Classifying Graph Lie Algebras

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Abstract

A Lie algebra is a linear object which has a powerful homomorphism with a Lie group, an important object in differential geometry. In previous work a construction is given that builds a Lie algebra on a Dynkin diagram, a commonly studied structure in Lie theory. We expand this definition to construct a Lie algebra given any simple graph, and consider the problem of determining its structure. We begin by defining an alteration on a graph which preserves its underlying graph Lie algebra structure, and use it to simplify the general graph. We then provide a decomposition move which further simplifies the Lie algebra structure of the general graph. Finally, we combine these two moves to classify all graph Lie algebras.

1 Introduction

A Lie group is a structure concerned with infinitesimal transformations, introduced by Sophus Lie in 1876 [4]. It is used for analyzing continuous symmetries of mathematical objects, giving it applications in differential geometry as well as theoretical physics. However, Lie groups can be studied more simply by considering the Lie algebras they give rise to. A Lie algebra is a linear object defined by a Lie group; every Lie algebra is locally defined by the Lie group which gives rise to it, but Lie algebras often are easier to study as a large amount of work has been done on linear objects.

Lie algebra is often considered in the context of representation theory, which studies algebraic structures by considering their elements as linear transformations [3]. Representations are often useful in determining important properties of algebraic structures. Additionally, if an algebraic structure is decomposable, we can study it in terms of its decomposed parts, which allows us to only consider irreducible representations, which are representations not describable as the direct sum of two nontrivial subrepresentations. Irreducible representations are the building blocks of representation theory, and as such studying them can prove useful.

Models of representations, studied by Bernstein, Gelfand, and Gelfand in [1], are representations that contain each irreducible exactly once. Models provide a classifiable structure which contains irreducibles, which can provide information about the irreducibles themselves. Some models of representations for simply-laced simple Lie algebras using Dynkin diagrams were studied by Khovanova in [6]. Her paper also introduces a Lie algebra corresponding to a Dynkin diagram, which is a subalgebra of the graph algebra of the same diagram.

The structures suggested in [6] can be extended to any simple graph. The aforementioned paper by Khovanova [6] builds a Lie algebra based on a Dynkin diagram, a graph commonly considered in Lie theory. In this paper we extend this work further by building and classifying Lie algebra structures defined by any simple graph. We begin by defining our graph Lie algebra and some other terminology used in the paper in 2. We then consider the basic structure of the graph Lie algebra in Section 3. We then define a graph alteration which preserves the graph Lie algebra structure in Section 4, and use it to simplify the general graph in Section 5. Then we determine a decomposition move that can be performed on many graphs in Section 6. Finally, in Section 7, we use all of this information to succinctly classify graph Lie algebras.

2 Background

A *simple graph* G is a set of $n = |G|$ vertices $v_1, \ldots, v_n$ and some number of edges where edges connect pairs of distinct vertices, and no pair of vertices is connected by multiple edges. For concision, we will call this a *graph*. A previous paper by Khovanova [5] defines the *graph algebra* $\mathcal{A}(G)$ as a unital algebra over $\mathbb{C}$ with n generators $e_1, \ldots, e_n$, calleed *vertex monomials*, with the following properties:

- For all $1 \leq i \leq n$, we have $e_i^2 = -1$.

- For all $i \neq j$, we have that $e_i e_j = -e_j e_i$ if v_i and v_j are connected, and $e_i e_j = e_j e_i$ otherwise.

Since an algebra is a vector space, we have a couple properties and operations of algebras which we are considering:

- The *dimension* of an algebra is the number of monomials it contains. For example, $\mathcal{A}(G)$ has dimension 2^n, one for each element of the power set $\{e_1, \ldots, e_n\}$.
- The *center* of an algebra is the set of elements which commute with every monomial. We say that *central* monomials are monomials which are elements of the center.
- The *direct sum* of two algebras is the set of possible sums of elements from both algebras. We can use the direct sum to decompose our graph algebra into the direct sum of smaller graph algebras, making them easier to classify.

In [5], it is proven that every graph algebra is isomorphic to the graph algebra of the disjoint union of a singleton vertices and b complete 2-vertex graphs for some a, b.

Whenever we have an algebra, we can extend it to a Lie algebra, but first we want to give a definition of a Lie algebra. We follow [3].

A *Lie algebra* is a vector space L equipped with addition and a *Lie bracket*, which is a bilinear operation $L \times L \to L$, denoted $(x, y) \to [x, y]$, satisfying the following properties for all $x, y, z \in L$:

- The bracket of two of the same element is 0: $[x, x] = 0$.
- The bracket satisfies the *Jacobi Identity*:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

We can build a Lie algebra structure $\mathfrak{L}'(G)$ on $\mathcal{A}(G)$, called an *extended graph Lie algebra*, by defining a Lie bracket of two elements x and y of $\mathcal{A}(G)$:

$$[x, y] = xy - yx.$$

Definition 1. The *graph Lie algebra* $\mathfrak{L}(G)$ is the minimal Lie subalgebra of $\mathfrak{L}'(G)$ generated by the set $\{e_1, \ldots, e_n\}$.

We can build a Lie algebra structure $\mathcal{A}(G)$ by defining the Lie bracket of two vectors:

Definition 2. The Lie algebra $\mathfrak{L}'(G)$ is defined as the vector space $\mathcal{A}(G)$ equipped with addition and a Lie bracket defined as the commutator of two elements:

$$[x, y] = xy - yx.$$

Then the *graph Lie algebra* $\mathfrak{L}(G)$ is the Lie subalgebra of $\mathfrak{L}'(G)$ generated by the set $\{e_1, \ldots, e_n\}$.

The graph Lie algebra is the object of our study, and our goal is to classify them and determine their structures. To do so, we define sets of alterations that can be performed on graphs resulting in graphs with an isomorphic graph Lie algebra. This allows us to more precisely classify graph Lie algebras. We also explicitly describe the graph Lie algebras of some famous graphs.

3 Structure of the Graph Algebra

Consider a set α of integers, each between 1 and n, inclusive. Then we define e_α as the product of the vertex monomials e_i such that α contains i. This allows us to write every monomial as e_α for some set α. Having this notation, we can now determine the result of multiplying two monomials e_α and e_β. We must first define the *symmetric difference* of two sets α and β as

$$\alpha \ominus \beta = (\alpha \cup \beta) \setminus (\alpha \cap \beta).$$

Using this definition, the product of any pair of monomials can be accurately described, as was mentioned in [5]:

Lemma 1. *For all monomials e_α, e_β, the product $e_\alpha e_\beta$ is equal to $\pm e_{\alpha \ominus \beta}$.*

Now it is proved in [5] that a monomial e_α is central if and only if for all vertices v_i, there are an odd number of integers j in α such that v_i is connected to v_j. This is the simplest way to identify central monomials, which is instrumental in decomposing algebraic structures determined by graphs.

