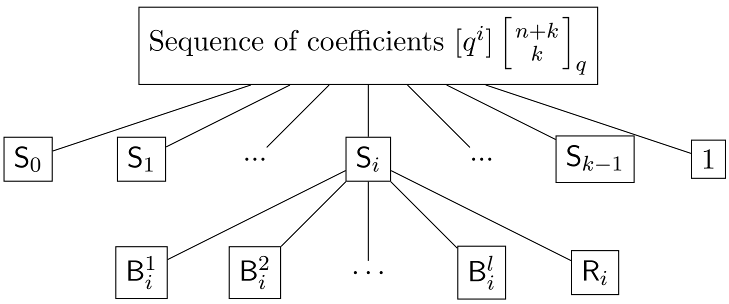

FIGURE 4. Decomposition of a q-binomial coefficient into sections modulo N. Here, edge connections denote concatenation (as per Definition 3.3) from left to right and $l := \lfloor \frac{n}{\pi'_N(k)} \rfloor$.

4. Proving $f_{k,R}$ is quasipolynomial

Using the definitions from Section 3, we investigate the structure of each individual section.

Definition 4.1. Fix a q-binomial coefficient $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$. Let $\mathcal{P}_{i,m}^{\text{bad}}$ be the set containing all pairs of partitions (λ, μ) such that

- $|\lambda| + |\mu| = mn + j$.
- λ has at most k parts each at most n, of which i are equal to n.
- μ has exactly i parts.

Lemma 4.2. *Fix n, m, k. For $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ we have the following identity for the associated functions $p^{(m)}_{\leq k}$:*

$$p^{(m)}_{\leq k}(j) = p_{\leq k}(mn + j) - \sum_{i \in [m]} \#\mathcal{P}^{\text{bad}}_{i,m}(j).$$

Proof. Let S be the set of partitions counted by $p^{(m)}_{\leq k}(j)$ and S' be defined similarly for $p_{\leq k}(mn+j)$. It is clear that $S \subseteq S'$, so we wish to show that $\sum_{i \in [m]} \#\mathcal{P}^{\text{bad}}_{i,m}(j)$ enumerates all of the additional partitions that leave the $n \times k$ box. Consider Figure 5 below.

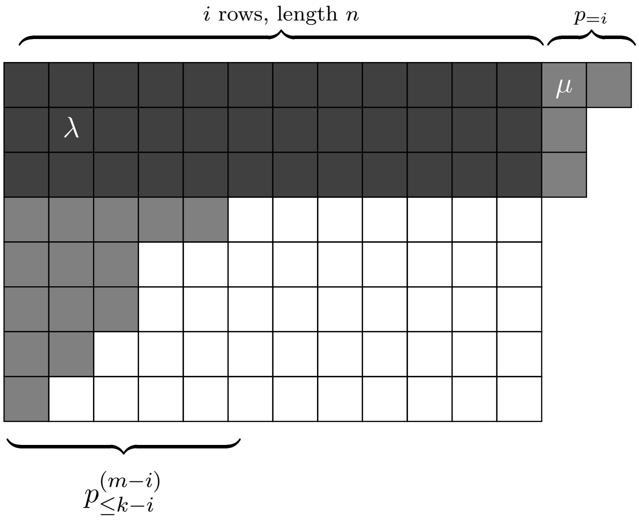

FIGURE 5. Classification of partitions in $\mathcal{P}^{\text{bad}}_{i,m}(j)$

Figure 5 depicts a pair $(\lambda, \mu) \in \mathcal{P}^{\text{bad}}_{i,m}(j)$. Here, λ is represented by the shaded boxes inside the $n \times k$ box. The darker boxes depict the i parts of λ that are exactly n, while the lighter gray boxes below depict the part of λ that can vary. Outside of the $n \times k$ boxes is μ, with precisely i parts. As labelled in the diagram, it is easy to see that λ are enumerated by $p^{(m-i)}_{\leq k-i}$ and μ are enumerated by $p_{=i}$. Construct a partition $\Pi = \lambda + \mu$ via part-wise addition. This is counted in S' by $p_{\leq k}$ but not in S since it *must* leave the $n \times k$ box. Thus, sending $(\lambda, \mu) \in \bigcup_{i \in [m]} \mathcal{P}^{\text{bad}}_{i,m}(j)$ to $\Pi = \lambda + \mu$ is a map ϕ from $\bigcup_{i \in [m]} \mathcal{P}^{\text{bad}}_{i,m}(j)$ to $|S' \setminus S|$. We claim ϕ is a bijection. It is not too difficult to see that ϕ is an injection: if $\lambda + \mu = \lambda' + \mu'$ then μ and μ' have the same number of parts and from this it is evident $\mu = \mu', \lambda = \lambda'$.

