The PRIMES 2016 Math Problem Set

Dear PRIMES applicant!

This is the PRIMES 2016 Math Problem Set. Please send us your solutions as part of your PRIMES application by December 1, 2015. For complete rules, see `http://web.mit.edu/primes/apply.shtml`

Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden! Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General math problems

Problem G1. Let N be a positive integer. A soon to be bankrupt casino lets you play the game $G(N)$. In the game $G(N)$, you roll a typical, fair, six-sided die, with faces labeled 1 through 6, up to N times consecutively. After each roll, you may either end the game and be paid the square of the most recent number you rolled, or roll the die again hoping for a better number — on the N-th roll you must take the money and cannot roll again. For example, in the game $G(2)$ you might first roll a 5, but, hoping for a 6, you roll again, only to be disappointed to roll a 1 on your second and final roll, and you walk away with \$1.

(a) Describe a strategy that maximizes the expected value of playing $G(N)$.

(b) What is this maximal expected value?

Solution. Let E_N be the expected gain in $G(N)$ under the optimal strategy. E.g., $E_1 = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}$. Now, consider $N > 1$. Suppose the result of the first roll is p. If the player decides to stop, his gain is p^2. Otherwise, he plays the game $G(N-1)$ and has expected gain E_{N-1}. Thus, if $p^2 \le E_{N-1}$, he should keep rolling, otherwise he should stop. So we get a recursion

$$E_N = \frac{[E_{N-1}^{1/2}]}{6} E_{N-1} + \frac{1}{6} \sum_{p=[E_{N-1}^{1/2}]+1}^{6} p^2.$$

In particular, we have $E_2 = \frac{245}{12}$, $E_3 = \frac{214}{9}$, $E_4 = \frac{1405}{54}$. Starting from E_4, the recursion is

$$E_N = \frac{5}{6} E_{N-1} + 6,$$

which means $E_N = 36 - a\left(\frac{5}{6}\right)^{N-4}$. Plugging in $N = 4$, we get $36 - a = \frac{1405}{54}$, which gives $a = \frac{539}{54}$.

Problem G2. (a) Let n be an even positive integer. Can one divide the numbers $1, ..., n$ into three nonempty groups, so that the sum of numbers in the first group is divisible by $n+1$, in the second one by $n+2$, and in the third one by $n+3$?

(b) For which odd positive integer n can one do this?

Solution. (a) No. Let the sum of the first group be $(n+1)a$, in the second $(n+2)b$, in the third $(n+3)c$. The total is $n(n+1)/2$, which is $2 + \frac{n}{2}$ modulo $n+2$. So we get that $c - a$ is $2 + \frac{n}{2}$ modulo $n+2$. On the other hand, we see that $a < \frac{n}{2}$ and $c < \frac{n}{2}$, hence $|c - a| < \frac{n}{2}$. But the numbers with smallest absolute value which are $2 + \frac{n}{2}$ modulo $n+2$ are $2 + \frac{n}{2}$ and $-\frac{n}{2}$, which gives a contradiction.

(b) For odd $n \geq 9$. If $n-1$ is divisible by 4, the a, b, c defined above can be taken to be $a = \frac{n-5}{4}$, $b = 1$, $c = \frac{n-1}{4}$. If $n-3$ is divisible by 4, then a, b, c can be taken to be $a = \frac{n-7}{4}$, $b = 1$, $c = \frac{n-3}{4}$. For $n \leq 7$, it is easy to see by inspection that the division into 3 groups as required is not possible.

Problem G3. Suppose you play a game whose goal is to collect three cards of the same suit. In your first move, you take three cards from a standard 52-card deck at random. Call them $C1, C2, C3$.

1. If $C1, C2, C3$ are all of the same suit, you win.
2. If $C1, C2, C3$ are all of different suits, you put them back, shuffle, and take three cards one more time. If now all are of the same suit, you win, otherwise, you lose.
3. If among $C1, C2, C3$, exactly two cards are of the same suit, you put the third card (the odd one out) back into the deck, shuffle, and pull out a card. If it is the same suit as the other two, you win, otherwise, you lose.

What is the chance of winning? (Write the answer as a fraction in lowest terms).

Solution. The probability to win at the first move is $p_1 = \frac{12 \cdot 11}{51 \cdot 50} = \frac{22}{425}$. The probability to get all three of different suits is $p_2 = \frac{39 \cdot 26}{51 \cdot 50} = \frac{169}{425}$. In this case, the probability of winning is $p_2 p_1 = \frac{3718}{180625}$. Finally, the probability of two of the same suit is $p_3 = \frac{234}{425}$, and then the probability of the third card being of the same suit is $p_4 = \frac{11}{50}$, so the probability of winning is $p_3 p_4 = \frac{1287}{10625}$. Altogether we get $p = p_1 + p_2 p_1 + p_3 p_4 = \frac{34947}{180625}$.

Problem G4.