3.1 Dynkin Diagrams

The work in [6] considers graph Lie algebras of Dynkin diagrams, and describes them in terms of more classically studied Lie algebras. This allows us to know the structure of any graph Lie algebra which can be expressed in terms of Dynkin diagrams. Letting θ denote the Cartan involution, we have the following theorem from [6]:

Theorem 2. *If G is the Dynkin diagram of a simply-laced Lie algebra $\mathfrak{g}$, then $\mathfrak{L}(G)$ is isomorphic to $\theta(\mathfrak{g})$.*

This gives us that $\mathfrak{L}(A_n) = \mathfrak{so}_{n+1}$, and $\mathfrak{L}(D_n) = \mathfrak{so}_n \times \mathfrak{so}_n$, where $\mathfrak{so}_n$ is the Lie algebra corresponding to the special orthogonal group of matrices of size n. Table 1 gives the dimensions of all graph Lie algebras of Dynkin diagrams.

graph G	A_n	D_n	E_6	E_7	E_8
dimension of $\mathfrak{L}(G)$	$(n^2+n)/2$	$n(n-1)$	36	63	120

Table 1: The dimensions of graph Lie algebras of Dynkin diagrams.

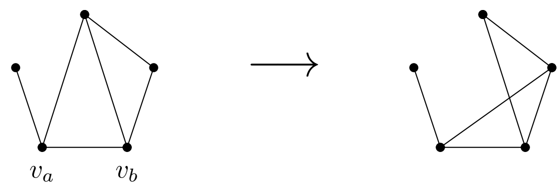


Figure 1: An example of swapping a graph. Here we swap about v_a with respect to v_b.

4 Manipulating Graphs

In this section we define a graph alteration that preserves a graph's underlying Lie algebra structure, which can in turn be used to greatly simplify our graph Lie algebra structure.

We first say that we *flip* one vertex with another by changing whether they are connected by either adding or removing an edge between them. We are now free to define a graph alteration with the intent of proving it preserves the Lie algebra structure of a graph:

Definition 3. We say G is *swapped* about a vertex v_a with respect to a vertex v_b connected to it by flipping v_a with every vertex (not including v_a) which is connected to v_b, as shown in Figure 1. The resulting graph is denoted by ${}_aG_b$.

Now for the idea of a swap to bn isomorphism on $\mathcal{A}(G)$ from $\mathcal{A}(G)$ to $\mathcal{A}({}_aG_b)$.

Let the monomials of $\mathcal{A}({}_aG_b)$ be denoted by s_α for subsets α of $\{1, \ldots, n\}$. We define the *swap isomorphism* ϕ on the monomials of $\mathcal{A}(G)$ as follows:

- $\phi(e_\alpha) = s_\alpha$, where $a \notin \alpha$,
- $\phi(e_\alpha e_a e_\beta) = s_\alpha s_a s_b s_\beta$, where $a \notin \alpha \cup \beta$.

Now to prove that ϕ is indeed an isomorphism, we begin by considering the graph algebra of G:

Theorem 3. *For all pairs of connected vertices e_a, e_b in G, the function ϕ is an isomorphism from $\mathcal{A}(G)$ to $\mathcal{A}({}_aG_b)$.*

Proof. We wish to prove that ϕ is an isomorphism from $\mathcal{A}(G)$ to $\mathcal{A}({}_aG_b)$. Clearly $\mathcal{A}(G)$ and $\mathcal{A}({}_aG_b)$ have the same dimension, so we can prove that it is bijective simply by proving injectivity. To do so, we note every monomial s_α is mapped to only by e_α when α does not contain a, and only by exactly one scalar multiple of $e_\alpha e_b$ when α contains a. Now it suffices to prove that our isomorphism holds, or that for all $\alpha, \beta \in \{1, \ldots, n\}$,

$$\phi(e_\alpha \cdot e_\beta) = \phi(e_\alpha) \cdot \phi(e_\beta).$$

To prove this, we simply need to consider whether or not α and β contain a:

Case 1. Say that neither α nor β contain a. Clearly the subalgebras $\mathcal{A}(G \setminus e_a)$ and $\mathcal{A}({}_aG_b \setminus e_a)$ are isomorphic, so in this case our isomorphism must hold.

Case 2. Say that α contains a but β does not. Then we can represent e_α as $e_{\alpha_1}e_ae_{\alpha_2}$. This gives us the following:

$$\phi(e_\alpha \cdot e_\beta) = s_{\alpha_1}s_as_bs_{\alpha_2}s_\beta = \phi(e_\alpha) \cdot \phi(e_\beta).$$

By the same logic, our isomorphism also holds when β contains a but α does not.

Case 3. Say that both α and β contain a. We can represent e_α as $e_{\alpha_1}e_ae_{\alpha_2}$ and e_β as $e_{\beta_1}e_ae_{\beta_2}$. We also define $k \in \{1, -1\}$ such that

$$e_ae_{\alpha_2}e_{\beta_1}e_a = -ke_{\alpha_2}e_{\beta_1}.$$

This gives us the following:

$$\begin{aligned}
\phi(e_\alpha \cdot e_\beta) &= \phi(e_{\alpha_1}e_ae_{\alpha_2}e_{\beta_1}e_ae_{\beta_2}) \\
&= -ks_{\alpha_1}s_{\alpha_2}s_{\beta_1}s_{\beta_2} \\
&= ks_{\alpha_1}s_{\alpha_2}s_as_bs_as_bs_{\beta_1}s_{\beta_2} \\
&= s_{\alpha_1}s_as_bs_{\alpha_2}s_{\beta_1}s_as_bs_{\beta_2} \\
&= \phi(e_\alpha) \cdot \phi(e_\beta).
\end{aligned}$$

Therefore, our isomorphism holds for all pairs of monomials, and by extension holds in general, proving the claim. □

Now in order to realize the benefit of swaps, we must also prove that they preserve the Lie algebra structure of a graph:

Lemma 4. *For all pairs of connected vertices v_a, v_b in a graph G, the graph Lie algebras $\mathfrak{L}(G)$ and $\mathfrak{L}({}_aG_b)$ are isomorphic.*

Proof. It suffices to prove that ϕ is an isomorphism between the two. Since we have defined the Lie bracket by only using operations in $\mathcal{A}(G)$, by Theorem 3 we have that $\phi : \mathfrak{L}'(G) \to \mathfrak{L}'({}_aG_b)$ is an isomorphism. Therefore it suffices to prove that ϕ is bijective, or equivalently that both of the following statements are true:

- For all vertices v_c in G, $\phi(e_c)$ is in $\mathfrak{L}({}_aG_b)$,
- For all vertices v_c in ${}_aG_b$, there exists some $x \in \mathfrak{L}(G)$ such that $s_c = \phi(x)$.