Now we show ϕ is a surjection. Take a "bad" partition $\Pi = \{\pi_1, \ldots, \pi_k\}$ with $|\Pi| = mn + j$ leaving the box. Such a partition must leave the box for the first i rows for some $i \in [m]$ (we cannot have $i > m$, since $|\Pi| = mn + j \leq (m+1)n$). Setting $\mu = \{\pi_\alpha - n : \pi_\alpha > n\}$ and $\lambda = \{\pi_\alpha : \pi_\alpha \leq n\} \cup \{n : \pi_\alpha > n\}$, we construct a pair (λ, μ). Both λ, μ satisfy the first two conditions of Definition 4.1 due to the construction. The first condition $|\lambda| + |\mu| = mn + j$ is also satisfied, as

$$|\lambda| + |\mu| = \sum_{\pi_\alpha > n} (\pi_\alpha - n) + n + \sum_{\pi_\alpha \leq n} \pi_\alpha = |\Pi| = mn + j.$$

Thus $(\lambda, \mu) \in \mathcal{P}^{\text{bad}}_{i,m}(j)$, and $\lambda + \mu = \Pi$.

Thus, we have a bijection ϕ from $\bigcup_{i \in [m]} \mathcal{P}^{\text{bad}}_{i,m}(j)$ to $|S' \setminus S|$, and it follows that

$$\sum_{i \in [m]} \#\mathcal{P}^{\text{bad}}_{i,m}(j) = |S' \setminus S|.$$

∎

The following is a restatement of Theorem 1.9 and is the main result.

Theorem 4.3. *Let $N \in \mathbb{N}$ be the modulus of $f_{k,R}$. Then $f_{k,R}$ is quasipolynomial, with quasi-period $\pi'_N(k)$. Additionally, all $\mathcal{L}^{(i)}_{k,R}$ are linear functions.*

Proof. Let $Q := \pi'_N(k)$. To prove the claim, the central idea of the argument is to show that in the section S_i of $\begin{bmatrix} Ql+r+k \\ k \end{bmatrix}_q$ we have $\mathsf{B}_i^1 = \mathsf{B}_i^2 = \ldots = \mathsf{B}_i^l$. From this fact, we make a simple argument that shows the claim. We use the q-binomial coefficient $\begin{bmatrix} Ql+k+r \\ k \end{bmatrix}_q$ so that we can read off that B_i have length Q, R_i has length r, and that l is the number of B_i in the decomposition of S_i.

To prove that $\mathsf{B}_i^1 = \mathsf{B}_i^2 = \ldots = \mathsf{B}_i^l$, we induct on the indices $m = 1, 2, \ldots, k-1$ of the sections, holding k fixed, and then induct on k.

In §2, we already showed that S_0 has the aforementioned property for all k by considering partitions with at most k parts. One can also show that $\mathsf{B}_i^1 = \mathsf{B}_i^2 = \ldots = \mathsf{B}_i^l$ holds when $k = 2$ by explicit computation of $p_{\leq 2}(j) = \lfloor \frac{j}{2} \rfloor + 1$. This establishes the base cases $m = *, k = 2$ and $m = 0, k = *$.

We show the claim holds for S_m assuming it holds for S_{m-1} and all smaller k. Using Lemma 4.2, we have for $n \in \mathbb{Z}/(Ql+r)\mathbb{Z}$,

$$\begin{aligned} p^{(m)}_{\leq k}(j) &\equiv p_{\leq k}(j+mr) - \sum_{i\in[m]} \#\mathcal{P}^{\mathrm{bad}}_{i,m}(j) \\ &\equiv p_{\leq k}(j+mr) - \sum_{i\in[m]} \sum_{\ell+\ell'=C_{i,m}(j)} p^{(m-i)}_{\leq k-i}(\ell)p_{=i}(\ell') \pmod N \end{aligned} \tag{3}$$

where $C_{i,m}(j) = |\lambda| + |\mu|$ for $(\lambda, \mu) \in \mathcal{P}^{\mathrm{bad}}_{i,m}(j)$ (or more explicitly $C_{i,m}(j) = (m-i)(Ql+r)+j$) and $\ell, \ell' \geq 0$. The functions $p_{=i}$ and $p^{(m-i)}_{\leq k-i}$ count μ and λ in $\mathcal{P}^{\mathrm{bad}}_{i,m}$ respectively. Note that

