A Connection Between Vector Bundles over Smooth Projective Curves and Representations of Quivers

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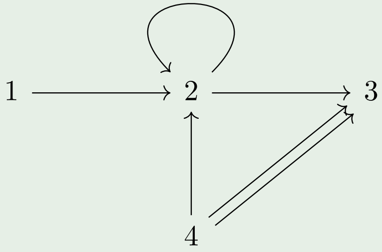
- Quiver
- Representation of Quiver
- Vector Bundle on Curve

Quiver

- Directed graph
- Multi-edges and self-loops allowed

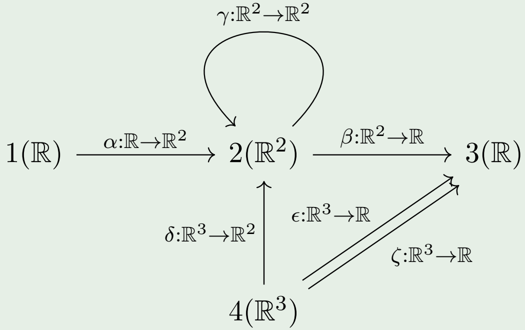
Example



Representation of Quiver

- Vector Space for each vertex
- Linear transformation for each edge

Example



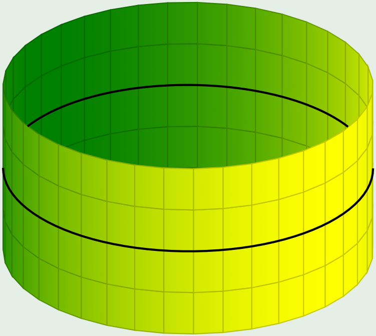
Vector Bundle on Curve

- Curve $\mathcal{C}$
- Vector space $\mathbb{F}^k$
- Copy of $\mathbb{F}^k$ for each $c \in \mathcal{C}$
- Continuously varying

Vector Bundle on Curve

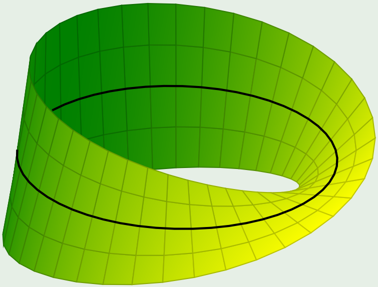
Example (Open Cylinder)

One copy of $\mathbb{R}$ for each point on circle



Example (Open Mobius Strip)

One copy of $\mathbb{R}$ for each point on circle.

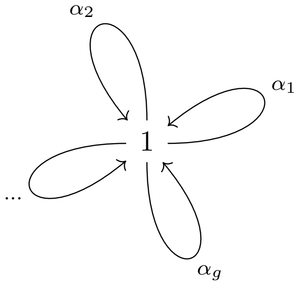


How are these related?

Conjecture (Schiffmann, 2016)

For any $g \in \mathbb{N}^+$ *and* $r \in \mathbb{N}$ *and* $d \in \mathbb{Z}$,

$$A_{g,r,d}(0) = A_{\Sigma_g,r}(1).$$



Definition Intermission

Definition

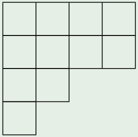
A **partition** λ is a finite non-increasing sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0.$$

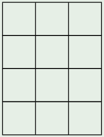
The **size** of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_l$.
The **length** of λ is $l(\lambda) = l$.
A **flat** partition is one in which all elements are equal.

Example

$\lambda = (4, 4, 2, 1)$



$\lambda = (3, 3, 3, 3)$



Partition Reformulation

Definition

For any partition π and any g, we define the rational function

$$B_\pi = \sum_{\pi_0,\ldots,\pi_s} q^{|\pi_0|-l(\pi_0)}(-1)^s \prod_{i=0}^{s} \frac{q^{(g-1)\langle \pi_i,\pi_i\rangle}}{b_{\pi_i}(q^{-1})}$$

where π_0 may be the empty partition but $\pi_1,\ldots,\pi_s$ are all nonempty, and $\pi_0 \cup \pi_1 \cup \cdots \cup \pi_s = \pi$.

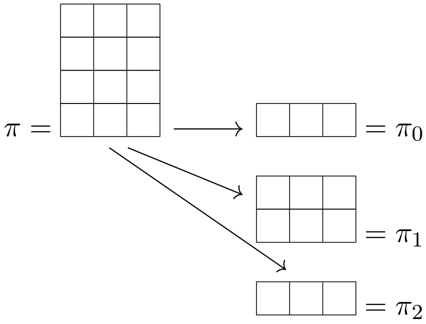
Conjecture

For all partitions π, B_π is a polynomial.

Flat Partition Case

Theorem

For all flat partitions π, B_π is a polynomial.



