Partially Directed nil-Temperley-Lieb Algebras

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nil-Temperley-Lieb (nTL) Algebras

Algebra based on a graph $G$. One generator per vertex: $x_1, x_2, x_3$. 

$x_2 x_1 x_3 x_2 = x_2 x_3 x_1 x_2$ is irreducible.
nil-Temperley-Lieb (nTL) Algebras

Algebra based on a graph $G$. One generator per vertex:

- $x_1, x_2, x_3$.
- $x_i^2 = 0$.
- For two adjacent vertices $i$ and $j$, $x_i x_j x_i = x_j x_i x_j = 0$.
- For two nonadjacent vertices $i$ and $j$, $x_i x_j = x_j x_i$.
- A monomial that does not equal 0 is called **irreducible**.

Graph:

1 -- 2 -- 3

Diagram:
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- A monomial that does not equal 0 is called irreducible.

Example

$x_3 x_1 x_2 x_3 = x_1 x_3 x_2 x_3 = 0$ is reducible.

$x_2 x_1 x_3 x_2 = x_2 x_3 x_1 x_2$ is irreducible.
Dimension of the Algebra

The **dimension** of the algebra is the number of distinct irreducible monomials.
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In $G_1$, these monomials are

$1, \ x_1, \ x_2, \ x_3, \ x_1x_2, \ x_1x_3, \ x_2x_3, \ x_3x_2, \ x_1x_2x_3, \ x_1x_3x_2$

and the dimension is 10. Not counted are repeated monomials ($x_2x_1 = x_1x_2$ and $x_3x_1 = x_1x_3$) and reducible monomials ($x_2x_3x_2 = 0$ and $x_3x_2x_3 = 0$).
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In $G_2$, there is an infinite irreducible monomial:

$$x_1x_2x_3x_1x_4x_5x_1x_2x_3x_1x_4x_5 \ldots$$

$$= x_1x_3x_2x_1x_5x_4x_1x_3x_2x_1x_5x_4 \ldots$$
Theorem

The nTL algebra on $G$ is finite iff $G$ is a Dynkin diagram.
Number the vertices 1 to $n$.

Dimension of the algebra known to be $C_{n+1}$, the $n + 1^{th}$ Catalan number.
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Dimension of the algebra known to be $C_{n+1}$, the $n + 1^{\text{th}}$ Catalan number.

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$(x_3 x_2 x_1) (x_5 x_4 x_3 x_2) (x_7 x_6)$

- This is the lexicographically smallest representation of the monomial.

If peaks don’t increase:

$x_4 x_3 x_2 x_1 x_4 x_3 = x_4 x_3 x_2 x_4 x_1 x_3 = x_4 x_3 x_4 x_2 x_1 x_3 = 0$
Motivation

- Map to the set of permutations on $n + 1$ elements: if $x_i$ is taken to the transposition of the $i^{th}$ and $i + 1^{th}$ elements.
  - By this construction, the elements of the algebra are 321-avoiding permutations.
Motivation

- Map to the set of permutations on $n + 1$ elements: if $x_i$ is taken to the transposition of the $i^{th}$ and $(i + 1)^{th}$ elements.
  - By this construction, the elements of the algebra are 321-avoiding permutations.
- Definitions similar to those of Coxeter groups. The elements of the algebra correspond to elements of Coxeter groups satisfying certain properties.
Partially Directed nTL Algebras

- Based on a graph $G$ with some directed and some undirected edges.
- $x_i^2 = 0$.
- For two nonadjacent vertices $i$ and $j$, $x_i x_j = x_j x_i$.
- For two vertices $i$ and $j$ connected by an undirected edge, $x_i x_j x_i = x_j x_i x_j = 0$. 
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The example has relations $x_2 x_3 x_2 = 0$ and $x_5 x_4 x_5 = 0$, but not $x_3 x_2 x_3 = 0$ or $x_4 x_5 x_4 = 0$. 
Dimensions of Partially Directed nTL algebras

Theorem

The nTL algebra on a partially directed graph $G$ is finite iff $G$ is a path graph with all directed edges going in the same direction.
Each monomial can be written uniquely as a series of decreasing runs with increasing valleys. For example, 
\[(x^5 x^4 x^3 x^2 x^1) (x^7 x^6 x^5 x^4 x^3) (x^6 x^5 x^4) (x^7)\].