$$p_{=i}(\ell) = p_{\leq i}(\ell - i),$$

an explicit bijection being given by taking $\lambda \vdash n$ counted by $p_{=i}$ and decreasing each part by one. Thus, we see $p_{=i}$ has period Q as $\pi_N(i)|\pi_N(k)|Q$. By inductive hypothesis, $p^{(m-i)}_{\leq k-i}$ also has period Q. Thus all functions in the above sum have period Q modulo N. We claim that the map $j \mapsto j+Q$ leaves $p^{(m)}_{\leq k}(j)$ unchanged modulo N, or that $p^{(m)}_{\leq k}(j+Q) - p^{(m)}_{\leq k}(j) \equiv 0 \pmod N$. Since $p_{\leq k}$ has period $\pi_N(k) \mid \pi'_N(k)$, the function $p_{\leq k}$ will vanish in the difference. So it suffices to show that $\#\mathcal{P}^{\mathrm{bad}}_{i,m}(j+Q) - \#\mathcal{P}^{\mathrm{bad}}_{i,m}(j) \equiv 0 \pmod N$. The expansion in (3) combined with the fact that $C_{i,m}(j+Q) - C_{i,m}(j) = Q$ and that the functions in the expansion have period dividing Q implies that

$$\#\mathcal{P}^{\mathrm{bad}}_{i,m}(j+Q) - \#\mathcal{P}^{\mathrm{bad}}_{i,m}(j) \equiv \sum_{i\in[m]} \sum_{\substack{\ell+\ell'\equiv j \\ \pmod Q}} p^{(m-i)}_{\leq k-i}(\ell)p_{=i}(\ell').$$

It follows immediately from the definition of $\pi'_N(k)$ that $\pi'_N(k)/\pi'_N(k-1) \in N\mathbb{Z}$. But note that $p^{(m-i)}_{\leq k-i}$ and $p_{=i}$ have periods dividing $\pi'_N(k-1)$ as it is always true that $i \in [k-1]$ for each sum, and hence are repeated at least N times when summed over $\mathbb{Z}/Q\mathbb{Z}$. Thus $p^{(m)}_{\leq k}(j)$ is Q-periodic since the sum is 0 modulo N, and by strong induction the same is true for each S_m. This completes the induction.

Then $l \mapsto l+1$ simply adds on another identical period in each S_m. Hence, we may write

$$\mathsf{S}_i = \left(\bigoplus_{j\in[l]} \mathsf{B}_i \right) \oplus \mathsf{R}_i,$$

where the B_i are identical modulo N. For short, we can now just write $\mathsf{S}_i = \mathsf{B}_i^{\oplus l} \oplus \mathsf{R}_i$. More importantly, this indicates that $f_{k,R}(Q(l+1)+r) - f_{k,R}(Ql+r)$ is a constant depending on r. Thus, we can write

$$f_{k,R}(n) = \mathcal{L}_{k,R}^{(i)}\left(\frac{n-i}{Q}\right),$$

which is precisely what we wanted. $\square$

The decomposition used in the Theorem 4.3 also allows us to prove the following observation about a special case of $p_{\leq k}^{(m)}$:

Corollary 4.4. *The last* $\binom{k+1-m}{2} - 1$ *entries of each component* B_m^i *of* S_m *are* 0 *modulo* N *for* $\begin{bmatrix} \pi'_N(k)l+k \\ k \end{bmatrix}_q$.

Proof. This is given for $m = 0$ by Corollary 2.8, so let $m > 0$. Similarly, $k = 2$ is trivial. We proceed by strong induction on k, m. By Lemma 4.2, for $j \in \mathbb{Z}/\pi'_N(k)l\mathbb{Z}$ we have

$$p_{\leq k}^{(m)}(j) \equiv p_{\leq k}(j) - \sum_{i\in[m]} \#\mathcal{P}_{i,m}^{\text{bad}}(j) \pmod N.$$

