Results of Triangles Under Discrete Curve Shortening Flow

Sam Cohen, Peter Rowley

January 7, 2016

Abstract

In this paper, we analyze the results of triangles under discrete curve shortening flow, specifically isosceles triangles with top angles greater than $\frac{\pi}{3}$, and scalene triangles. By considering the location of the three vertices of the triangle after some small time ϵ, we use the definition of the derivative to calculate a system of differential equations involving parameters that can describe the triangle. Constructing phase plane diagrams and then analyzing them, we find that the singular behavior of discrete curve shorting flow on isosceles triangles with top angles greater than $\frac{\pi}{3}$ is a point, and for scalene triangles is a line segment.

1 Introduction

Discrete curve shortening flow (DCSF) is an analog of curve shortening flow (CSF). CSF is applied to smooth curves defined by a smooth map $F : S^1 \mapsto \mathbb{R}^2$. (Here, S^1 is the unit circle.) We define the curvature $k(t)$ at each point as the reciprocal of the radius of the circle which best approximates the curve at that point. The normal vector N is the outward-facing unit vector normal to the curve at that point. We thus define the motion of the curve by the differential equation

$$\frac{dF}{dt} = -kN$$

The Gage-Hamilton-Grayson Theorem on continuous CSF states that the shape of every curve will converge to a circle under the flow and that every curve will eventually go to a single point [1][2].

In DCSF only finitely many points define a curve, namely, vertices. The curve, instead of being continuous, is made up of finitely many line segments connecting the vertices, which form a polygon. Because of this, its definition is slightly different, but analogous to that of CSF.

We call the position vector of the ith vertex x_i and the angle at that vertex α_i. $k(x_i)$, the curvature, is defined by $k(x_i) = \pi - \alpha_i$. $\vec{n}(x_i)$, the normal vector, is defined as the outward facing unit vector in the direction of the angle bisector. Based on these definitions, we define the motion of each point by the differential equation

$$\frac{dx_i}{dt} = -k(x_i)\vec{n}(x_i)$$

Under these definitions, we will determine what shapes result from DCSF on curves with three vertices, or triangles. Some work has already been done on this, in particular in the case of isosceles triangles with top angles less than $\frac{\pi}{3}$, and it has been determined that the end behavior of these triangles under DCSF is a line segment [3]. We will determine if this is perhaps a special case, and, like in CSF, the end result of DCSF on triangles will be a point.

2 Isosceles Triangles

Theorem 1. *The end behavior of DCSF applied to isosceles triangles with top angles greater than $\frac{\pi}{3}$ is a point.*

We will begin by deriving differential equations for parameters which describe the triangle.

Define an isosceles triangle ABC with $AB = AC = x$ and $\angle BAC = \alpha$.

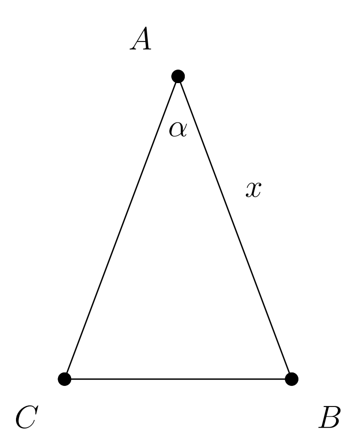

Figure 1: Isosceles triangle

Then the coordinates are $A(0,0)$, $B(x\sin\frac{\alpha}{2}, -x\cos\frac{\alpha}{2})$, and $C(-x\sin\frac{\alpha}{2}, -x\cos\frac{\alpha}{2})$.

Consider the linearized approximations of the coordinates of A, B, and C after a small time $\epsilon > 0$ under the differential equation $\frac{dx}{dt} = -k(x)\vec{n}(x)$. We calculate the normal vector and curvature at each point:

$$\vec{n}(A) = \langle 0, 1\rangle$$
$$\vec{n}(B) = \langle \cos\frac{\pi-\alpha}{4}, -\sin\frac{\pi-\alpha}{4}\rangle$$
$$\vec{n}(C) = \langle -\cos\frac{\pi-\alpha}{4}, -\sin\frac{\pi-\alpha}{4}\rangle$$
$$k(A) = \pi - \alpha$$
$$k(B) = \frac{\pi+\alpha}{2}$$
$$k(C) = \frac{\pi+\alpha}{2}$$

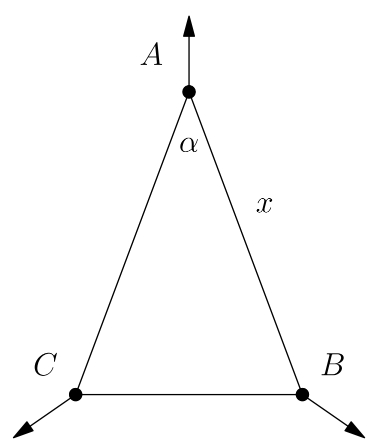

Figure 2: Isosceles triangle normal vectors

We can use these to get the coordinates:

$$\tilde{A}(0, \epsilon(\alpha-\pi))$$
$$\tilde{B}(x\sin\frac{\alpha}{2} - \epsilon\frac{\pi+\alpha}{2}\cos\frac{\pi-\alpha}{4}, -x\cos\frac{\alpha}{2} + \epsilon\frac{\pi+\alpha}{2}\sin\frac{\pi-\alpha}{4})$$
$$\tilde{C}(-x\sin\frac{\alpha}{2} + \epsilon\frac{\pi+\alpha}{2}\cos\frac{\pi-\alpha}{4}, -x\cos\frac{\alpha}{2} + \epsilon\frac{\pi+\alpha}{2}\sin\frac{\pi-\alpha}{4})$$