The first condition is trivial for all $c \neq a$, as $\phi(e_c) = s_c$. When $a = c$, we have

$$2\phi(e_a) = 2s_a s_b = [s_a, s_b] \in \mathfrak{L}({}_aG_b).$$

The second condition is trivial for all $c \neq a$, as $s_c = \phi(e_c)$. When $c = a$, we have $\phi([e_b, e_a]/2) = s_a$, completing the proof. □

This theorem allows us to perform swaps on graphs while maintaining their underlying graph Lie algebra structure, which is useful for classifying graph Lie algebras.

5 Simplifying Graphs using Swaps

To more easily perform a larger number of swaps, we introduce the *lit-only sigma game*, as seen in [2]. A *lighting* is a graph, with some vertices being in a *lit* state and the rest being in an *unlit* state. In this game, we are allowed to *toggle* any lit vertex v, which changes the state of every vertex connected to v, not including v itself. We say that two lightings are *equivalent* if there is a sequence of toggles which changes one of the lightings into the other.

Given a set S of vertices in G, we wish to define a lighting on which we perform toggles which correspond to sequences of swaps on G. If we remove the vertices of S and the edges containing them from G, the resulting graph is the disjoint union of some number of connected components. We call one of these components *homogeneous* if every vertex in the component is either connected to every vertex in S or no vertex in S. Consider the subgraph which is the disjoint union of all homogeneous components. The compact lighting of S, denoted $G(S)$, is this subgraph, where a vertex is lit if in G it is connected to each vertex in S. If S contains a single vertex v, we can also write this as $G(v)$. An example of a compact lighting is seen in Figure 2.

Now say we have some lighting χ, which is equivalent to $G(S)$. Then the *endgame of* G *by* χ, denoted $G(S, \chi)$, is the result when edges in G connecting elements of S to lit vertices in $G(S)$ are removed, and edges connecting elements of S to lit vertices in χ are added. We have an important lemma concerning the relationship between this game and the process of performing swaps on graphs:

Lemma 5. *For all graphs G, there is a sequence of swaps resulting in any given endgame of G.*

Proof. It suffices to prove that for all sets S of vertices in G, there is a sequence of swaps from G to the endgame of G by χ, where toggling a single vertex in $G(S)$, say v, results in χ. To perform the sequence of swaps described in the claim, we simply swap about each vertex in S with respect to v. □

Now define an *extended star* as the disjoint union of at least 3 path graphs, along with one vertex O and a set of edges connecting O to exactly one end vertex in each path graph. We also define a *central vertex* in a tree as a vertex with degree at least 3, and for any central vertex v in a tree G, every path graph which

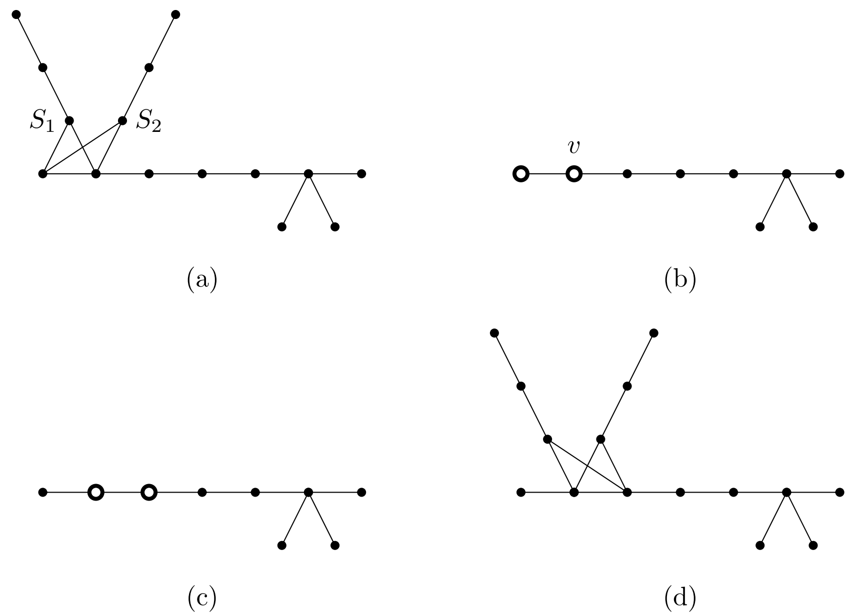


Figure 2: Note first that (b) is the compact lighting of the set $S = \{S_1, S_2\}$ in (a). Since the subgraph $G \setminus S$ contains only one homogeneous component, (b) contains only one connected component. We can then toggle v in (b), resulting in (c), proving that (b) and (c) are equivalent. Then (d) is the endgame of (a) by (c), so by Lemma 5 there must be a sequence of swaps on (a) resulting in (d).

is a connected component of $G \setminus v$ is called a *leg* of v. We define an *arm* more generally as a connected component of $G \setminus v$ which is a path graph for any vertex v in a tree G. For concision, we always let O denote the central vertex of an extended star, and let L_1, L_2, ... denote the vertices of a leg L in increasing order of distance from O, where L_1 is connected to O. In this section, we play with toggles on a few graphs, and prove that there is a sequence of swaps on any connected graph resulting in an extended star.

In order to succinctly prove that sequences of swaps exist between graphs, especially extended stars, we can find endgames of graphs, which regularly involves toggling sequence consecutive vertices. Therefore when we toggle the sequence of connected vertices L_x, ..., L_y, we write $L_{x \to y}$ for the sake of concision.

We wish to also introduce some terms concerning the properties of lightings of extended stars. Let χ be such a lighting. We say χ is *normalized* when O is lit and each leg contains at most one lit vertex. We also say that the *height* of a leg in a normalized lighting is the shortest distance from the lit vertex of L to O, where the distance between two vertices is the number of edges in the shortest path from one to the other. If no vertex in L is lit, we assign it height 0. In order to make this definition more useful, we prove a lemma which makes this lighting more prevalent:

Lemma 6. *All lightings of extended stars with at least one lit vertex are equivalent to some normalized lighting.*

Proof. To prove this, we define a sequence of toggles resulting in a normalized lighting. First, if O is unlit, we toggle the closest lit vertex to O until O is lit. Then consider some leg L with multiple lit vertices, assuming one exists. We toggle the second closest lit vertex to O in L until exactly 1 vertex in L is lit. Performing this algorithm on each leg with multiple lit vertices results in a normalized lighting. □

Define a *shortened star* as an extended star where each leg has length at most 4. We can use swaps to simplify our study of extended stars to that of shortened stars in many cases:

Lemma 7. *Say there are at least two vertices at distance 2 from some central vertex O in a tree G. Then if there is a leg L of O with length $l > 4$, there is a sequence of swaps which removes it and replaces it with two legs of length 4 and $l - 4$.*

Proof. By Lemma 5, it suffices to prove that $G(L_5)$ is equivalent to a lighting of the same graph in which the only lit vertices are O and L_6 (assuming it exists). From this we only need to consider the connected component of $G(L_5)$ containing O, since in the other component it is already true that only L_6 is lit. From the claim, there must some path O, M_1, M_2 where M_1 is not in L. There must also be some vertex N connected to O which is neither L_1 nor M_1, since O is central. To prove the stated equivalence we perform toggles on $G(L_5)$ according to the following sequence of vertices:

$$L_{4\to 1}, O, M_1, N, O, L_1, M_{2\to 1}, O, L_{2\to 1}, L_{3\to 2}, L_{4\to 3}, N, O, L_{1\to 2}, M_{1\to 2}, O, L_1, M_1, O.$$