Noting that $\binom{a}{2} < \binom{b}{2}$ when $a < b$ we see that for $j \in [\pi'_N(k) - \binom{k+1-m}{2} + 1, \pi'_N(k) - 1]$ that $p_{\leq k}(j) \equiv 0 \pmod N$ by Corollary 2.8. Therefore, for such j we have the simplified form $p_{\leq k}^{(m)}(j) \equiv -\sum_{i\in[m]} \#\mathcal{P}_{i,m}^{\text{bad}}(j) \pmod N$. We wish to show that $\sum_{i\in[m]} \#\mathcal{P}_i^{\text{bad}} \equiv 0 \pmod N$. To do this, we use the expansion of $\#\mathcal{P}_{i,m}^{\text{bad}}(j)$ from the main theorem:

$$\begin{aligned}
\#\mathcal{P}_{i,m}^{\text{bad}}(j) &\equiv \sum_{\ell+\ell'=j} p_{\leq k-i}^{(m-i)}(\ell) p_{=i}(\ell') \pmod N \\
&\equiv \sum_{\substack{\ell+\ell'\equiv 0 \\ (\text{mod } \pi'_N(k))}} p_{\leq k-i}^{(m-i)}(\ell) p_{=i}(\ell') \pmod N
\end{aligned}$$

as the last $\binom{(k-i)+1-(m-i)}{2} - 1 = \binom{k+1-m}{2} - 1$ entries of $p_{\leq k-i}^{(m-i)}$ are 0 by inductive hypothesis, and our specifically chosen j makes it so that each new term added must then be 0 mod N. The final sum is the same as the one considered in the main theorem, which was shown to be 0 modulo N. $\square$

Remark 4. Using the symmetry of the q-binomial coefficients, we can notice that a similar claim holds for the first entries of B_m^i. Explicitly, the first $\binom{k+1-((k-1)-m)}{2} - 1 = \binom{m+2}{2} - 1$ entries are 0 modulo N.

5. The generating function of $f_{k,R}$

The result from the previous section allows for the generating function for $f_{k,R}$ to be explicitly calculated.

Theorem 5.1. *For a modulus* $N \in \mathbb{N}$, *we have*

$$F_{k,R}(x) := \sum_{n\geq 0} f_{k,R}(n)x^n = \frac{1}{(1-x^Q)^2} \sum_{i\in\mathbb{Z}/Q\mathbb{Z}} (1-x^Q)b_i x^i + m_i x^{Q+i},$$

where $\mathcal{L}_{k,R}^{(i)}$ *has constant term* b_i *and slope* m_i *and* $Q = \pi'_N(k)$.

Proof. For simplicity, let $\mathcal{L}_i = \mathcal{L}_{k,R}^{(i)}$, and $Q = \pi'_N(k)$ as above. Then we let $F_{\mathcal{L}_i}(x) = \sum_{j\geq 0} \mathcal{L}_i(j)x^{jQ+i}$, so that

$$F_{k,R}(x) = \sum_{i\in\mathbb{Z}/Q\mathbb{Z}} F_{\mathcal{L}_i}(x).$$

Fortunately, each term is simple to find. We have

$$\begin{aligned}\sum_{i\in\mathbb{Z}/Q\mathbb{Z}} F_{\mathcal{L}_i}(x) &= \sum_{i\in\mathbb{Z}/Q\mathbb{Z}} x^i \left(\sum_{j\geq 0} b_i x^{jQ} + \sum_{j\geq 0} m_i j x^{jQ}\right)\\ &= \sum_{i\in\mathbb{Z}/Q\mathbb{Z}} x^i \left(\frac{b_i}{1-x^Q} + \frac{m_i x^Q}{(1-x^Q)^2}\right)\\ &= \frac{1}{(1-x^Q)^2} \sum_{i\in\mathbb{Z}/Q\mathbb{Z}} (1-x^Q) b_i x^i + m_i x^{Q+i}\end{aligned}$$

which proves the theorem.

□

Letting $Q' = \pi'_N(k-1)$, it turns out that one can often rewrite this as

$$F_{k,R}(x) = \frac{1}{(1-x^Q)^2}\left(\sum_{i\in\mathbb{Z}/Q\mathbb{Z}} (1-x^Q) b_i x^i + \sum_{i\in\mathbb{Z}/Q'\mathbb{Z}} m_i x^{Q+i} \frac{x^Q - 1}{x^{Q'}-1}\right).$$

This stems from the fact that the slopes of the functions $\mathcal{L}^{(i)}_{k,R}$ often have a smaller period (in i, where k, R are fixed) than the actual quasiperiod itself, namely Q'. This is formalized by Theorem 5.2, and an example is given in Figure 6.