In a couples therapy session, n couples are to be seated at a round table (in $2n$ chairs), but no person is allowed to sit next to his/her spouse. How many seat assignments are there? What is the number of seatings for 5 couples?

Solution. Pick k out of the n couples. Then the number of seatings so that these k couples sit together is computed as follows: there are $2n$ ways to seat the first couple (up to order), and then the number of seatings of the rest of the k couples equals $(2n-k-1)...(2n-2k+1)$. So the total is

$$N_k = 2^k \cdot 2n \cdot (2n-k-1)!$$

Thus, using the inclusion and exclusion formula, we get that the answer is

$$N = \sum_{k \geq 0} (-1)^k \binom{n}{k} N_k$$

For 5 couples we get

$10 \cdot (9! - 5 \cdot 2 \cdot 8! + 10 \cdot 4 \cdot 7! - 10 \cdot 8 \cdot 6! + 5 \cdot 16 \cdot 5! - 32 \cdot 4!) = 1,125,120.$

Problem G5. A zero-one matrix A is said to *contain* another zero-one matrix P if some submatrix[1] of A can be transformed to P by changing some ones to zeroes. Otherwise A is said to *avoid* P.

Consider the following pattern avoidance game, denoted by PAG(n, P): Starting with the $n \times n$ all zeroes matrix, two players take turns changing zeroes to ones. If any player's turn causes the matrix to contain the pattern P, then that player loses.

If no dimension of P exceeds n, then PAG(n, P) will always have a winner. Define $W(n, P)$ to be the winner of PAG(n, P) if both players employ optimal strategies.

(a) Determine $W(n, P)$ for every $n \geq k$ when P is a k by 1 matrix with every entry equal to 1.

(b) Determine $W(n, P)$ for every $n \geq 2$ when P is a 2 by 2 identity matrix: $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

(a) If P is a $k \times 1$ matrix with all ones, then $W(n, P) = 2$ if $n \geq k$ and $n(k-1)$ is even, but $W(n, P) = 1$ if $n \geq k$ and $n(k-1)$ is odd.

Proof: Regardless of where anyone plays their turn, no player can add the k^{th} one to any row. If there are any rows with fewer than k ones, then the current player can take a turn. Otherwise they will have to add the k^{th} one to some row. Therefore if $n \geq k$, then the first player wins if $n(k-1)$ is odd, but the second player wins if $n(k-1)$ is even.

(b) $W(n, P) = 1$

Proof: The first player should play in the top left corner. After this, both players can only play in the first column or first row. There are only $2n - 1$ entries in this column and row, so the second player will have to play somewhere that forms the forbidden matrix P.

Problem G6. Suppose that n pine trees grow at points $T_1, ..., T_n$ of the plane (no three on the same line). A cyclic order C of $T_1, ..., T_n$ (i.e., an order up to cyclic permutation) is called *visible* if there exists a point P in the plane from which an observer sees the trees $T_1, ..., T_n$ in the order C. Show that if $n \geq 7$ then there exists a cyclic order which is not visible. What about $n = 6$?

Solution. It suffices to solve the problem for $n = 7$. The orders change at lines connecting the points. There are $n(n-1)/2 = 21$ such

[1] A submatrix is obtained from a matrix by crossing out some rows and some columns.

lines, so the number of parts into which they subdivide the plane is at most $1+21\cdot 22/2=232$. At the same time, the number of cyclic orders is $6!=720$. Since $720>232$, there exists a cyclic order which is not visible.

For $n=6$ the number of cyclic orders is 120 and the number of regions is 121, so the argument needs to be imporved. To do so, note that we can erase the intervals of the lines which are between the points, since when P crosses the line T_iT_j between T_i and T_j, the order of T_i and T_j does not change. When we do so, the number of regions goes below 120, and the same argument applies.

Problem G7. A permutation s of n elements has order 2016 (i.e., the smallest number of times you need to repeat s to get to the original position is 2016). What is the smallest possible value of n? Give an example of such s for the minimal n. (Hint: consider the cycle decomposition of s).

Solution. $n=48$. We have $2016=2^5\cdot 3^2\cdot 7$. The order of a permutation is the least common multiple of orders of its cycles. So there is a cycle of order divisible by 32, a cycle of order divisible by 9, and a cycle of order divisible by 7. If these are different cycles, then $n\geq 32+9+7=48$. Otherwise, it will have to be even larger. But for $n=48$ we can take three cycles of lengths 32, 9, and 7.

Advanced math problems

Problem M1. There are n piles with coins. In one move you can pick several piles and take the same number of coins from those piles. Given a set of piles, its *piles number* is the smallest number of moves you need to remove all coins from all the piles. For example, if you have three piles with 1, 2, and 3 coins each, you can remove all the coins in three moves by treating one pile at a time. But the piles number is 2, as the smallest number of moves is 2.