Note that since O is toggled an even number of times, no unnamed vertex connected to O remains lit after the given sequence of toggles. □

We may now deal primarily with shortened stars, so we define a class of lightings such that the endgames by these lightings are easier to swap into extended stars. We call a normalized lighting *semi-reduced* if at most one vertex has height greater than 1, and *reduced* if it is semi-reduced and every leg has height at most 2. We can prove that lightings of shortened stars are frequently equivalent to reduced or semi-reduced lightings:

Lemma 8. *For all lightings χ of a shortened star G, there is some semi-reduced lighting equivalent to χ.*

Proof. Begin by normalizing χ. Then, if two legs L, M have heights greater than 2, we can reduce their heights by 1 by toggling our lighting according to the sequence O, $L_{1\to l-1}$, $M_{1\to m-1}$. From this we have that the number of legs with height greater than 1 is a decreasing monovariant so long as there are at least two, proving the claim. □

While it is a helpful fact in itself that all lightings of shortened stars are equivalent to semi-reduced lightings, in many cases we can prove something a bit stronger:

Lemma 9. *If there is a leg with length less than 4 in a shortened star G, then any lighting on G is equivalent to some reduced lighting.*

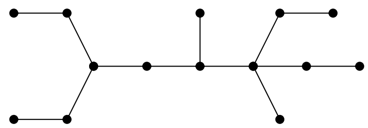
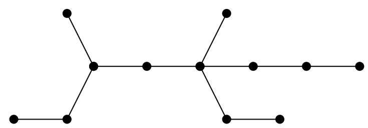
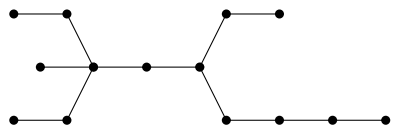
Name	Example	Description
Type-1		A tree with three central vertices, at least two of which are connected, such that the central vertex on the path between the other two has degree 3 and a leg of length 1.
Type-2		A tree with exactly two central vertices.
Type-3		A Type-2 intermediate graph where at least one central vertex has degree 3 and legs of lengths 2 and 4.

Figure 3: Descriptions of the intermediate graphs used in our sequences of swaps.

Proof. By Lemma 9, our lighting is equivalent to some semi-reduced lighting χ. We only need to consider the case where there is a leg L with length $l > 2$ and height $h > 2$. If $h = l$, we toggle according to the sequence $L_{h\to 2}$, O. Otherwise, L must have height 3 and length 4, and there is some other leg M with length less than 4. Now if M has height 0, toggle according to the sequence O, $L_{1\to 2}$, $M_{1\to m}$, O, L_1, $M_{1\to m-1}$. Otherwise, it has height 1, in which case we toggle according to the sequence $L_{3\to 1}$, M_1, O, $L_{4\to 1}$, $M_{2\to 1}$, O. In this case, if $m = 3$, we then also toggle according to the sequence $M_{3\to 1}$, O, resulting in a reduced lighting. □

Now endgames of graphs by reduced or semi-reduced lightings often have common characteristics, which allows us to more easily study them. We define a few types of *intermediate graphs* which we encounter throughout our manipulations in Figure 3. We also define the *extensivity* of an intermediate graph to be the furthest distance between any two central vertices.

Now the goal of this section is to prove that there exists a sequence of swaps on any connected graph resulting in an extended star. By attempting to prove this claim using induction, we encounter $(n+1)$-vertex graphs with n-vertex subgraphs which are shortened stars. Therefore it makes sense to consider swaps on such graphs:

Lemma 10. *For all connected $(n+1)$-vertex graphs G with an induced n-vertex subgraph which is a shortened star, there is a sequence of swaps on G resulting in either a Type-1 intermediate graph or a graph with at most 2 central vertices.*

Proof. Let v be the vertex which when removed turns the graph to a shortened star. Now consider the lighting $G(v)$. We now have two cases regarding this lighting:

Case 1. Say $G(v)$ is equivalent to some reduced lighting χ. Say that there is a leg L with length $l > 1$ and height 2. Then toggle the vertex O and take the endgame of this lighting. Then swap about O with respect to v, then swap about v with respect to the vertices in the sequence $L_{2\to l}$, resulting in a Type-1 intermediate graph. If χ does not contain a leg of height 2, simply take $G(v, \chi)$ and swap about O with respect to v.

Case 2. Now say $G(v)$ is not equivalent to a reduced lighting. Then by Lemma 9, every leg has length 4, and by Lemma 8, we have that $G(v)$ is equivalent to some semi-reduced lighting χ which is not reduced. Now say there is a leg L of height 4 in χ. Then toggling according to the sequence $L_{4\to 2}$, O results in a reduced lighting, contradiction. Therefore there is a leg L of height 3, and every other leg has height at most 1. Now say there is a leg M with height 0. Toggling our lighting according to the sequence O, $L_{1\to 2}$, $M_{1\to 4}$, O, L_1, $M_{1\to 3}$ gives L height 1 and M height 3. This makes the number of legs with height 1 an increasing monovariant so long as there exist legs of height 3 and 0. Therefore there is a sequence of toggles resulting in a graph where one leg N has height 3 and the rest have height 1. Toggling this according to the sequence O, $N_{1\to 2}$ gives us a lighting whose endgame satisfies the conditions in the claim.

□

We now improve Lemma 10 by considering more graphs than the ones it originally describes:

Lemma 11. *For all connected $(n+1)$-vertex graphs with n-vertex subgraphs which are extended stars or path graphs, there exists a sequence of connected swaps resulting in a Type-1 intermediate graph or a graph with at most 2 central vertices.*

Proof. Say that v is a vertex such that $G \setminus v$ is either an extended star or a path graph. To prove the claim, we simply need to consider possible structures of this subgraph:

Case 1. Say $G \setminus v$ is a path graph. Then continue to toggle the second rightmost lit vertex in $G(v)$ until exactly one vertex is lit, at which point its endgame will have exactly one central vertex.

Case 2. Say $G \setminus v$ is an extended star with multiple legs of length at least 2. By Lemma 7, there is a sequence of swaps on G such that $G \setminus v$ is a shortened star. By Lemma 10, this is sufficient to prove the claim.