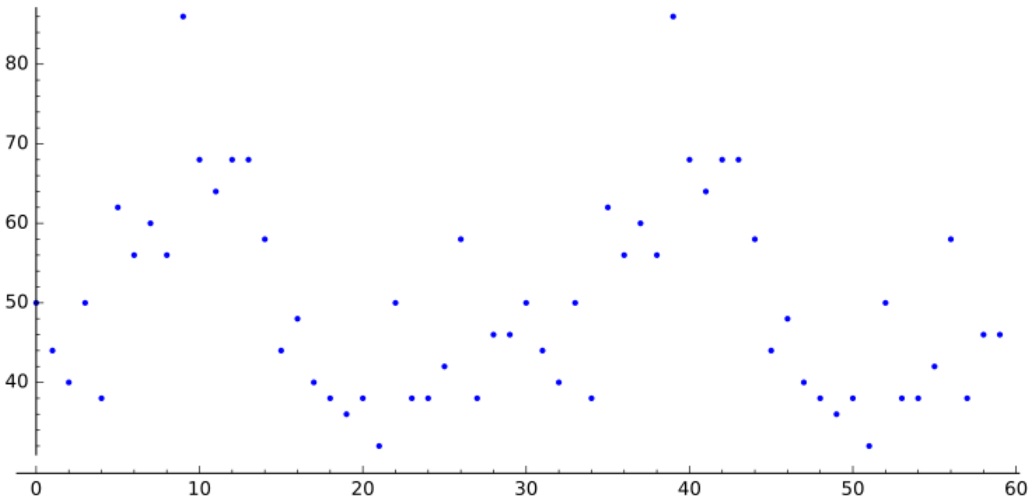

FIGURE 6. **An illustration of Theorem 5.2.** This figure shows the slopes of the functions $\mathcal{L}^{(i)}_{4,1}$ when $N = 5$. The x axis is i, while the y axis is the slope. Notice that the slopes have a period that is half of the actual minimal quasiperiod (in this case, given by the function π_5) and $\pi_5(4)/\pi_5(3) = 60/30 = 2$, as claimed.

Theorem 5.2. *Fix a pair (k, N) such that*

$$\sum_{i\in\mathbb{Z}/\pi_N(k-1)\mathbb{Z}} p_{\leq k-1}(i) \equiv 0 \pmod{N}$$

by Theorem 2.9 or Corollary 2.10 and let R be arbitrary. Then the slope of $\mathcal{L}^{(i)}_{k,R}$ is equal to that of $\mathcal{L}^{(i')}_{k,R}$ where $i' \equiv i + \pi'_N(k-1) \pmod{\pi'_N(k)}$.

Proof. In order to prove the theorem, we actually make a deeper claim. Consider the q-binomial coefficients $\begin{bmatrix} n+k \\ k\end{bmatrix}_q, \begin{bmatrix} \widetilde{n}+k \\ k\end{bmatrix}_q$ where $\widetilde{n} = n + \pi'_N(k-1)$ and decompose the coefficients into S_i and

$\widetilde{\mathsf{S}}_i$ respectively. Then when we make the decompositions $\mathsf{S}_i = \mathsf{B}_i^{\oplus l} \oplus \mathsf{R}_i$ and $\widetilde{\mathsf{S}}_i = \widetilde{\mathsf{B}}_i^{\oplus l} \oplus \widetilde{\mathsf{R}}_i$, we want to show that B_i is a cyclic shift of $\widetilde{\mathsf{B}}_i$. From Lemma 4.2, we have

$$p_{\leq k}^{(m)}(j) \equiv p_{\leq k}(j+mn) - \sum_{i\in[m]} \sum_{\ell+\ell'=C_{i,m}(j)} p_{\leq k-i}^{(m-i)}(\ell) p_{=i}(\ell') \pmod N$$

where $C_{i,m}(j) = (m-i)n + j$. Now we take $n \mapsto \widetilde{n} = n + \pi'_N(k-1)$ and obtain a function $\widetilde{p}_{\leq k}^{(m)}(j)$ for $\widetilde{\mathsf{S}}_1$. If we take $j \mapsto \widetilde{j} = j + m\pi'_N(k-1)$, we claim that

$$p_{\leq k}^{(m)}(\widetilde{j}) = \widetilde{p}_{\leq k}^{(m)}(j).$$

This is equivalent to B_i being a cyclic shift of $\widetilde{\mathsf{B}}_i$. Using the Lemma 4.2 again, we see this is equivalent to

$$p_{\leq k}(\widetilde{j}+mn) - \sum_{i\in[m]} \#\mathcal{P}_{i,m}^{\text{bad}}(\widetilde{j}) \equiv p_{\leq k}(j+m\widetilde{n}) - \sum_{i\in[m]} \#\mathcal{P}_{i,m}^{\text{bad}}(j) \pmod N.$$