Find the piles number (with proof) for the following sets of piles:

- 1, 2, 3, 10, 20, 30, 100, 200, 300.
- 1, 2, 3, 11, 12, 13, 101, 102, 103.
- 1, 3, 4, 7, 11, 18, 29, 47, 76, 123.
- Any sequence of natural numbers of length n where each term starting from the third one is the sum of two previous terms.

Solution. First observation: we can do our moves in any order.

Second observation: It is obvious that three piles in an arithmetic progression a, $2a$, and $3a$ can be removed in two moves. The first move removes a coins from the first and the last pile. The second move removes $2a$ coins from the second and the last pile. Also any three piles that are different can't be removed in one move.

Third observation. Suppose our piles are in the increasing order. If k-th pile has more coins than all the first i piles, where $k > i$, then there exists a move that includes the k-th pile, without including the first j piles. Indeed, as we can do our moves in any order, we can start with moves that included any of the first j piles. The sum of amounts removed in all of those moves is not more than the total of the coins in the first j piles. That means the k-th pile is still not empty and we need one more move.

Fourth observation. When we have two piles with the same number of coins we can treat them the same way. That means we can ignore all the piles with repeated number of coins.

1. We need two moves to process the first three piles 1, 2, 3, two more moves to process the next three piles 10, 20, 30, and two more moves to process the last three piles 100, 200, 300. Now we need to prove that we can't do better. Suppose we do the moves that include any of the first three piles first. We need at least two moves to empty them. If we sum up the number of coins removed from one pile in these moves, the total is not more than 6. That means after these moves, the next three piles can't be the same or empty. That means we need at least two more moves that included one of the next three piles. Similarly, we

need at least two more moves to finish the last three piles. The piles number is 6.

2. In the first move we remove 1 coin from piles indexed 1, 3, 4, 6, 7, 9. We are left with 0, 2, 2, 10, 12, 12, 100, 102, 102. In the next move we remove two coins from piles indexed 2, 3, 5, 6, 8, 9. We are left with 0, 0, 0, 10, 10, 10, 100, 100, 100. There are two different numbers. We can finish this in two more moves. To prove that this number is optimal, we observe that we need at least two moves that include one of the first three piles. By the third observation there exists a move that includes pile 4 and doesn't include the first three pile. That means we need one more move. Using the third observation again for pile 7, we need one more move. The piles number is 4.

4. Suppose the first two piles are the same. Then we can process the first pile the same way as the second pile. This means we can ignore the first pile, and assume that our sequence is of length $n-1$ with the first two piles that are different.

Suppose the first two piles are different. Denote the piles sequence a_i. On the first move we remove a_{n-1} cookies from the last two piles. The second to the last pile is emptied, and the last pile becomes the same as one of the other piles. Thus, we essentially discarded the two last piles. We need to look at the ending condition. If n is odd, we will have one pile at the end that requires one move. If n is even, we will have two piles with the different number of coins, so we will need two moves. Thus the answer for this strategy is $\lfloor n/2 \rfloor + 1$.

Let us prove that it is optimal. First we show by induction that $a_n - a_1 = \sum_{i=1}^{n-2} a_i$. By the third observation there must be a move that involves the n-th pile and doesn't involve any of the first $n-2$ piles. By observation 1 we can start with this move. We will not jeopardize the optimality if we use this move to get rid of the two piles with the largest numbers of coins. Continuing this reasoning, our strategy must be one of the optimal ones.

The answer is $\lfloor n/2 \rfloor + 1$ if the first two piles are different and $\lfloor (n-1)/2 \rfloor + 1$, if they are the same.

3. It follows from number 4. We need 5 moves.

Problem M2. Let $f_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$.

(a) Show that $f_n(x) > 0$ for all real x if n is even, and that f_n has a unique real root x_n for n odd.

Hint: use the relationship between f_n and its derivative.

(b) Show that all complex roots of f_n are simple (i.e., if a is a root of f_n then $f_n'(a) \neq 0$).

(c) Let $n = 2k+1$ for positive integer k, and $c = \lim_{k\to\infty} \frac{x_n}{n}$ (it is known to exist). Find c. (Represent the answer as a root of an equation, and compute it to 4th digit precision). *Hint: Use the relationship between $f_n(x)$ and the function e^x.*

Solution: (a) We have $f_n'(x) = f_n(x) - \frac{x^n}{n!}$. Thus for n even, if $f_n(x) = 0$ then $f_n'(x) < 0$. This can't happen for the largest root, so there is no roots at all. For n odd, there must be a root, and all roots are negative. So at each root, $f_n'(x) > 0$. This can't happen if there is more than one root. So there is only one root.

(b) If $f_n(a) = 0$ then $a \neq 0$ so $f_n'(a) = -a^n/n! \neq 0$.