We will now use the Pythagorean distance formula to find $(\tilde{x})^2$, where $\tilde{x}$ is the length of $\tilde{A}\tilde{B}$. This gives

$$(\tilde{x})^2 = (x\sin\frac{\alpha}{2} - \epsilon\frac{\pi+\alpha}{2}\cos\frac{\pi-\alpha}{4})^2 + (-x\cos\frac{\alpha}{2} + \epsilon(\frac{\pi+\alpha}{2}\sin\frac{\pi-\alpha}{4} + \pi - \alpha))^2$$

We know that using this approximation method, we approximate $(\tilde{x})^2 = x^2 + \epsilon\frac{dx^2}{dt}$. Thus, the real value of $\frac{dx^2}{dt}$ can be found using the equation $\frac{dx^2}{dt} = \lim_{\epsilon\to 0}\frac{(\tilde{x})^2 - x^2}{\epsilon}$. From there, we can use the fact that $\frac{dx^2}{dt} = 2x\frac{dx}{dt}$ to find $\frac{dx}{dt}$. As both ϵ and $(\tilde{x})^2 - x^2$ go to zero as ϵ goes to zero, we can use L'Hopital's rule. This gives us:

$$\frac{dx^2}{dt} = \lim_{\epsilon\to 0}\frac{(\tilde{x})^2 - x^2}{\epsilon} = \lim_{\epsilon\to 0}\frac{\frac{d(\tilde{x})^2}{d\epsilon}}{\frac{d\epsilon}{d\epsilon}} = \lim_{\epsilon\to 0}\frac{d(\tilde{x})^2}{d\epsilon}$$

$$= -x(\pi+\alpha)\cos\frac{\pi-\alpha}{4}\sin\frac{\alpha}{2} - x((\pi+\alpha)\sin\frac{\pi-\alpha}{4} + 2(\pi-\alpha))\cos\frac{\alpha}{2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-x(\pi+\alpha)\cos\frac{\pi-\alpha}{4}\sin\frac{\alpha}{2} - x((\pi+\alpha)\sin\frac{\pi-\alpha}{4} + 2(\pi-\alpha))\cos\frac{\alpha}{2}}{2x}$$

$$= -\frac{\pi+\alpha}{2}(\cos\frac{\pi-\alpha}{4}\sin\frac{\alpha}{2} + \sin\frac{\pi-\alpha}{4}\cos\frac{\alpha}{2}) + \cos\frac{\alpha}{2}(\alpha-\pi)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\frac{\pi+\alpha}{2}\sin\frac{\pi+\alpha}{4} + \cos\frac{\alpha}{2}(\alpha-\pi)} \tag{1}$$

We approximate that $\tilde{x} = x - \epsilon\frac{dx}{dt} = x + \epsilon(\frac{\pi+\alpha}{2}\sin\frac{pi+\alpha}{4} + \cos\frac{\alpha}{2}(\pi-\alpha))$. We will approximate the value of $\sin\frac{\tilde{\alpha}}{2}$ using the calculated approximations of $\tilde{x}$ and the x-coordinate of $\tilde{B}$. We will then use the approximation $\sin\frac{\tilde{\alpha}}{2} = \sin\frac{\alpha}{2} + \epsilon\frac{d\sin\frac{\alpha}{2}}{dt}$. This gives the approximation

$$\sin\frac{\tilde{\alpha}}{2} = \frac{x\sin\frac{\alpha}{2} - \epsilon\frac{\pi+\alpha}{2}\cos\frac{\pi-\alpha}{4}}{x - \epsilon\frac{dx}{dt} = x + \epsilon(\frac{\pi+\alpha}{2}\sin\frac{\pi+\alpha}{4} + \cos\frac{\alpha}{2}(\pi-\alpha))}$$