As $\widetilde{j} + mn = j + m\widetilde{n}$, we need only consider the sums $\sum_{i\in[m]} \#\mathcal{P}_{i,m}^{\text{bad}}(n)$. Expanding, we want to show that $\sum_{i\in[m]} \sum_{\ell+\ell'=C_{i,m}(j)} p_{\leq k-i}^{(m-i)}(\ell) p_{=i}(\ell')$ is invariant modulo N under $j \mapsto \widetilde{j}$. As $\pi'_N(k)/\pi'_N(k-1) \in N\mathbb{Z}$, we need only consider the term $\sum_{\ell\in\mathbb{Z}/\pi'_N(k-1)\mathbb{Z}} p_{\leq k-1}^{(m-1)}(\ell)$ and show it is 0 modulo N. This is easier than it seems: if we expand using Lemma 4.2, we see

$$\sum_{\ell\in\mathbb{Z}/\pi'_N(k-1)\mathbb{Z}} p_{\leq k-1}^{(m-1)}(\ell) \equiv \sum_{\ell\in\mathbb{Z}/\pi'_N(k-1)\mathbb{Z}} p_{\leq k-1}(\ell) + \sum_{\ell\in\mathbb{Z}/\pi'_N(k-1)\mathbb{Z}} (\text{vanishing terms}) \pmod N.$$

The remaining terms will vanish since $\pi'_N(k-1)/\pi'_N(k-2) \in N\mathbb{Z}$. This is easy to show by simply counting the number of times terms are repeated and noting that they are all repeated some multiple of N using this fact. Precisely, using the decomposition in the main theorem the remaining terms can be written as a triple sum

$$\sum_{\ell\in\mathbb{Z}/\pi'_N(n-1)\mathbb{Z}} \sum_{i\in[m-1]} \sum_{\ell'\leq C_{i,m-1}(\ell)} f_i(\ell') = \sum_{i\in[m-1]} \left(\sum_{\ell\in\mathbb{Z}/\pi'_N(n-1)\mathbb{Z}} \sum_{\ell'\leq C_{i,m-1}(\ell)} f_i(\ell') \right)$$

where $f_i(\ell') = p_{\leq k-1-i}^{(m-1-i)}(C_{i,m-1}(\ell) - \ell') p_{=i}(\ell')$ has period dividing $\pi'_N(k-2)$. Fixing i, we can restrict ourselves to looking at the inner sums over ℓ and ℓ' and note that $C_{i,m}(\ell)$ is of the form $K + \ell$ for some constant K. The value of K is irrelevant since $\pi'_N(k-1)/\pi'_N(k-2) \in N\mathbb{Z}$ – this means the sum as a whole is unchanged modulo N if the innermost sum is replaced with $\sum_{\ell'\leq\ell} f_i(\ell')$. In this equivalent form modulo N, each value of f_i over its period $\pi'_N(k-2)$ is repeated $\binom{\pi'_N(k-1)/\pi'_N(k-2)+1}{2}$ times, and since N is odd and the quotient $\pi'_N(k-1)/\pi'_N(k-2) \in N\mathbb{Z}$ each inner sum vanishes modulo N so we can ignore all of these terms and focus on $\sum_{\ell\in\mathbb{Z}/\pi'_N(k-1)\mathbb{Z}} p_{\leq k-1}(\ell)$. This must go to zero modulo N due to the restrictions on the pair (k, N), so we are done.

Thus, $p_{\leq k}^{(m)}(\widetilde{j}) = \widetilde{p}_{\leq k}^{(m)}(j)$. We conclude that $\widetilde{\mathsf{B}}_i$ is a cyclic shift of B_i and the result follows since the slopes of $\mathcal{L}_{k,R}^{(i)}$ depend only on the number of occurrences of the residue R in each B_i (which clearly is the same under a cyclic shift). □

6. Asymptotics for the quasi-period

Given the complex nature of the definition for $\pi'_N(k)$ it is worth investigating asympototics to understand how quickly $f_{k,R}(n)$ and its generating function grow in complexity.

First we investigate asymptotics for $\pi_p(k)$ for each prime p. We have the expansion

$$\pi_p(k) = p^{b_p([k])} L_p([k]),$$

where $b_p([k])$ is as previously defined and $L_p([k]) = \mathrm{lcm}([k])/p^{\nu_p(\mathrm{lcm}([k]))}$. Note that $\mathrm{lcm}([k]) = e^{\psi(k)}$ where $\psi(k)$ is the Chebyshev function. It is known that $\psi(k) = x + o(x)$, so $\mathrm{lcm}([k]) = e^{k+o(k)}$. Let

$$\Pi(k) := \sum_{i\in[k]} p^{\nu_p(i)}.$$

We first consider the asymptotics of this function in Lemma 6.1.