(c) The polynomial in question is the Taylor polynomial of the exponential function. We have

$$e^{x_n} = \frac{x_n^{n+1}}{(n+1)!}(1 + \frac{x_n}{n+2} + \frac{x_n^2}{(n+2)(n+3)} + ...),$$

hence, taking logs,

$$x_n = \log \frac{x_n^{n+1}}{(n+1)!} + \log(1 + \frac{x_n}{n+2} + \frac{x_n^2}{(n+2)(n+3)} + ...).$$

Plugging in cn instead of x_n, neglecting the second summand on the right hand side, and using Stirling's formula, we find

$$cn \cong (\log(-c) + 1)n,$$

i.e. $c = 1 + \log(-c)$. Solving this equation gives an approximate value $c \simeq -0.2785$.

Problem M3. Let p be a prime.

(a) Find the number of square matrices A of size n over the field $\mathbb{F}_p$ of p elements such that $A^p = A$.

(b) Suppose that $p \geq 3$. Find the number of square matrices of size n over $\mathbb{F}_p$ such that $A^2 + 1 = 0$ (where 1 is the identity matrix and 0 is the matrix of all zeros). You may have to consider two cases for p.

Solution. (a) The matrix has eigenvalues $0, 1, 2, ..., p-1$ with eigenspaces of dimension $n_0, n_1, n_2, ..., n_{p-1}$. The group $GL_n(\mathbb{F}_p)$ acts on such arrangements transitively, with stabilizer $GL_{n_0}(\mathbb{F}_p) \times ... \times GL_{n_{p-1}}(\mathbb{F}_p)$. So the number of matrices is

$$N = \sum_{n_0,...,n_{p-1}:\sum n_i = n} \frac{\prod_{j=0}^{n-1}(p^n - p^j)}{\prod_{i=0}^{p-1}\prod_{j=0}^{n_i-1}(p^{n_i} - p^j)}.$$

(b) Let $p = 4k+3$. By quadratic reciprocity, -1 is a non-square modulo p. So the eigenvalues of A generate a quadratic extension $\mathbb{F}_{p^2}$, and n has to be even ($n = 2m$), otherwise there is no such matrices.

In fact, A defines a structure of an $\mathbb{F}_{p^2}$-vector space on $\mathbb{F}_p^{2m}$. So the stabilizer of A is $GL_m(\mathbb{F}_{p^2})$, thus the answer is

$$N = \frac{|GL_{2m}(\mathbb{F}_p)|}{|GL_m(\mathbb{F}_{p^2})|} = \prod_{i=1}^{m}(p^{2m} - p^{2i-1}).$$

If $p = 4k + 1$, -1 is a square and the eigenvalues of A are in $\mathbb{F}_p$, so the problem is analogous to (a). The answer is

$$N = \sum_{n_0,n_1:n_0+n_1=n} \frac{\prod_{j=0}^{n-1}(p^n - p^j)}{\prod_{i=0}^{1}\prod_{j=0}^{n_i-1}(p^{n_i} - p^j)}.$$

Problem M4. Suppose we are given integers $m, n > 0$, and a collection S of (distinct) subsets of some ambient set $\mathcal{A}$, each of size at most m. Assume $|S| > (n-1)^m m!$. Prove that there exist n sets $A_1, \ldots, A_n \in S$ such that the intersections $A_i \cap A_j$ are the same for all pairs (i, j) with $i \neq j$.

Solution: It's enough to prove that if $|S| > (n-1)m$ then either S contains n disjoint sets, or there are at least $|S|/[(n-1)m]$ sets in S which all have a common element (then use induction by m). Indeed, assume S does not contain n disjoint sets. Assume k is the maximal number of disjoint sets it contains (k is at most $n-1$). Consider those k sets whose union contains at most $(n-1)m$ elements. Any other set must intersect this union (otherwise there will be $k+1$ disjoint sets), so by pigeonhole principle there are at least $|S|/[(n-1)m]$ sets sharing an element.

Problem M5. Find the number of colorings of the faces of the cuboctahedron (`https://en.wikipedia.org/wiki/Cuboctahedron`) in n colors, up to rotations (i.e. two colorings equivalent by rotation are regarded as the same).

Solution: Using Polya's enumeration theorem one gets

$$N = \frac{1}{24}(n^{14} + 3n^8 + 6n^7 + 8n^6 + 6n^5).$$

Problem M6. Let $D_i, i \geq 1$ be disks of radii $r_i < 1$ contained in the unit disk D, such that $D = \cup_{i\geq 1} D_i$.

(a) Show that for each $0 < a < 1$ the series $\sum_i r_i^a$ is divergent.

(b) Show that $\sum_i r_i$ is divergent.

(c) For any $a > 1$, can you pick D_i so that $\sum_i r_i^a$ is convergent?

(d) Can you solve (a),(b) if the union of the disks D_i is not necessarily the whole D but a subset $D' \subset D$ of full area (i.e., area π)?