Case 3. Say that $G \setminus v$ is an extended star with at most one leg with length greater than 2. Normalize the lighting $G(v)$. If every leg has height at most 1, then take its endgame and swap about O with respect to v. Otherwise, there is some leg L with height h greater than 1. If every leg other than L has the same height k, then toggle O if $k = 1$, then toggle according to the sequence $L_{h\to 1}$. Then continue to toggle the second closest lit vertex to O until 1 vertex is lit, when the endgame of our lighting has exactly two central vertices. Otherwise, there is always a leg M other than L with height 1. Then performing the sequence of toggles O, $L_{1\to h-1}$, M_1 reduces the height of L, making it a decreasing monovariant so long as it is greater than 1. Once L has height 1, simply take the endgame of our lighting and swap about O with respect to v.

□

We now have another lemma which can help simplify our graph in more general cases:

Lemma 12. *Say that for some tree G there is a pair v_a, v_b of vertices with arms L, M of length k for some odd integer k. The graph $G \setminus v_a$ consists of some number of connected components. Consider one of these components other than L which does not contain v_b, containing a vertex v_c connected to v_a in G. There exists a sequence of swaps removing the edge from v_a to v_c and adding an edge from v_b to v_c.*

Proof. Consider the path $P = P_1, P_2, \ldots, P_x$ from the degree 1 vertex of L to the degree 1 vertex of M. Consider the graph $G(v_c)$. We toggle according to the sequence $P_{k+1\to x}, P_{k\to x-1}, \ldots, P_{1\to x-k}$. The endgame of the resulting lighting is the graph described in the claim. □

Now that we have simplified our graphs into these two categories, we consider another transformation that can help to simplify them further. We must first consider again lightings on shortened stars:

Lemma 13. *Any normalized lighting χ of a tree such that two legs L, M of some vertex O have height 1 and length k for some even integer k is equivalent to χ with vertices L_1, M_1 unlit.*

Proof. If $k = 2$, toggle the lighting according to the sequence $A_{1\to 2}, B_{1\to 2}, O, A_1, B_1, O$. Otherwise $k = 4$, and we toggle according to the following sequence:

$$A_{1\to 4}, B_{1\to 4}, O, A_{1\to 3}, B_{1\to 3}, O, A_{1\to 2}, B_{1\to 2}, O, A_1, B_1, O.$$

Since every leg of even length in a shortened star has length 2 or 4, these two cases are sufficient to prove the claim. □

Using this alteration on lightings, we can prove that there is a sequence of swaps on a tree which performs a useful transformation:

Lemma 14. *If in some tree G there is a pair of legs L, M of some central vertex O_1 which both have length $k \in \{2, 4\}$, there is a sequence of swaps removing the edges from L_1 to O_1 and M_1 to O_1 and adding edges from L_1 to O_2 and M_1 to O_2 for any central vertex $O_2 \neq O_1$.*

Proof. It is sufficient to consider the case when the path from O_1 to O_2 contains no other central vertices. Call this path $P = P_1, P_2, \ldots, P_x$. First swap about $P_1 = O_1$ with respect to the vertices in $P_{2\to x}$. Then consider the lighting $G(P_x)$. By Lemma 13, since all sequences of toggles are invertible, $G(P_x)$ is equivalent to the resulting lighting when L_1 and M_1 are changed to unlit in $G(P_x)$, which we denote χ. Then there must be a sequence of swaps resulting in $G(P_x, \chi)$. Then swap about P_1 with respect to the vertices in the sequence $P_{x\to 2}$. □

Now we wish to consider only trees with at most two central vertices, so we must prove that there is a sequence of swaps on Type-1 intermediate graphs resulting in these:

Lemma 15. *There is a sequence of swaps from any Type-1 intermediate graph G resulting in a graph with at most 2 central vertices.*

Proof. Call our 3 central vertices v_a, v_b, v_c, where v_c is on the path from v_a to v_b, and v_a is connected to v_c. By Lemma 7 we can perform a sequence of swaps resulting in a Type-1 intermediate graph where no leg has length greater than 4. Now if both v_a and v_b have legs of length 3, by Lemma 12 we can move every other leg in v_a to v_b, completing the proof. Additionally, if one of v_a, v_b has a leg of length 1 we can move each of its legs to v_c. Otherwise, there exists an external center v with no legs of even length. By Lemma 14, we can move all but at most one length 2 leg and one length 4 leg from v to the other external center. Then either we are done or v has degree 3. Now let S be the set of vertices at a distance 2 from v which are not in legs of v. Consider the lighting $G(S)$. Let L and M be the legs of length 2 and 4, respectively, and let v_d be the vertex connected to v not in a leg of v. We toggle according to the following sequence:

$$v_d,\ L_{1\to 2},\ M_{1\to 2},\ v_d,\ L_1,\ M_1,\ M_3,\ v,\ M_2,\ M_4,\ M_1,\ M_3,\ M_2.$$

Now if G has an extensivity of 2, or $v = v_a$, our endgame has 3 central vertices, 2 of which have legs of length 1, which by Lemma 12 is sufficient to prove the claim. Otherwise, since v and v_c have legs of length 1 in our endgame, by Lemma 12 the leg L can be removed from v and connected to v_c. Now the vertex $v_d = M_2$ has exactly two legs N, P of lengths 2 and 3. Now let v_e be the vertex connected to v_d in neither N nor P, and let the set T consist of the vertices connected to v_e other than v_d. We take $G(T)$ and perform the following sequence of toggles:

$$v_e,\ v_d,\ N_{1\to 2},\ P_{1\to 2},\ v_d,\ N_1,\ v_e,\ v_d,\ P_2,\ P_1.$$

This gives us one of two graphs, depending on the extensivity of the original graph G:

Case 1. If G has an extensivity of 3, we arrive at a graph with 3 central vertices, at least 2 of which have legs of length 1. Then by Lemma 12 we are done.

Case 2. If G has an extensivity greater than 3, we arrive at a graph with 4 central vertices:

- The vertex v_a,
- The vertex v_c, which has degree 4 and a leg with length 1,
- The vertex v_b, which has degree 3, a leg of length 2, and is connected to v_d,
- The vertex v_d, which has degree 3, with a leg of length 1 and a leg of length 2.

By Lemma 12, we can remove the leg of length 2 from v_d and add it to v_c through some sequence of swaps. In the resulting graph, v_b has degree 3, and two legs of length 2. By Lemma 14, we can remove these legs and add them to v_c, resulting in a graph with 2 central vertices, v_a and v_c.

□

It now suffices to consider trees with at most 2 central vertices:

Lemma 16. *For all Type-2 intermediate graphs G, there is a sequence of swaps resulting in an extended star or a Type-3 intermediate graph.*

Proof. First, by Lemma 7 we can remove legs with length greater than 4 and replace them with legs of length at most 4. We then have a couple of cases concerning the legs of odd length in G:

Case 1. Say both central vertices have a leg of the same odd length. Then by Lemma 12 there is a sequence of swaps resulting in an extended star.

Case 2. Say that there is a central vertex with no legs of odd length. Then by Lemma 14 we can move all but at most one leg of length 2 and one leg of length 4 from this central vertex to the other, giving us a graph satisfying the conditions in the claim.