Lemma 6.1.

$$\Pi(k) = \sum_{i\in[k]} p^{\nu_p(i)} \sim \frac{p-1}{p} k \log_p(k).$$

Proof. This can be done by observing that $\sum_{i\in[k]} p^{\nu_p(i)} = \sum_{i=0}^{\lfloor \log_p(k)\rfloor} \#V_{i,p}p^i$, where $V_{i,p} = \{j | j \in [k], \nu_p(j) = p^i\}$. Now we can take $\#V_{i,p} = \left\lfloor \frac{k}{p^i}\right\rfloor - \left\lfloor \frac{k}{p^{i+1}}\right\rfloor$, yielding

$$\sum_{i=0}^{\lfloor \log_p(k)\rfloor} \#V_{i,p}p^i = \sum_{i\geq 0}\left(\left\lfloor \frac{k}{p^i}\right\rfloor - \left\lfloor \frac{k}{p^{i+1}}\right\rfloor\right)p^i = (p-1)\left(\sum_{i\geq 0}\left\lfloor \frac{k}{p^i}\right\rfloor p^{i-1}\right).$$

We now want to show $\sum_{i\geq 0}\left\lfloor \frac{k}{p^i}\right\rfloor p^{i-1} \sim \frac{k\log_p(k)}{p}$ for large k. That is, we want to show that $\lim_{k\to\infty} \frac{k\log_p(k)}{\sum_{i\geq 0}\left\lfloor \frac{k}{p^i}\right\rfloor p^i} = 1$. We obtain upper and lower bounds for the limit by considering the sum when $k = p^i, p^i - 1$, namely

$$k\log_p(k) \geq \sum_{i\geq 0}\left\lfloor \frac{k}{p^i}\right\rfloor p^i \geq k\left(\log_p(k) - \frac{p}{p-1}\right),$$

and dividing we see

$$\frac{k\log_p(k)}{\sum_{i\geq 0}\left\lfloor \frac{k}{p^i}\right\rfloor p^i} \leq \frac{k\log_p(k)}{k(\log_p(k) - \frac{p}{p-1})} = 1 + \frac{\frac{p}{p-1}k}{k\log_p(k) - \frac{p}{p-1}k}.$$

Thus, the limit is bounded above by 1. The lower bound clearly goes to 1, and we conclude that

$$\sum_{i\in[k]} p^{\nu_p(i)} \sim \frac{p-1}{p} k \log_p(k).$$

□

Lemma 6.2. $\nu_p(\mathrm{lcm}([k])) = \lfloor \log_p(k)\rfloor$.

Now we can make an asymptotic for the log of $\pi_p(k)$.

Theorem 6.3. *We have*

$$\log_p(\pi_p(k)) \sim \log_p \log_p(k) + \frac{\psi(k)}{\ln p}.$$

Proof. Using Lemma 6.2, we see

$$\pi_p(k) = p^{b_p([k])}L_p([k]) = \mathrm{lcm}([k])p^{b_p([k]) - \lfloor \log_p(k)\rfloor}.$$

We know that this can be simplified to

$$\begin{aligned}\log_p \pi_p(k) &= \frac{\psi(k)}{\ln p} + \big(b_p([k]) - \lfloor \log_p(k)\rfloor\big)\\ &= \frac{\psi(k)}{\ln p} + \big(\lceil \log_p \Pi(k)\rceil - \lfloor \log_p(k)\rfloor\big). \qquad (3)\end{aligned}$$

We use Lemma 6.1 to show $\log_p \Pi(k) \sim \log_p \left(\frac{p-1}{p} k \log_p(k)\right)$. Ignoring constant terms, we improve this asymptotic to

$$\log_p \Pi(k) \sim \log_p \log_p(k) + \log_p(k),$$

and in the limit floors become irrelevant so the $\log_p(k)$ term is cancelled in (3), yielding the desired asymptotic. $\square$

Lemma 6.4. $\log_p \left(\frac{\pi'_p(k)}{\pi_p(k)}\right) \sim k - \log_p(k) - \log_p \log_p(k)$

Proof. Note that

$$\frac{\pi'_p(k)}{\pi_p(k)} = p^{\#S_k},$$

where

$$S_k = \left\{ i : i \leq k, \frac{\pi_p(i)}{\pi_p(i-1)} \notin p\mathbb{Z} \right\}.$$