Hint. Consider the intersection of D_i with the circle of radius $1-t$ centered at the origin, or (for (d)) the annulus between this circle and the unit circle.

Solution. (a) We may assume that $r_i \to 0$ as $i \to \infty$, otherwise there is nothing to prove. Let $t > 0$ be a small number, and consider the circle $x^2 + y^2 = (1-t)^2$. Let $N > 1$ be fixed. What is the maximal portion of this circle that can be covered by a disk of radius r, where $Nt < r < 1$? Clearly, the optimal position is when the circle of radius r touches the unit circle. So say it has the equation $(x-1+r)^2+y^2 = r^2$. Subtacting, we get

$$2x(1-r) - (1-r)^2 = (1-t)^2 - r^2.$$

After simplifications, this gives

$$x = 1 - \frac{t - t^2/2}{1-r}$$

for the upper intersection point of the two circles. Thus, the angle covered is

$$\alpha = 2\arccos\frac{x}{1-t} = 2\arccos\frac{1 - \frac{t-t^2/2}{1-r}}{1-t}.$$

On the other hand, if $r \le Nt$ then for the upper intersection point we have $y \le r$. So, since the circle of radius $1-t$ is covered completely, for sufficiently small t we have

$$\sum_{i:r_i>Nt} \arccos\frac{1 - \frac{t-t^2/2}{1-r_i}}{1-t} + \sum_{i:r_i\le Nt} \arcsin\frac{r_i}{1-t} \ge \pi.$$

Now use the inequalities

$$\arccos(1-u) < Cu^{1/2},\ \arcsin(u) < Cu,$$

for small $u > 0$ and some $C > 0$. Sending $t \to 0$ and using this inequality, we get

$$\sum_{i:r_i>Nt} (tr_i)^{1/2} + \sum_{i:r_i\le Nt} r_i \ge K$$

for small enough t and some fixed $K > 0$. This means that for any $1/2 < a < 1$,

$$\sum_i t^{1-a} r_i^a \ge L$$

for small t and a fixed constant $L > 0$. So

$$\sum_i r_i^a \ge Lt^{a-1}$$

for small t and a fixed constant $L > 0$. Sending t to zero, we get the statement.

(b) Suppose for the sake of contradiction that $\sum_i r_i < \infty$.

Then $\sum_{i:r_i \le Nt} r_i \to 0$ as $t \to 0$, so for sufficiently small t we have from the above:

$$\sum_{i:r_i > Nt} (tr_i)^{1/2} \ge K/2.$$

We can replace t with r_i/N. So we get

$$N^{-1/2} \sum_{i:r_i > Nt} r_i \ge K/2.$$

Keeping N fixed and sending t to zero, we get

$$N^{-1/2} \sum_i r_i \ge K/2$$

Thus $\sum_i r_i \ge N^{1/2} K/2$ for any N, a contradiction.

(c) Start with a disk D_1 of radius $1 - t_1$ centered at the origin, for a small enough t_1. Then cover its boundary with disks $D_2, ... D_{n+1}$ centered at vertices of a regular n-gon, with radii equal to the sides of the n-gon, and n being the smallest number so that these disks fit into the unit disk. Take the largest disk centered at the origin that's covered completely (of radius $1 - t_2$ for some $t_2 < t_1$), and repeat the procedure.

(d) The argument is the same, replacing the circle of radius $1 - t$ with the band between this circle and the unit circle. (This argument can also be used for (a),(b),(c)).

Alternative solutions of M6, by P. Suwara.

Part a)

Follows from b) by a simple comparison test.

Part b)

The reader might notice that changing disks to other shapes and r_i to their diameters does not change the first proof below.

Projection on a diameter. Let $\pi : D \to I$ be an orthogonal projection of D onto one of its diameters I. Define functions (one could say "random variables") $f_i : I \to \mathbb{R}$ by formula

$$f_i(x) = \begin{cases} 1 & x \in \pi(D_i) \\ 0 & x \notin \pi(D_i) \end{cases}.$$

We see that

$$\int_I f_i(x)\,dx = |\pi(D_i)| = 2r_i.$$

Define $f : I \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by

$$f = \sum_i X_i$$

We now have

$$\int_I f(x)\,dx = \sum_i \int_I f_i(x)\,dx = 2\sum_i r_i.$$

Therefore it suffices to prove that $f(x) = \infty$ for almost all $x \in I$.

We now give another interpretation of f. Define $L(x)$ to be the line in $\mathbb{R}^2$ which is perpendicular to I and passes through point $x \in I$. Define $J(x) = L(x) \cap D$, the intersection of this line with the disk D. Then $f_i(x) = 1$ if and only if $J(x) \cap D_i$ is nonempty. If follows that $f(x)$ is the number of disks D_i that intersect $J(x)$.