We also know that

$$\frac{d\sin\frac{\alpha}{2}}{dt} = \frac{1}{2}\cos\frac{\alpha}{2}\frac{d\alpha}{dt} = \lim_{\epsilon\to 0}\frac{\sin\frac{\tilde{\alpha}}{2} - \sin\frac{\alpha}{2}}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\frac{x \sin \frac{\alpha}{2} - \epsilon \frac{\pi+\alpha}{2} \cos \frac{\pi-\alpha}{4}}{x - \epsilon \frac{dx}{dt} = x + \epsilon(\frac{\pi+\alpha}{2} \sin \frac{pi+\alpha}{4} + \cos \frac{\alpha}{2}(\pi - \alpha))} - \sin \frac{\alpha}{2}}{\epsilon}$$

$$= \frac{\frac{\pi+\alpha}{2}(\sin \frac{\pi+\alpha}{4} \sin \frac{\alpha}{2} - \cos \frac{\pi-\alpha}{4}) + \frac{1}{2} \sin \alpha(\pi - \alpha)}{x}$$

$$\Rightarrow \frac{d\alpha}{dt} = \frac{(\pi+\alpha)(\sin \frac{\pi+\alpha}{4} \sin \frac{\alpha}{2} - \cos \frac{\pi-\alpha}{4}) + \sin \alpha(\pi - \alpha)}{x \cos \frac{\alpha}{2}}$$

$$= \frac{(\pi+\alpha)(\sqrt{2}(\sin \frac{\alpha}{4} + \cos \frac{\alpha}{4}) \sin \frac{\alpha}{4} \cos \frac{\alpha}{4} - \frac{1}{\sqrt{2}}(\sin \frac{\alpha}{4} + \cos \frac{\alpha}{4})) + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}(\pi - \alpha)}{x \cos \frac{\alpha}{2}}$$

$$= \frac{\frac{1}{\sqrt{2}}(\pi+\alpha)(2 \sin^2 \frac{\alpha}{4} \cos \frac{\alpha}{4} - \cos \frac{\alpha}{4} + 2 \cos^2 \frac{\alpha}{4} \sin \frac{\alpha}{4} - \sin \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}(\pi - \alpha)}{x \cos \frac{\alpha}{2}}$$

$$= \frac{\frac{1}{\sqrt{2}}(\pi+\alpha)(\cos \frac{\alpha}{4}(2 \sin^2 \frac{\alpha}{4} - 1) + \sin \frac{\alpha}{4}(2 \cos^2 \frac{\alpha}{4} - 1)) + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}(\pi - \alpha)}{x \cos \frac{\alpha}{2}}$$

$$= \frac{\frac{1}{\sqrt{2}}(\pi+\alpha)(\sin \frac{\alpha}{4} \cos \frac{\alpha}{2} - \cos \frac{\alpha}{4} \cos \frac{\alpha}{2}) + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}(\pi - \alpha)}{x \cos \frac{\alpha}{2}}$$

$$\Rightarrow \boxed{\frac{d\alpha}{dt} = \frac{\frac{1}{\sqrt{2}}(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2}(\pi - \alpha)}{x}} \tag{2}$$

Lemma 1.1. *$\frac{\dot{\alpha} x}{\dot{x}} > \alpha - \pi$ for α near π*

Proof.

$$\frac{\dot{\alpha} x}{\dot{x}} = \frac{\frac{1}{\sqrt{2}}(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2}(\pi - \alpha)}{-\frac{\pi+\alpha}{2} \sin \frac{\pi+\alpha}{4} + \cos \frac{\alpha}{2}(\alpha - \pi)}$$

$$\frac{\frac{1}{\sqrt{2}}(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2}(\pi - \alpha)}{(-\frac{\pi+\alpha}{2} \sin \frac{\pi+\alpha}{4} + \cos \frac{\alpha}{2}(\alpha - \pi))(\alpha - \pi)} < 1$$

$$\frac{1}{\sqrt{2}}(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2}(\pi - \alpha) < (-\frac{\pi+\alpha}{2} \sin \frac{\pi+\alpha}{4} + \cos \frac{\alpha}{2}(\alpha - \pi))(\alpha - \pi)$$

$$\frac{1}{\sqrt{2}} \frac{\pi+\alpha}{\pi-\alpha}(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2} < \frac{\pi+\alpha}{2} \sin \frac{\pi+\alpha}{4} - \cos \frac{\alpha}{2}(\alpha - \pi)$$

It is sufficient to prove that

$$\lim_{\alpha \to \pi} \left(\frac{1}{\sqrt{2}} \frac{\pi+\alpha}{\pi-\alpha}(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) + 2 \sin \frac{\alpha}{2} \right) < \lim_{\alpha \to \pi} \left(\frac{\pi+\alpha}{2} \sin \frac{\pi+\alpha}{4} - \cos \frac{\alpha}{2}(\alpha - \pi) \right).$$

This is because as these functions are continuous, the strict inequality implies that there is a small interval near pi where this inequality is true.

$$\lim_{\alpha \to \pi} \left(\frac{1}{\sqrt{2}} \frac{\pi+\alpha}{\pi-\alpha}(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4}) \right) + 2 < \pi$$

$$\frac{1}{\sqrt{2}} \lim_{\alpha \to \pi} \left(\frac{(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4})}{\pi - \alpha} \right) + 2 < \pi$$

$\beta = \pi - \alpha$:

$$\lim_{\alpha \to \pi} \left(\frac{(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4})}{\pi - \alpha} \right) = \lim_{\beta \to 0} \left(\frac{(2\pi - \beta)(\sin \frac{\pi-\beta}{4} - \cos \frac{\pi-\beta}{4})}{\beta} \right)$$