Proving Polynomialness

Example

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + \dots$$

Example

$$\frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \dots$$

Example

$$\begin{aligned}\frac{1-q^2}{1-q} &= (1-q^2)(1+q+q^2+q^3+\dots) \\ &= (1+q+q^2+q^3+\dots) - (q^2+q^3+q^4+q^5+\dots) \\ &= 1+q\end{aligned}$$

Proving Polynomialness

$$\pi = (a, a, a, \dots, a)$$

$$l(\pi) = b$$

$$(-1)^b q^{(g-1)ab} B_\pi = c_0 q^0 + c_1 q^1 + c_2 q^2 + \dots$$

Theorem

For some N, for all $n \geq N$, the coefficient $c_n = 0$.

Corollary

$\deg(-1)^b q^{(g-1)ab} B_\pi = b(b+1)(g-1)a + b(a-1)$.

Combinatorial Interpretation

$$c_n = \sum_p \text{sign}(p), \text{ where } p = (p^0, p^1, \ldots, p^s) \text{ and } \text{sign}(p) = (-1)^s$$

Example ($a = 2$, $b = 5$, $g = 2$, $n = 50$)

$$p = ((15, 10, 5), (11, 6))$$

- $p^1, \ldots, p^s$ are nonempty
- sum of lengths of partitions is $b = 5$
- sum of sizes of partitions, plus $(a-1)l(p^0)$, is $n = 50$
- each p^i is a d-stair partition for $d = 2(g-1)a + 1 = 5$.

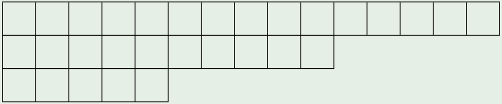
Definition Intermission

Definition

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is **d-stair** if every consecutive difference is at least d, and $\lambda_l \geq d$.

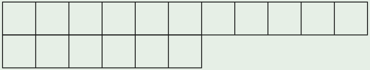
Example

$p^0 = (15, 10, 5)$ is 5-stair.



Example

$p^1 = (11, 6)$ is 5-stair.



Bijection

Need to cancel out all tuples λ of the same size

$$n = (a-1)l(\lambda^0) + |\lambda^0| + |\lambda^1| + \cdots + |\lambda^s|$$

for large enough n.
Idea: pair each tuple λ with another tuple of opposite sign - i.e. with one more or one less partition.

1. Pair {tuples with zeroth partition nonempty} with part of {tuples with zeroth partition empty}
2. Pair off remaining elements of {tuples with zeroth partition empty}

Zeroth Partition Nonempty

Example ($a = 4$, $b = 3$, $g = 2$, $n = 40$)

$$((9), (19, 9)) \longrightarrow ((), (12), (19, 9))$$

Example ($a = 4$, $b = 3$, $g = 2$, $n = 42$)

$$((18, 9), (9)) \longrightarrow ((), (21, 12), (9))$$

We pair

$$\{p : |p^0| > 0\}$$

with

$$\{p : |p^0| = 0 \land p^1_{l(p^1)} \geq d + a - 1\} \text{ where } d = 2(g-1)a + 1$$

Main Bijection

What's left is

$$\{p : |p^0| = 0 \wedge p^1_{l(p^1)} < d + a - 1\}$$

Two operations:

- Unroll
- Tuck

Main Bijection — Unroll or Tuck

Example ($a = 2$, $d = 5$)

tuck

$$((), (5), (9), (30, 16)) \qquad ((), (5), (30, 16, 9))$$

unroll

We apply whichever comes first.

Main Bijection — Can't unroll or tuck?

Example ($a = 2$, $d = 5$)

$$? \xleftarrow{\textbf{tuck}} ((), (5), (9), (13), (17)) \xrightarrow{\textbf{unroll}} ?$$

- every partition is a singleton
- differences less than $d = 2(g-1)a + 1$
- first value less than $d + a - 1$

$\therefore$ size is bounded by

$$\begin{aligned} &|p^0| + |p^1| + \cdots + |p^b| \\ &\leq 0 + (d + a - 2) + \cdots + (d + a - 2 + (b-1)(d-1)) \\ &= b(b+1)(g-1)a + b(a-1). \end{aligned}$$

Flat Partition Case

Either

- size is small
- everything cancels out

Theorem

For all flat partitions π, B_π is a polynomial.

Future Directions

Conjecture (General Case of B_π)

For all partitions π, B_π is a polynomial.

$$\begin{matrix} p_0^1 & p_1^1 & \cdots & p_s^1 \\ p_0^2 & p_1^2 & \cdots & p_s^2 \\ p_0^3 & p_1^3 & \cdots & p_s^3 \\ \vdots & \vdots & \ddots & \vdots \end{matrix}$$

Acknowledgements

I would like to thank:

- My mentor, Vishal Arul
- MIT PRIMES-USA
- My parents

For any questions about quivers and vector bundles and curves, the chances are quite high that I don't know the answer.

Example

Q: What is the analogy between representations of quivers and vector bundles on curves?
A: I don't know.