Case 3. Say that one central vertex, which we call v_a, has some number of legs of length 3 but no legs of length 1, and the other, say v_b, has some number of legs with length 1 but none of length 3. By Lemma 12 we can move every leg except one leg of length 1 from v_b to a vertex v_c connected to a degree 1 vertex of a length 3 leg connected to v_a. In the resulting graph there is a distance of two between v_a and v_c, the two resulting central vertices. Now by Lemma 7 we can remove all legs of length greater than 4 from v_a and replace them with legs of length at most 4. If this results in v_a having a leg of length 1, by Lemma 12 we are done. Otherwise, say there is a leg L of length 3 connected to v_a. Consider the set S of vertices connected to v_c not in the path from v_c to v_a. Letting v_d be the vertex connected to both v_a and v_c, and M be some vertex other than L_1 and v_d connected to v_a, which must exist since v_a is central, we perform the following sequence of toggles on $G(S)$:

$$v_c,\, v_d,\, v_a,\, M,\, L_1,\, v_a,\, v_d,\, v_c,\, L_{2\to 1},\, v_a,\, M,\, v_d,\, v_a,\, L_{1\to 2}.$$

We then take the endgame of this lighting. The result is equivalent to removing a leg of length 3 from v_a, and adding a leg of length 2 to v_a and adding a leg of length 1 to v_c. This makes the number of legs with odd length in v_a a decreasing monovariant. Therefore there is a sequence of swaps resulting in a graph where one central vertex has no legs of odd length, which by Lemma 14 is sufficient to prove the claim.

□

We now simply need to consider swaps on Type-3 intermediate graphs, the final thread in our tapestry of swaps:

Lemma 17. *For all Type-3 intermediate graphs G, there is a sequence of swaps resulting in an extended star.*

Proof. First by Lemma 7 we only need to consider Type-3 intermediate graphs where every leg has length at most 4. Say that v_a has degree 3 and legs L and M with lengths 2 and 4, and let v_b be our other central vertex. We have a different set of cases based on the extensivity of G and the legs of its central vertices:

Case 1. Say the extensivity of G is $x > 1$. Let S be the set of vertices connected to v_b not on the path $P = P_1, \ldots, P_k$ from v_b to v_a, and perform the sequence of toggles on $G(S)$:

$$P_{1\to k},\ L_{1\to 2},\ M_{1\to 2},\ v_a,\ L_1,\ M_1,\ M_3,\ v_a,\ M_2,\ M_4,\ M_1,\ M_1,\ M_3.$$

The endgame H of this graph has an extensivity of two and 2 central vertices, including v_a, which has degree 3 and two legs N and Q with lengths 2 and $q > 1$, and v_c. Now let T be the set of vertices connected to v_c not on the path $R = R_1, R_2, R_3$ from v_c to v_a, and perform the following sequence of swaps on $H(T)$:

$$R_{1\to 3},\ N_{1\to 2},\ Q_{1\to 2},\ v_a,\ N_2,\ Q_2,\ R_{2\to 1},\ Q_{2\to 1},\ v_a,\ N_1,\ R_2,\ v_a,\ Q_{1\to 2}.$$

Then the endgame of this lighting is an extended star.

Case 2. Say G has extensivity 1 and some leg of odd length. Let S be the set of vertices connected to v_b other than v_a. We perform the following sequence of toggles on $G(S)$:

$$v_b,\ v_a,\ L_{1\to 2},\ M_{1\to 2},\ v_a,\ L_1,\ M_1,\ M_3,\ v_a,\ M_2,\ M_4,\ M_1,\ M_3,\ M_2.$$

The endgame H of the resulting lighting is a tree with extensivity 2 and two central vertices v_c and v_d, where v_c has degree 3 and legs of length 1 and 2. Now say that v_d has a leg N of length 3. Since it is central it must also have some other leg P. Let T be the set of vertices connected to v_c not on the path from v_c to v_d, and let v_e be the common vertex between v_c and v_d. Perform the following sequence of toggles on $H(T)$:

$$v_c,\ v_e,\ v_d,\ N_1,\ P_1,\ v_d,\ v_e,\ v_c,\ N_{2\to 1},\ v_d,\ v_e,\ P_1,\ v_d,\ N_{1\to 2}.$$

The endgame of this lighting is the graph H with one leg of length 3 removed from v_c and replaced with a leg of length 2, and one leg of length 1 added to v_d. This makes the number of legs of length 3 a decreasing monovariant while maintaining an extensivity of 2 and the existence of a leg of length 1 on v_d. Therefore there is a sequence of swaps on G resulting in a tree with 2 central vertices and extensivity 2 which has no legs of length 3, and a leg of length 1 on at least 1 central vertex. If there exists a leg of length 1 on both central vertices, by Lemma 12 we are done. Otherwise one central vertex v_c has no legs of odd length, and by Lemma 14 we can move at most one leg of length 2 and one leg of length 4 from v_c to v_d, giving us either an extended star or a Type-3 intermediate graph with extensivity greater than 1, which has already been considered.

Case 3. Say G has extensivity 1 and no leg of odd length. By Lemma 14 either both central vertices have an odd number of length 2 and length 4 legs or there is a sequence of swaps resulting in an extended star. Let S be the set of vertices connected to v_b other than v_a. We perform the following sequence of toggles on $G(S)$:

$$v_b,\ v_a,\ L_{1\to 2},\ M_{1\to 2},\ v_a,\ L_1,\ M_1,\ M_3,\ v_a,\ M_2,\ M_4,\ M_1,\ M_3,\ M_2.$$

The endgame of the resulting lighting has two central vertices, one of which has an even number of legs of even length and an odd number of legs of odd length, which by Lemma 14 is sufficient to prove the claim.

□

Theorem 18. *For all connected graphs, there exists a sequence of swaps resulting in either a path graph, a shortened star, or an extended star where 1 leg has length greater than 4 and the rest have length 1.*

Proof. We recursively perform swaps. Pick a 1-vertex subset of our graph. This is one of the two graphs described. Now say we have an n-vertex subgraph which is one of the graphs described in the claim. Pick a vertex connected to at least 1 of these vertices. Then by Lemmas 11, 15, 16, 17 there is a sequence of swaps resulting in a tree with at most 1 central vertex. Then by Lemma 7 we are done. □

6 Decomposing Graph Lie Algebras

In this section, we once again consider the mathematical structure of our graph Lie algebras, and prove that in many cases they can be decomposed nicely into other graph Lie algebras.