By L'Hopital's Rule:

$$= \lim_{\beta \to 0} \left(\frac{\frac{1}{4}(2\pi - \beta)(-\sin \frac{\pi-\beta}{4} - \cos \frac{\pi-\beta}{4}) - (\sin \frac{\pi-\beta}{4} - \cos \frac{\pi-\beta}{4})}{1} \right) = \frac{-\pi}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \lim_{\alpha \to \pi} \left(\frac{(\pi+\alpha)(\sin \frac{\alpha}{4} - \cos \frac{\alpha}{4})}{\pi - \alpha} \right) + 2 = 2 - \frac{\pi}{2} < \pi$$

□

Proof of Theorem 1. In order for the triangle with $\alpha > \frac{\pi}{3}$ to go to a single point, x needs to go to 0 faster than α can go to π. Looking at the phase portrait for clues, we can see that this will happen if the magnitude of the slope of the path at any point representing such a triangle is greater than than the magnitude of the slope of the point to the point $(0, \pi)$. This would mean that the path is moving toward a point lower than $(0, \pi)$, so x would go to 0 faster than α would go to π.

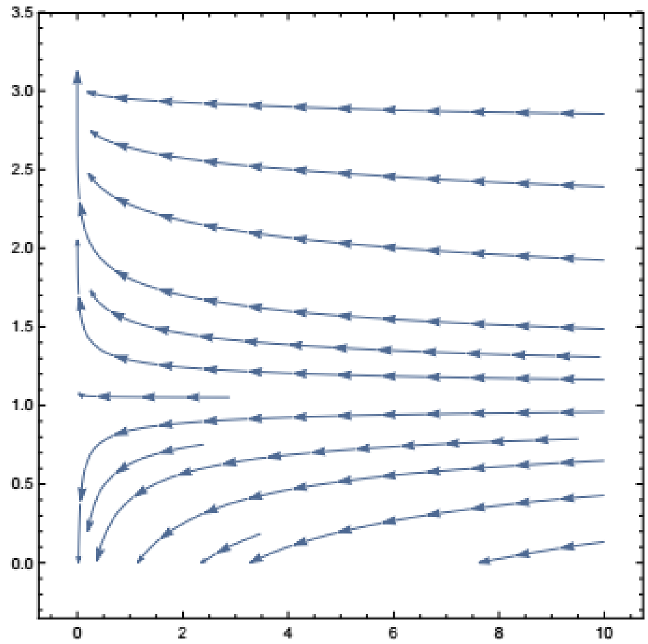

Figure 3: The phase plane diagram of α vs x

In order to prove this now, we want to show that

$$\frac{\dot{\alpha}}{\dot{x}} > \frac{\alpha - \pi}{x}$$

Or

$$\frac{\dot{\alpha}x}{\dot{x}} > \alpha - \pi$$

But we have already shown this for α near π, and this is really all we need to show. Because for $\alpha > \frac{\pi}{3}$, $\dot{\alpha} > 0$, and therefore α approaches π, so we only need to prove that $\frac{\dot{\alpha}x}{\dot{x}} > \alpha - \pi$ close to π when the curve shortening is almost complete in order to determine what the final result will be. And because $\frac{\dot{\alpha}x}{\dot{x}} > \alpha - \pi$, x goes to 0 faster than α goes to π, the isosceles triangles with $\alpha > \frac{\pi}{3}$ will become points. $\square$

3 Scalene Triangles

Theorem 2. *The end behavior of DCSF applied to scalene triangles is a line segment.*

We will first derive $\frac{dc}{dt}$, $\frac{d\alpha}{dt}$, and $\frac{d\beta}{dt}$ to prove this theorem.

We will now describe the behavior of a general triangle. We will label the triangle as seen below.

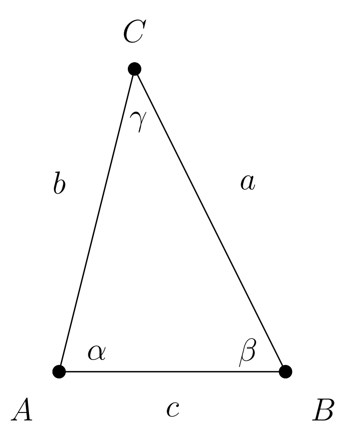

Figure 4: General triangle

We will need three parameters to uniquely describe a general triangle. Though we have many choices for what parameters to use, we will use α, β, and c.

Similar to the isosceles triangle, we will derive the differential equations by using the linear approximation of A, B, and C after a small time ϵ and using the limit as ϵ goes to zero to find the actual equations for $\frac{d\alpha}{dt}$, $\frac{d\beta}{dt}$, and $\frac{dc}{dt}$.

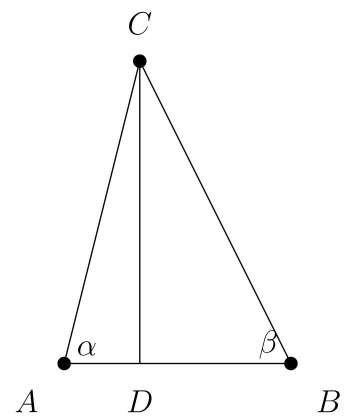

Figure 5: Height

We will say that C is at $(0,0)$. We know that the height CD from C to AB is $A\sin\beta$. By the Law of Sines, $\frac{A}{\sin\alpha} = \frac{C}{\sin\gamma}$. Thus, as $\gamma = \pi - (\alpha+\beta)$ and $\sin\theta = \sin(\pi-\theta)$, we know that $CD = A\sin\beta = \frac{c\sin\beta\sin\alpha}{\sin(\alpha+\beta)}$.