Suppose there is a finite number of D_i that intersect $J(x)$ for a fixed x. We show that endpoints of $J(x)$ are points of tangency of D and some of the disks D_i. Take any sequence of points $y_i \in J(x)$ converging to a chosen endpoint p of $J(x)$. Since a finite number of disks D_i covers $J(x)$, therefore there is a disk D_j such that $y_i \in D_j$ for infinitely many i. Take subsequence $z_i \in J(x) \cap D_j$. Now, z_i converge to $p \in \partial J(x) \subset \partial D$, but also $z_i \in D_j$, so $p \in \bar{D}_j$. In fact, since $p \in \partial D$, therefore p cannot be in the interior of D_j, so $p \in \partial D_j$. Therefore p is a point of tangency of D_j and D.

But the number of disks D_i is countable and each of them can have at most 1 point of tangency with D, so for all x besides at most countable number of them, $f(x) = \infty$. Finally

$$\sum_i r_i = \frac{1}{2} \int_I f(x)\, dx = \infty.$$

Limiting circles. This proof is based on the hint. We consider circles C_t of radius $1 - t$ concentric with D. Each such circle is covered by disks D_i. Assume $\sum_i r_i$ is convergent.

Lemma 0.1. *Let C_R be a circle of radius R in the plane. Let D_r be a disk of radius r in the plane. Then $|C_R \cap D_r| < 2\pi r$, that is, D_r covers an arc of C_R of length less than $2\pi r$.*

Proof. If $R \leq r$, then C_R has length less than $2\pi r$, so there is nothing to prove.

Assume $R > r$. If C_R and D_r do not intersect, there is nothing to prove.

Otherwise, let C_R and ∂D_r intersect at points p, q. These points divide C_R into two arcs. The longer arc contains a diameter of C_R, which is longer than the diameter of D_r, so it cannot be contained in D_r. Therefore the shorter arc is contained in D_r, denote this arc A. Let D' be the disk with diameter I. D' contains the arc A since D' intersects C_R and does not contain the longer arc $C_R \setminus A$ (because of the same diameter argument as above).

Denote by l the length of I and by 2θ the angle of arc A. The length of A is equal to $2\theta R$, the length of I is equal to $l = 2R\sin\theta$. Now, since sin is concave on $[0, \pi/2]$, $\sin(0) = 0$ and $\sin(\pi/2) = 1$, therefore we have

$$\sin(x) = \sin\left(\frac{\pi - 2x}{\pi} \cdot 0 + \frac{2x}{\pi} \cdot \frac{\pi}{2}\right) \geq \frac{\pi - 2x}{\pi} \cdot \sin(0) + \frac{2x}{\pi} \cdot \sin\left(\frac{\pi}{2}\right) = \frac{2x}{\pi}$$

And it follows that

$$\pi l/2 = \pi R \sin\theta \geq 2R\theta = |A|$$

and finally

$$2\pi r > 2r\pi/2 \geq l\pi/2 = \pi l/2 \geq |A|$$

since I is contained in D_r, and therefore not longer than $2r$. □

Take some $0 < t_1 < 1$. We know C_{t_1} is covered by disks D_i. Let $J_1 = \{i \in \mathbb{N} : D_i \cap C_{t_1} \neq \varnothing\}$ be the set of indices of disks intersecting C_{t_1}. Let $T = \{i \in \mathbb{N} : D_i \text{ is tangent to } D\}$ enumerate disks that are tangent to D. Let $T_1 = T \cap J_1$. The reader should notice that in b) we could use compactness of a circle to prove J_1 can be shrunk to be finite and still contain enough indices to cover C_{t_1}, but this argument no longer works in d). Therefore we choose a longer path.

Notice that for each $i \in T_1$, D_i is tangent to D and intersects C_{t_1}, so it has diameter $2r_i$ which is at least t_1. Therefore, if T_1 is an infinite set, then $\sum_i r_i$ is infinite, contradiction. Therefore T_1 is finite.

Notice that for each i we have $\lim_{t\to 0} |D_i \cap C_t| = 0$. Since T_1 is finite, there is $t_1 > t_2 > 0$ such that $|C_{t_2} \setminus \bigcup_{i\in T_1} D_i| \geq \pi$. Notice that C_{t_2} is intersected by at most finitely many disks D_i for $i \in J_1 \setminus T_1$. Otherwise, for each such disk we would have $2r_i \geq t_1 - t_2$ and it would follow that $\sum_i r_i = \infty$. But for $i \in J_1 \setminus T_1$ one can find t_2 large enough so that $C_{t_2} \cap D_i = \varnothing$. It follows that for such t_2 we get

$$\left| C_{t_2} \setminus \bigcup_{i\in J_1} D_i \right| \geq \pi.$$

We now describe the inductive procedure to construct $1 > t_1 > t_2 > \ldots > 0$ such that for $J_k = \{i \in \mathbb{N} : D_i \cap C_{t_k} \neq \varnothing\}$, we have

$$\left| C_{t_k} \setminus \bigcup_{i\in J_{k-1}} D_i \right| \geq \pi.$$

As for $k = 2$, we take $T_k = T \cap J_k$, by the same arguments T_k is finite and for sufficiently small t we have $C_t \cap D_i = \varnothing$ for $i \in J_k \setminus T_k$, and for sufficiently small t_{k+1} we also have $|C_{t_k} \setminus \bigcup_{T_k} D_i| \geq \pi$ so t_k is as wished.