6.1 Decomposing Graph Algebras

In order to decompose a graph algebra, we must first gather more information about its center. Consider a monomial e_α that is central in $\mathcal{A}(G)$. We then define a value f_α as a scalar multiple of e_α such that $f_\alpha^2 = 1$:

$$\begin{cases} f_\alpha = ie_\alpha & \text{if } e_\alpha^2 = -1, \\ f_\alpha = e_\alpha & \text{otherwise.} \end{cases}$$

We can then define a central idempotent $c_\alpha = (1 + f_\alpha)/2$. This idempotent allows us to write our graph algebra as the direct sum of two other algebras:

$$\mathcal{A}(G) = c_\alpha \cdot \mathcal{A}(G) \oplus (1 - c_\alpha) \cdot \mathcal{A}(G)$$

In fact, a theorem from [5] makes this decomposition even more useful, by expressing each addend as its own graph Lie algebra:

Theorem 19. *For all central monomials e_α in $\mathcal{A}(G)$ and integers $i \in \alpha$, the algebras $c_\alpha \cdot \mathcal{A}(G)$ and $(1 - c_\alpha) \cdot \mathcal{A}(G)$ are isomorphic to $\mathcal{A}(G \setminus v_i)$.*

Now proving a similar fact for graph Lie algebras requires us also to consider which monomials are contained in it, whereas a graph algebra of an n-vertex graph always contains 2^n monomials, each the product of the elements of some subset of the power set of the set of vertex monomials. We begin this consideration by constructing new monomials one by one:

Definition 4. The set $C = \{C_0, \dots, C_d\}$ of *construction sets* of G, where $d = \dim \mathfrak{L}(G) - n$, is defined as follows:

- The set C_0 is defined as the set of vertex monomials of $\mathfrak{L}(G)$.

- The set C_k is defined as $C_{k-1} \cup e_\alpha$ for some monomial e_α in $\mathfrak{L}(G) \setminus C_{k-1}$ generated by taking the Lie bracket of two elements of C_{k-1} and multiplying by $1/2$.

Now every element in each of these sets must be in $\mathfrak{L}(G)$, by the nature of their construction. Therefore, we have that C_d forms a basis of $\mathfrak{L}(G)$, since $|C_d| = \dim \mathfrak{L}(G)$. Having defined these construction sets we can now better describe the monomials in $\mathfrak{L}(G)$:

Lemma 20. *If there exists a vertex v_i in G and a monomial e_α in $\mathfrak{L}(G)$ such that $e_\alpha \mathfrak{L}(G) \subseteq \mathfrak{L}(G)$ and $e_i e_\alpha \in \mathfrak{L}(G \setminus v_i)$, then for all monomials e_β in $\mathfrak{L}(G)$, exactly one of e_β, $e_\alpha e_\beta$ is in $\mathfrak{L}(G \setminus v_i)$.*

Proof. Define a function of monomials in $\mathfrak{L}(G)$ as follows:

$$\begin{cases} e'_\beta = e_\beta e_\alpha & \text{if } i \in \alpha, \\ e'_\beta = e_\beta & \text{otherwise.} \end{cases}$$

Now let C'_k be the resulting set when every monomial e_β in C_k is replaced with e'_β, and let C' be the set $\{C'_0, \ldots, C'_d\}$. We can now induct on the indices of elements of C'. First, every element of C'_0 is in $\mathfrak{L}(G \setminus v_i)$, since every element other than e'_i is a vertex monomial which is not e_i , and $e'_i = e_i e_\alpha$, which from the problem statement is in $\mathfrak{L}(G \setminus v_i)$. Now say that every element of C'_k is in $\mathfrak{L}(G \setminus v_i)$. The monomial $C_k \setminus C_{k-1}$ is the product of two anticommuting monomials e_β, e_γ in C_k. Since e_α is a central monomial, we have

$$[e'_\beta, e'_\gamma] = 2e'_\beta e'_\gamma = 2(e_\beta e_\gamma)',$$

completing our inductive step. Therefore every element of C'_d is in $\mathfrak{L}(G \setminus v_1)$. Now since $e_\alpha \mathfrak{L}(G)$ is a subset of $\mathfrak{L}(G)$, the set of monomials in $\mathfrak{L}(G)$ which do not contain e_i is the set C'_d. Suddenly we're done, since no monomial in $\mathfrak{L}(G \setminus v_1)$ contains e_i. □

This allows us to prove our decomposition move:

Theorem 21. *For all central monomials e_α and vertices v_i in G such that $e_\alpha \mathfrak{L}(G)$ is a subset of $\mathfrak{L}(G)$ and $e_\alpha e_i$ is in $\mathfrak{L}(G \setminus v)$, we have*

$$\mathfrak{L}(G) = \mathfrak{L}(G \setminus v_i) \oplus \mathfrak{L}(G \setminus v_i).$$

Proof. To prove this theorem we describe a decomposition, and then prove that it is equivalent to the one given. Consider the value c_α as defined in subsection 6.1. The values c_α and $1 - c_\alpha$ are central idempotents whose product is 0. Therefore our Lie algebra is the direct sum of $c_\alpha \cdot \mathfrak{L}(G)$ and $(1 - c_\alpha) \cdot \mathfrak{L}(G)$. We prove that $\mathfrak{L}(G \setminus v_i)$ is isomorphic to $c_\alpha \mathfrak{L}(G)$.

Consider the function $f : \mathfrak{L}(G \setminus v_i) \to c_\alpha \cdot \mathfrak{L}(G)$ where $f(x) = c_\alpha x$. This is the function that we prove is an isomorphism. We begin by proving f is injective. Say $c_\alpha x = c_\alpha y$ for some $x, y \in \mathfrak{L}(G \setminus v)$. Consider some monomial e_β. The only monomials which contain e_β when multiplied by c_α are e_β and $e_\alpha e_\beta$. However, by Lemma 20, exactly one of these is in $\mathfrak{L}(G \setminus v_i)$. Therefore the coefficient of e_α in $c_\alpha x$ is a constant nonzero scalar multiple of the coefficient of some monomial in x. Therefore y must have the same coefficient of this monomial. Therefore the coefficients of every monomial in $\mathfrak{L}(G \setminus v_i)$ are equal in x and y, so $x = y$.

We now wish to prove that f is surjective. Consider the value $c_\alpha x$ for some x in $\mathfrak{L}(G)$. Since from the problem statement every monomial in $\mathfrak{L}(G)$ is either a monomial in $\mathfrak{L}(G \setminus v_i)$ or an element of $\mathfrak{L}(G \setminus v)$ multiplied by f_α as defined in subsection 6.1, we can say $x = a + f_\alpha b$ where a and b are in $\mathfrak{L}(G \setminus v_i)$. However, since $c_\alpha f_\alpha = c_\alpha$, we have $c_\alpha(a + b) = c_\alpha x$, so our function is surjective, and therefore a bijection.