Thus, $AD = CD\cot\alpha = \frac{c\sin\beta\sin\alpha}{\sin(\alpha+\beta)} \times \frac{\cos\alpha}{\sin\alpha} = \frac{c\sin\beta\cos\alpha}{\sin(\alpha+\beta)}$ and $BD = CD\cot\beta = \frac{c\sin\beta\sin\alpha}{\sin(\alpha+\beta)} \times \frac{\cos\beta}{\sin\beta} = \frac{c\cos\beta\sin\alpha}{\sin(\alpha+\beta)}$. We now know enough to find the coordinates of A and B:

$$A = \left(-\frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)}, -\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\right)$$

$$B = \left(\frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)}, -\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\right)$$

In order to find $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, we must find the curvature and normal vector at A, B, and C. The curvature is relatively simple:

$$k(A) = \pi - \alpha$$

$$k(B) = \pi - \beta$$

$$k(C) = \pi - \gamma = \alpha + \beta$$

To find the normal vectors, we must first find the angles they form.

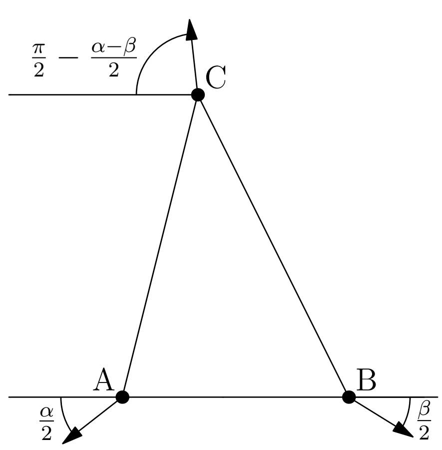

Figure 6: Normal vectors

We can see that, as the normal vectors are angle bisectors, they form angles of $\frac{\alpha}{2}$, $\frac{\beta}{2}$, and $\frac{\pi}{2} - \frac{\alpha-\beta}{2}$ with the horizontal, as shown above. Thus, we can find the normal vectors:

$$\vec{n}(A) = \left\langle -\cos\frac{\alpha}{2}, -\sin\frac{\alpha}{2}\right\rangle$$

$$\vec{n}(B) = \left\langle \cos\frac{\beta}{2}, -\sin\frac{\beta}{2}\right\rangle$$

$$\vec{n}(C) = \left\langle -\cos\left(\frac{\pi}{2} - \frac{\alpha-\beta}{2}\right), \sin\left(\frac{\pi}{2} - \frac{\alpha-\beta}{2}\right)\right\rangle = \left\langle -\sin\frac{\alpha-\beta}{2}, \cos\frac{\alpha-\beta}{2}\right\rangle$$

Notice that though the configuration shown has $\alpha > \beta$, $\vec{n}(C)$ is correct for all configurations as if $\alpha < \beta$, $\sin\frac{\alpha-\beta}{2}$ will be negative.

We can now use the curvatures and the normal vectors to find $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$:

$$\tilde{A} = \left(-\frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon(\pi-\alpha)\cos\frac{\alpha}{2}, -\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon(\pi-\alpha)\sin\frac{\alpha}{2}\right)$$

$$\tilde{B} = \left(\frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)} - \epsilon(\pi-\beta)\cos\frac{\beta}{2}, -\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon(\pi-\beta)\sin\frac{\beta}{2}\right)$$

$$\tilde{C} = \left(\epsilon(\alpha+\beta)\sin\frac{\alpha-\beta}{2}, -\epsilon(\alpha+\beta)\cos\frac{\alpha-\beta}{2}\right)$$

Using $\tilde{A}$ and $\tilde{B}$, we can construct an expression for $(\tilde{c})^2$ using the Pythagorean distance formula:

$$(\tilde{c})^2 = \left(c - \epsilon\left((\pi-\alpha)\cos\frac{\alpha}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right)\right)^2 + \left(\epsilon\left((\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right)^2$$

$$= c^2 - 2c\epsilon\left((\pi-\alpha)\cos\frac{\alpha}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right) + \epsilon^2\left((\pi-\alpha)^2 + (\pi-\beta)^2 + (\pi-\alpha)(\pi-\beta)\cos\frac{\alpha+\beta}{2}\right)$$

By the chain rule,

$$\frac{dc}{dt} = \frac{1}{2c}\frac{dc^2}{dt} = \frac{1}{2c}\lim_{\epsilon\to 0}\frac{(\tilde{c})^2 - c^2}{\epsilon}$$

$$= \frac{1}{2c}\lim_{\epsilon\to 0}\left(-2c\left((\pi-\alpha)\cos\frac{\alpha}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right) + \epsilon\left((\pi-\alpha)^2 + (\pi-\beta)^2 + (\pi-\alpha)(\pi-\beta)\cos\frac{\alpha+\beta}{2}\right)\right)$$

$$\Rightarrow \boxed{\frac{dc}{dt} = -(\pi-\alpha)\cos\frac{\alpha}{2} - (\pi-\beta)\cos\frac{\beta}{2}} \tag{3}$$

Note that for all $\alpha, \beta < \pi$ this gives a negative value.