More precisely, the condition in S_k can be re-written in terms of the p-adic valuation as $\nu_p(\pi_p(i)) = \nu_p(\pi_p(i-1))$. But this valuation is just $b_p([i])$, so we really have $\#S_k = k - b_p([k])$. Now we can write

$$\#S_k = k - \left\lceil \log_p \left(\Pi(k)\right) \right\rceil$$

where we already have an asymptotic formula for $\Pi(k)$. We can obtain

$$\#S_k \sim k - \log_p \left(\frac{p-1}{p} k \log_p(k)\right).$$

This implies that

$$\begin{aligned} \log_p \left(\frac{\pi'_p(k)}{\pi_p(k)}\right) &= \#S_k \\ &\sim k - \log_p \left(\frac{p-1}{p} k \log_p(k)\right). \end{aligned}$$

Ignoring constants in the above asymptotic, we obtain the desired asymptotic. $\square$

By understanding π_p, we can easily derive formulas for π'_p by simply accounting for a power of p as above. That is,

$$\begin{aligned} \log_p \pi'_p(k) &\sim \log_p \pi_p(k) + k - \log_p(k) - \log_p \log_p(k) \\ &\sim \frac{\psi(k)}{\ln p} + k - \log_p(k). \end{aligned}$$

where we have already bounded π_p via Theorem 6.3. For prime powers, we have the formula

$$\pi'_{p^e}(k) = \left(\pi_p(k) p^{e-1} \left(\frac{\pi'_p(k)}{\pi_p(k)}\right)^e\right), \tag{4}$$

which then yields an estimate through Theorem 6.3 and Lemma 6.4. Namely,

$$\log_p \left(\pi'_{p^e}(k)\right) \sim \frac{\psi(k)}{\ln p} + e\big(k - \log_p(k)\big) + (1-e) \log_p \log_p(k).$$

We know that

$$\pi'_N(k) = \operatorname*{lcm}_{p|N} \left(\pi'_{p^{\nu_p(N)}}(k)\right),$$

so these asymptotics may be combined for the general case.

7. Conclusion and Future Directions

We have shown that the function $f_{k,R}(n)$ is quasipolynomial modulo any $N \in \mathbb{N}$, from which an explicit formula for the generating function $F_{k,R}(x)$ follows. Additionally, the structure of the coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has been described in terms of the sections S_i of that q-binomial coefficient, and the repeating period in each section has been shown to retain some of the properties of $\mathcal{S}$.

A good future direction is to determine the minimal quasiperiod of $f_{k,R}(n)$. It is expected to lie somewhere between $\pi_N(k)$ and $\pi'_N(k)$ but it is unclear how the function actually behaves.

It is also interesting to investigate symmetry in the minimal period of the slopes of the functions $\mathcal{L}^{(i)}_{k,R}$ — if we let this period be P, we mean precisely that the slope of $\mathcal{L}^{(i)}_{k,R}$ matches that of $\mathcal{L}^{(P-i)}_{k,R}$ for $0 \leq i \leq P$. Figure 6 gives a counterexample (the slopes for $0 \leq i < 30$ are not symmetric in this way), but in many examples the pattern holds true.

8. Thanks

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List of Abbreviations and Symbols

$\begin{bmatrix} n \\ k \end{bmatrix}_q$: A q-binomial coefficient. 1

$f_{k,R}(n)$: Equal to $f_{k,R}(n) = \#\left\{\alpha : [q^\alpha] \begin{bmatrix} n \\ k \end{bmatrix}_q \equiv R \pmod{N}\right\}$. 2

$F_{k,R}(x)$: Generating function of $f_{k,R}$. 2

$p_{\leq k}(n)$: Number of partitions with at most k parts of n. 2

$\pi_N(k)$: Minimal period of $p_{\leq k}$, equal to $|\mathcal{S}|$. 2

$\mathcal{S}$: Repeating sequence of residues of $p_{\leq k}$ mod N. 3

$\pi'_N(k)$: A quasiperiod of $f_{k,R}(n)$. 2

S_i: A section of $\begin{bmatrix} n \\ k \end{bmatrix}_q$. 6

$p^{(i)}_{\leq k}(j)$: Gives jth entry in S_i. 6

B^j_i: Periodic portion of the section S_i. 7

R_i: Incomplete period of S_i. 7

$\oplus$: Concatenation operator. 7

$\mathcal{P}^{\mathbf{bad}}_{i,m}$: Counts partitions leaving $n \times k$ box as in Lemma 4.2. 7

$\mathcal{L}^{(i)}_{k,R}$: Linear function describing $f_{k,R}(n)$ for $n \equiv i \pmod{\pi'_N(k)}$. 6

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