Finally, we wish to prove that f is an isomorphism. Since we have proven it is a bijection, it suffices to prove that it preserves the addition and Lie bracket operators. It clearly preserves addition, as $c_\alpha(x + y) = c_\alpha x + c_\alpha y$. Also, since c_α is a central idempotent, we have $c_\alpha[x, y] = [c_\alpha x, c_\alpha y]$. Therefore f is an isomorphism. By the same logic, we can prove that $g(x) = (1 - c_\alpha)x$ is an isomorphism from $\mathfrak{L}(G \setminus v_i)$ to $(1 - c_\alpha) \cdot \mathfrak{L}(G)$, completing the proof. ☐

Now before we can find a graph that we can decompose with this theorem, we must make a brief remark concerning anticommuting elements in $\mathfrak{L}(G)$:

Lemma 22. *If G is connected and has at least two vertices, then for all monomials $e_\alpha \in \mathfrak{L}(G)$ there exists a vertex v_i in G such that $[e_i, e_\alpha] \neq 0$.*

Proof. If α is a one element set, then v_i can be any vertex connected to the vertex whose vertex monomial is e_α. Otherwise, the construction set C_0 does not contain e_α, but some construction set does, implying $e_\alpha = [e_\beta, e_\gamma]/2$ for some β, γ. Therefore $[e_\beta, e_\alpha] \neq 0$. Now e_β is the product of some number of vertex monomials, each of which either commutes or anticommutes with e_α. If each of them commuted with e_α, we would have $[e_\beta, e_\alpha] = 0$, contradiction. Therefore there is some vertex monomial e_i such that $[e_i, e_\alpha] \neq 0$. ☐

We now introduce some notation concerning Lie brackets, which simplifies our description of objects obtained by taking many Lie brackets:

Definition 5. Define a *nested Lie bracket* of a nonempty set $x_1, ..., x_k$ recursively as

$$[x_1, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k],$$

where $[x_1, x_2]$ has its usual definition. When $k = 1$, define $[x_1] = x_1$.

We now wish to find a couple of extended stars which can be decomposed with this theorem:

Lemma 23. *If an extended star G has two legs L, M of length $2k-1$, where S is the set of vertices in these legs at an odd distance from O, and p is the product of the vertex monomials of the vertices in S, then*

$$p\mathfrak{L}(G) \in \mathfrak{L}(G).$$

Proof. It suffices to prove that for all monomials e_α the product pe_α is in $\mathfrak{L}(G)$. By Lemma 22, there exists some integer i such that $[e_\alpha, e_i] = 2e_\alpha e_i$. We first prove that the monomial pe_i is in $\mathfrak{L}(G)$. To do so, we need to define a sequence of vertices which will be used in the proof. Let S_x be the sequence

$$L_{1\to 2x+1}, M_{1\to 2x+1}, O, L_{2\to 2x}, M_{2\to 2x}, O.$$

Now if $k > 1$ we say that S is the sequence S_{k-1}, …, S_1, L_1, M_1, and otherwise we say it is simply L_1, M_1. Now say that $pe_i \in \mathfrak{L}(G)$. Then we can prove the claim by taking the following nested Lie bracket:

$$[e_\alpha, e_i, pe_i] = 2e_\alpha e_i pe_i - 2pe_i e_\alpha e_i = -4pe_\alpha.$$

Therefore it suffices to prove that $\mathfrak{L}(G)$ contains pe_i. We consider a few cases concerning the location of the vertex v_i:

Case 1. Say that $v_i = O$. Let N be some leg other than L and M. We take the nested Lie bracket of the sequence of vertex monomials of the sequence O, N_1, S, N_1 of vertices.

Case 2. Say v_i is the point L_j. Let N be some leg other than L and M. We take the nested Lie bracket of the sequence of vertex monomials of the following sequence of vertices:

$$O, N_1, S, N_1, L_1, O, L_{2\to 1}, L_{3\to 2}, \ldots, L_{j\to j-1}.$$

Note that this construction works equally well when v_i is on M.

Case 3. Say v_i is the vertex N_j on some leg N. We then take the nested Lie bracket of the sequence of vertex monomials of the sequence $N_{j\to 1}$, O, S, O, $N_{1\to j-1}$ of vertices.

□

We can now perform a decomposition move on many extended stars by the following theorem:

Theorem 24. *If G is an extended star with two legs L, M of length $2k-1$, then its graph Lie algebra is isomorphic to the direct sum of two copies of the graph Lie algebra of $G \setminus L_{2k-1}$.*

Proof. First define v_i as L_{2k-1}. We consider the product p of the vertex monomials of the vertices in L and M which are an odd distance from O, which is a central monomial. By Theorem 21 and Lemma 23, it suffices to prove that pe_i is in $\mathfrak{L}(G \setminus L_{2k-1})$. If $k = 1$ this is trivial, as pe_i is a vertex monomial. Otherwise, let S_x be the following sequence of vertices:

$$L_{1\to 2x+1}, M_{1\to 2x}, O, L_{2\to 2x}, M_{2\to 2x-1}, O.$$

Now say that S is the sequence S_{k-1}, ..., S_1. Now say N is some leg other than L and M. We take the nested Lie bracket of the vertex monomials of the following sequence of vertices:

$$O, N_1, S, N_1, M_1, O, M_{1\to 2k-1}.$$

This results in pe_i, proving the claim. □

Now that we have considered how to manipulate graphs using both swaps and decomposition moves, we are free to make a couple of general statements about graph Lie algebras in the next section.

7 Classifying Graph Lie Algebras and Future Work

In this section, we condense our work from previous sections and consider possible next steps. First, using our work up to this point we can derive a nice, elegant theorem which classifies graph Lie algebras:

Theorem 25. *All graph Lie algebras of connected graphs are isomorphic to the direct sum of some number graph Lie algebras of path graphs and shortened stars, each of which has at most one leg of length 1 and one leg of length 3.*

Proof. Consider the graph Lie algebra $\mathfrak{L}(G)$ of some graph G. By Theorem 18, there is a sequence of swaps on G resulting in the disjoint union H of some number of shortened stars, path graphs, and extended stars with 1 leg with length greater than 1. Since the graph Lie algebra of the disjoint union of two graphs is the direct sum of their graph Lie algebras, by Theorem 24 we are done. □

Now that we have this theorem, graph Lie algebras can be studied with much more ease, allowing for further advances in Lie theory in the future. This could transpire in one of two ways. First, it could be proven that all graph Lie algebras are isomorphic to the direct sum of Dynkin diagrams, proving that all graph Lie algebras are semisimple, making their study quite uninteresting. However, using the methods we have described in this paper, it is not possible to determine whether all graph Lie algebras are semisimple. Consider the shortened star with four legs of length two. Assuming we can take the Lie brackets of every pair of elements in any set, we can build the construction sets of any graph and in doing so determine its dimension. Using a Java program, we used this method to prove that its dimension was 255. This makes simplification to a Dynkin diagram impossible using our methods, which can only maintain the dimension of the graph being considered or divide it by two, and Table 1 shows that no Dynkin diagram has a graph Lie algebra with dimension 255. In fact, there exist many cases which exhibit this. This leaves open the possibility that some graph Lie algebras are not semisimple, and are their own unique structures. If this is the case, the results in this paper can be used to much more easily understand the set of new graphs which are formed by defining graph Lie algebras. If these new Lie algebras do exist, the quicker we can understand them the sooner we will be able to spot them in nature, as many Lie algebras have been in fields such as particle physics, and the sooner we can bring them into the real world.

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