Now we derive $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$. Using the formulas for $\tilde{A}$ and $\tilde{B}$ shown above, we can calculate the vector components of the three sides of $\tilde{A}\tilde{B}\tilde{C}$.

$$\tilde{A}\tilde{B} = \left\langle c - \epsilon\left((\pi-\alpha)\cos\frac{\alpha}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right), \epsilon\left((\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right\rangle$$

$$\tilde{A}\tilde{C} = \left\langle \frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} - (\pi-\alpha)\cos\frac{\alpha}{2}\right), \frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right\rangle$$

$$\tilde{B}\tilde{C} = \left\langle \epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} - (\pi-\beta)\cos\frac{\beta}{2}\right) - \frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)}, \frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\right\rangle$$

We know that

$$\tilde{A}\tilde{B}\cdot\tilde{A}\tilde{C} = ||\tilde{A}\tilde{B}|| \times ||\tilde{A}\tilde{C}|| \times \cos\tilde{\alpha}$$

so

$$\cos\tilde{\alpha} = \frac{\tilde{A}\tilde{B}\cdot\tilde{A}\tilde{C}}{||\tilde{A}\tilde{B}|| \times ||\tilde{A}\tilde{C}}||$$

$$= \frac{\left(c - \epsilon\left((\pi-\alpha)\cos\frac{\alpha}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right)\right)\left(\frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} - (\pi-\alpha)\cos\frac{\alpha}{2}\right)\right)}{\sqrt{(\tilde{c})^2\left(\left(\frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} - (\pi-\alpha)\cos\frac{\alpha}{2}\right)\right)^2 + \left(\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right)^2\right)}}$$

$$+ \frac{\left(\epsilon\left((\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right)\left(\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right)}{\sqrt{(\tilde{c})^2\left(\left(\frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} - (\pi-\alpha)\cos\frac{\alpha}{2}\right)\right)^2 + \left(\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\right)^2\right)}}$$

We also know that

$$\frac{d}{dt}\cos\alpha = -\sin\alpha\frac{d\alpha}{dt} = \lim_{\epsilon\to 0}\frac{\cos\tilde{\alpha} - \cos\alpha}{\epsilon}$$

so

$$\frac{d\alpha}{dt} = \frac{-1}{\sin\alpha}\lim_{\epsilon\to 0}\frac{\cos\tilde{\alpha} - \cos\alpha}{\epsilon}$$

As $\lim_{\epsilon\to 0}\cos\tilde{\alpha} - \cos\alpha = 0$, we can use L'Hopital's Rule. This gives

$$\frac{d\alpha}{dt} = \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\cos\tilde{\alpha}$$

Going through the calculations (using Mathematica), we get that

$$\frac{d\alpha}{dt} = -\frac{\left((\alpha+\beta)\sin\alpha\cos\alpha\cos\frac{\alpha-\beta}{2} + (\pi-\alpha)\sin\alpha\cos\alpha\sin\frac{\alpha}{2} - (\pi-\alpha)\sin^2\alpha\cos\frac{\alpha}{2} + (\alpha+\beta)\sin^2\alpha\sin\frac{\alpha-\beta}{2}\right)\sin(\alpha+\beta)}{c\sin\alpha\sin\beta}$$

$$+ \frac{\left((\pi-\alpha)\frac{\sin\alpha\sin\beta\sin\frac{\alpha}{2}}{\sin(\alpha+\beta)} - (\pi-\beta)\frac{\sin\alpha\sin\beta\sin\frac{\beta}{2}}{\sin(\alpha+\beta)})\right)\sin(\alpha+\beta)}{c\sin\alpha\sin\beta}$$

Simplifying, we get

$$\boxed{\frac{d\alpha}{dt} = \frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\sin(\alpha+\beta)}{c\sin\beta} + \frac{(\pi-\alpha)\sin\frac{\alpha}{2} - (\pi-\beta)\sin\frac{\beta}{2}}{c}} \tag{4}$$