Notice that if D_i intersects $C_{t_{k-1}}$ and $C_{t_{k+1}}$, then it intersects C_{t_k}. Therefore we have

$$\left| C_{t_k} \setminus \bigcup_{i\in J_1\cup\ldots\cup J_{k-1}} D_i \right| \geq \pi.$$

But C_{t_k} is covered with D_i, so if we denote $S_k = J_k \setminus (J_1 \cup \ldots \cup J_{k-1})$ then

$$\left| C_{t_k} \cap \bigcup_{i\in S_k} D_i \right| \geq \pi$$

and since S_k are disjoint, it follows that, using the lemma,

$$\begin{aligned}2\pi\sum_i r_i \geq& 2\pi\sum_k\sum_{i\in S_k} r_i\\ \geq& \sum_k\sum_{i\in S_k}|C_{t_k}\cap D_i|\\ \geq& \sum_k\left|C_{t_k}\cap\bigcup_{i\in S_k}D_i\right|\\ \geq& \sum_k \pi\\ =& \infty\end{aligned}$$

and the proof is finished.

Part c)

Let $Y_n = \{(i/2^n, j/2^n) : i, j \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$ be a square lattice in the plane. Notice that disks of radius 2^{-n} centered at points of Y_n cover the plane since squares with side length 2^{-n} centered at vertices of Y_n form a tiling of the plane and are contained in these disks.

Let $Z_n = \{x \in Y_n : 1 - 10\cdot 2^{-n} \leq \|x\| \leq 1 - 2\cdot 2^{-n}\}$ for $n \geq 0$. Denote $B(a, b) = \{x \in \mathbb{R}^2 : 1 - a < \|x\| < 1 - b\}$. In other words, $Z_n = Y_n \cap B(10\cdot 2^{-n}, 2\cdot 2^{-n})$. Let $D_{n,i}$ be disks centered at vertices of Z_n of radius 2^{-n}. Let (D_i) be the sequence of all these disks for $n = 1, 2, \ldots$. We claim that $|Z_n| \leq C2^n$ for some constant $C > 0$. It quickly follows that for $a > 1$

$$\sum_i r_i^a = \sum_n |Z_n|2^{-an} = C\sum_n 2^{(1-a)n} < \infty$$

since $1 - a < 0$ and this is a geometric series. Secondly, we claim that D_i cover D.

Let us now prove the first claim. Notice that disks of radius 2^{-n-1} centered at points of Z_n are disjoint and contained in $B(10\cdot 2^{-n} + 2^{-n-1}, 2\cdot 2^{-n} - 2^{-n-1})$, in particular in $B(11\cdot 2^{-n}, 0)$, which (for $n \geq 4$) has area

$$\pi(1 - (1 - 11\cdot 2^{-n})^2) \geq 22\pi\cdot 2^{-n}$$

and since these small disks have area $2^{-2n-4}\pi$, therefore there is at most

$$\frac{22\pi\cdot 2^{-n}}{2^{-2n-4}\pi} = 352\cdot 2^{-n}$$

of these small disks, so

$$|Z_n| \leq 352\cdot 2^{-n},$$

as wished.