There are \(n + 1\) choices for the run with valley \(x^1\): 
\[1, x^1, x^2 x^1, \ldots, x^n x^{n-1} \ldots x^2 x^1\].

Similarly, there are \(n\) choices for the run with valley \(x^2\), \(n-1\) choices for the run with valley \(x^3\), and so on.
Each monomial can be written uniquely as a series of decreasing runs with increasing valleys. For example,

\[(x_5 x_4 x_3 x_2 x_1) (x_7 x_6 x_5 x_4 x_3) (x_6 x_5 x_4) (x_7).\]

There are \(n + 1\) choices for the run with valley \(x_1\):

1, \(x_1\), \(x_2 x_1\), \ldots, \(x_n x_{n-1} \ldots x_2 x_1\).

Similarly, there are \(n\) choices for the run with valley \(x_2\), \(n - 1\) choices for the run with valley \(x_3\), and so on.
Theorem

There are \((n + 1) \times n \times (n - 1) \times \ldots \times 2 = (n + 1)!\) elements in the maximally directed algebra.
Maximally Directed nTL Algebras

Theorem

There are \((n + 1) \times n \times (n - 1) \times \ldots \times 2 = (n + 1)!\) elements in the maximally directed algebra.

Mapping the generator \(x_i\) to the transposition of \(i\) and \(i + 1\) in the set of permutations on \(n + 1\) elements, each irreducible monomial corresponds to a different element of the set of permutations on \(n + 1\) elements.
Peaks and Valleys

Every decreasing run has a peak and valley: $x_5x_4x_3x_2x_1$. 
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Every partially directed nTL algebra is a subalgebra of the maximally directed nTL algebra. Thus,

**Theorem**

*The monomials of a partially directed nTL algebra are sequences of decreasing runs with increasing valleys.*
**Conditions on the Peaks**

<table>
<thead>
<tr>
<th>Theorem</th>
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<tr>
<td>If there is an undirected edge from ( i ) to ( i + 1 ) and there are two peaks with (from left to right) ( p_1 \geq i + 1 ) and ( p_2 = i + 1 ), there must be a peak of ( i ) between ( p_1 ) and ( p_2 ).</td>
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Conditions on the Peaks

**Theorem**

*If there is an undirected edge from $i$ to $i + 1$ and there are two peaks with (from left to right) $p_1 \geq i + 1$ and $p_2 = i + 1$, there must be a peak of $i$ between $p_1$ and $p_2$."

For example, when there is an undirected edge between 3 and 4 ($i = 3$), $x_5x_4x_3x_2x_1x_3x_2x_4$ is irreducible, but $x_5x_4x_3x_2x_1x_2x_4$ is not.

This theorem completely describes the irreducible monomials in the partially directed nTL algebras.
Corollary

There is no condition on the peaks of the maximally directed algebra.
Conditions on the Peaks

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In the nTL algebra, peaks must be increasing.
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Corollary

In the algebra based on the “undirected-directed” graph shown, peaks must strictly increase or remain higher than $k$. 
Special Cases

Dimension: $C_n + C_{n+1} - 1$, where $C_n$ is the $n^{th}$ Catalan number.
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Dimension: $\binom{2n}{n} = (n + 1)C_n$
Future research

- Find a general formula to calculate the dimension of any partially directed nTL algebra.
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- Further study which permutations are represented by a partially directed nTL algebra.
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- Further study which permutations are represented by a partially directed nTL algebra.
- A directed edge between $i$ and $j$ means changing the relation $x_i x_j x_i = x_j x_i x_j = 0$ to $x_i x_j x_i = 0$. What if we changed it to $x_i x_j x_i = x_j x_i x_j$?
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