We also know that

$$\tilde{B}\tilde{A}\cdot\tilde{B}\tilde{C} = ||\tilde{A}\tilde{B}|| \times ||\tilde{A}\tilde{C}|| \times \cos\tilde{\beta}$$

so

$$\cos \tilde{\beta} = \frac{\tilde{B}\tilde{A} \cdot \tilde{B}\tilde{C}}{||\tilde{A}\tilde{B}|| \times ||\tilde{A}\tilde{C}||}$$

$$= \frac{\left(\epsilon\left((\pi-\alpha)\cos\frac{\alpha}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right) - c\right)\left(\epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right) - \frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)}\right)}{\sqrt{(\tilde{c})^2\left(\left(\epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right) - \frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)}\right)^2 + \left(\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\right)^2\right)}}$$

$$+ \frac{\left(\epsilon\left((\pi-\alpha)\sin\frac{\alpha}{2} - (\pi-\beta)\sin\frac{\beta}{2}\right)\right)\left(\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\right)}{\sqrt{(\tilde{c})^2\left(\left(\epsilon\left((\alpha+\beta)\sin\frac{\alpha-\beta}{2} + (\pi-\beta)\cos\frac{\beta}{2}\right) - \frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)}\right)^2 + \left(\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} - \epsilon\left((\alpha+\beta)\cos\frac{\alpha-\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\right)^2\right)}}$$

Also,

$$\frac{d}{dt}\cos\beta = -\sin\beta\frac{d\beta}{dt} = \lim_{\epsilon\to 0}\frac{\cos\tilde{\beta} - \cos\beta}{\epsilon}$$

so

$$\frac{d\beta}{dt} = \frac{-1}{\sin\beta}\lim_{\epsilon\to 0}\frac{\cos\tilde{\beta} - \cos\beta}{\epsilon}$$

By L'Hopital's Rule,

$$\frac{d\beta}{dt} = \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\cos\tilde{\beta}$$

Calculating this using Mathematica, we get

$$\frac{d\beta}{dt} = -\frac{\left((\alpha+\beta)\sin\beta\cos\beta\cos\frac{\alpha-\beta}{2} + (\pi-\beta)\sin\beta\cos\beta\sin\frac{\beta}{2} - (\pi-\beta)\sin^2\beta\cos\frac{\beta}{2} - (\alpha+\beta)\sin^2\beta\sin\frac{\alpha-\beta}{2}\right)\sin(\alpha+\beta)}{c\sin\alpha\sin\beta}$$

$$+ \frac{\left(-(\pi-\alpha)\frac{\sin\alpha\sin\beta\sin\frac{\alpha}{2}}{\sin(\alpha+\beta)} + (\pi-\beta)\frac{\sin\alpha\sin\beta\sin\frac{\beta}{2}}{\sin(\alpha+\beta)}\right)\sin(\alpha+\beta)}{c\sin\alpha\sin\beta}$$

Simplifying, we get

$$\boxed{\frac{d\beta}{dt} = \frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\sin(\alpha+\beta)}{c\sin\alpha} + \frac{(\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2}}{c}} \tag{5}$$

Looking at the phase space defined by the differential equations for α, β, and c, we note that both $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ are $\frac{1}{c}$ times a formula in only α and β, so in the phase space the α and β components have the same ratio independent of the value of c. Thus, we can analyze any cross section of this phase space parallel to the $\alpha - \beta$ plane, one of which is shown below, in order to analyze how α and β change.

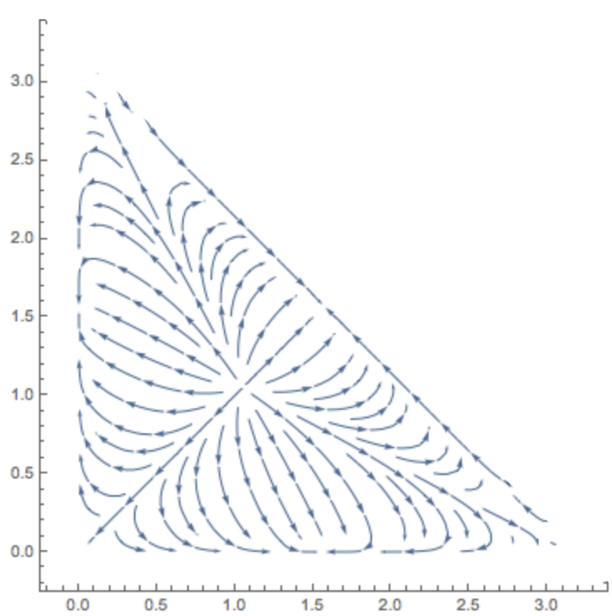

Figure 7: Cross section of 3D phase space diagram: α axis is horizontal, β axis is vertical

Lemma 2.1. *Every triangle can be represented by a point inside each of the six triangles formed by the medians of from the triangle defined by $\alpha = 0$, $\beta = 0$, and $\alpha + \beta = \pi$ in the α-β phase space.*

Proof. We note that this phase plane has 6 symmetries, as any ordering of the 3 angles of the triangle is congruent. Because of this, the phase plane can split into six subsections of equal area that all represent triangles with equivalent behavior under curve shortening flow. As the cross section of the phase space is a triangle defined by $\alpha = 0$, $\beta = 0$, and $\alpha + \beta = \pi$ (or $\gamma = 0$), we will prove that the six smaller triangles formed by the medians of the larger triangle can be these subsections.

The medians of the triangular phase plane are $\alpha = \beta$, $\alpha = \gamma$ and $\beta = \gamma$. We can see that which side of a median a point is on determines definitively the relationship between the two variables equated in that median. For example, every point below $\alpha = \beta$ has the property that $\alpha > \beta$, and every point above $\beta = \gamma$ has $\beta > \gamma$. Thus, each point not on a median has an ordering of α, β, and γ determined uniquely by which smaller triangle it is in.