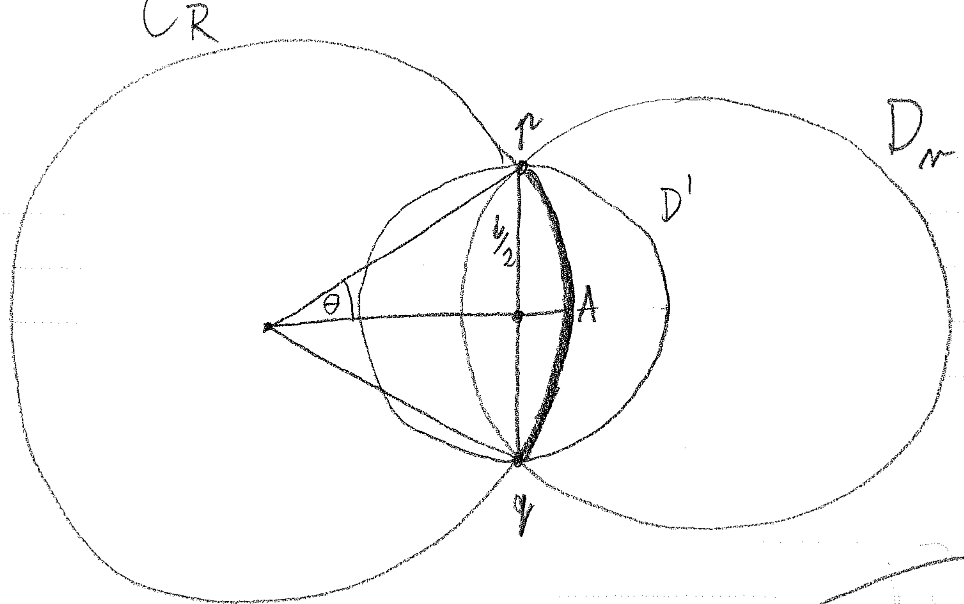
We prove the second claim by proving that disks $D_{n,i}$ cover the band $B_n = B(8 \cdot 2^{-n}, 4 \cdot 2^{-n})$ if $n > 3$ or the disk $B_n = B(0, 4 \cdot 2^{-n})$ if $n \neq 3$. Indeed, all the points inside B_n are at least $2 \cdot 2^{-n}$ apart from boundary of B_n. Take a point $x \in B_n$. It is contained in some square S of side 2^{-n} with vertices in Y_n. This square S is contained in a disk D' of radius $2 \cdot 2^{-n}$ centered at x, which is contained in $B(10 \cdot 2^{-n}, 2 \cdot 2^{-n})$ since $x \in B(8 \cdot 2^{-n}, 4 \cdot 2^{-n})$. So square S is contained in $B(10 \cdot 2^{-n}, 2 \cdot 2^{-n})$, therefore all its vertices are in Z_n, so S is covered by disks $D_{n,i}$ and therefore x is covered by these disks. Which finishes the proof of this claim.

And it finishes the proof of part c).

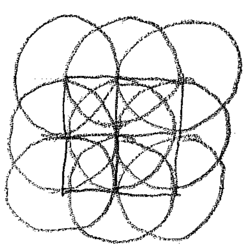
Part d)

The "Projection on a diameter" proof generalizes easily since by Fubini's theorem, for almost all x we have that almost all points of $J(x)$ are covered by disks D_i.

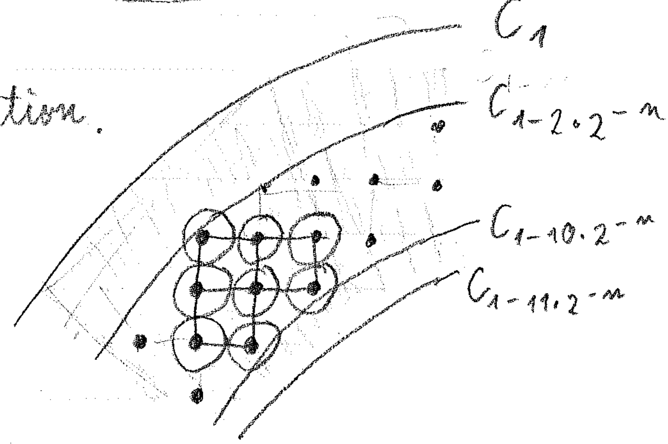
The "Limiting circles" proof generalizes easily since we only used the fact that the full measure of almost all C_t are covered, which follows from Fubini's theorem. It also can be refined to work with bands, but with some, possibly unpleasant, technicalities.



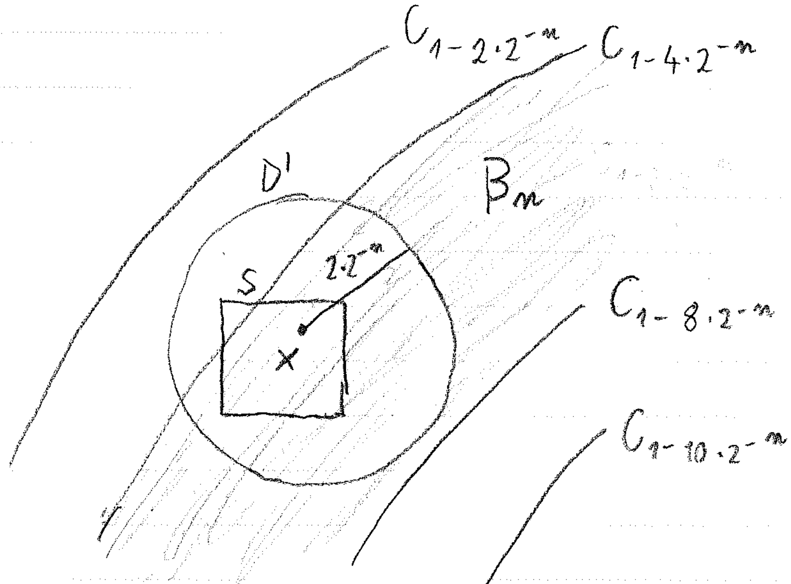
Lemma 1 illustration.



Disks centered at Y_n



Smaller circles centered at Z_n



Neighbourhood of $x \in B_n$