Thus, all scalene triangles have one point representing them in each of the six smaller triangles of the phase plane, representing each permutation of the angles in the triangle. It will suffice, then, to prove that scalene triangles in one of these regions will go to line segments. We will use the region with $\gamma > \alpha > \beta$, the lower left triangle of the phase plane.

□

Lemma 2.2. *$\frac{d\alpha}{dt} > \frac{d\beta}{dt}$ in the region of interest.*

Proof. Define

$$f(x) = (\pi - x)\sin\frac{x}{2}.$$

First we will prove that for all points (α, β) in this region, $f(\alpha) > f(\beta)$.

We find that

$$f'(x) = \frac{1}{2}(\pi - x)\cos\frac{x}{2} - \sin\frac{x}{2}$$

and

$$f''(x) = -\frac{1}{4}(\pi - x)\sin\frac{x}{2} - \cos\frac{x}{2}.$$

As $\alpha, \beta < \frac{\pi}{2}$ for all points in the region, $f''(x) < 0$ for all $x = \alpha$ and $x = \beta$, so $f'(x)$ is monotonically decreasing.

Since $f'(\frac{\pi}{3}) = \frac{\pi}{6} - \frac{1}{2} > 0$, f' is positive for all $x \leq \frac{\pi}{3}$. As $\alpha > \beta$, any point with $\alpha, \beta \leq \frac{\pi}{3}$ will have $f(\alpha) > f(\beta)$. In addition, as $\beta < \frac{\pi}{3}$ in the region and $f(\frac{\pi}{3}) = \frac{\pi}{3}$, $f(\beta) < \frac{\pi}{3}$.

However, as f' is decreasing, it becomes negative between $\frac{\pi}{3}$ and $\frac{\pi}{2}$. Thus, after $\frac{\pi}{3}$ $f(x)$ starts increasing and then reaches a maximum and afterward decreases. If $f(x) \leq \frac{\pi}{3}$ for any $\frac{\pi}{3} < x < \frac{\pi}{2}$, then $f(\frac{\pi}{2}) \leq \frac{\pi}{3}$. However, $f(\frac{\pi}{2}) = \frac{\pi}{2\sqrt{2}} > \frac{\pi}{3}$, so all α between $\frac{\pi}{3}$ and $\frac{\pi}{2}$ have $f(\alpha) > \frac{\pi}{3}$. Thus, $f(\alpha) > f(\beta)$ for all (α, β) in the region.

Substituting the expressions for $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$, we get that we are trying to prove that

$$\frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\sin(\alpha+\beta)}{c\sin\beta} + \frac{(\pi-\alpha)\sin\frac{\alpha}{2} - (\pi-\beta)\sin\frac{\beta}{2}}{c} >$$

$$\frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\sin(\alpha+\beta)}{c\sin\alpha} + \frac{(\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2}}{c}$$

in the region.

This is equivalent to

$$\frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + f(\alpha)\right)\sin(\alpha+\beta)}{\sin\beta} + f(\alpha) - f(\beta) > \frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + f(\beta)\right)\sin(\alpha+\beta)}{\sin\alpha} + f(\beta) - f(\alpha)$$

As $f(\alpha) > f(\beta)$,

$$\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + f(\alpha)\right)\sin(\alpha+\beta) > \left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + f(\beta)\right)\sin(\alpha+\beta).$$

Also, as $\alpha, \beta < \frac{\pi}{2}$, sin is monotonically increasing, so since $\alpha > \beta$, $\sin\alpha > \sin\beta$. This means that

$$\frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + f(\alpha)\right)\sin(\alpha+\beta)}{\sin\beta} > \frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + f(\beta)\right)\sin(\alpha+\beta)}{\sin\alpha}.$$

Finally, $f(\alpha) > f(\beta)$, so $f(\alpha) - f(\beta) > f(\beta) - f(\alpha)$. This means that $\frac{d\alpha}{dt} > \frac{d\beta}{dt}$ in the region.

□

Lemma 2.3. *$\frac{d\beta}{dt} < 0$ in the region of interest.*

Proof. First we see that if $c\frac{d\beta}{dt} < 0$, $\frac{d\beta}{dt} < 0$, as c is always positive. Using this, we are now trying to prove that

$$\frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\sin(\alpha+\beta)}{\sin\alpha} + (\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2} < 0$$

Using the same $f(x)$ as defined above, we see this is equivalent to

$$\frac{(f(\beta) - f(\pi-\alpha-\beta))\sin(\alpha+\beta)}{\sin\alpha} + f(\beta) - f(\alpha) < 0$$

We already know that $f(\alpha) > f(\beta)$, so $f(\beta) - f(\alpha) < 0$. Now, in the region $\sin(\alpha) > 0$ and $\sin(\alpha+\beta) > 0$, so if we can prove that $f(\beta) < f(\pi - \alpha - \beta)$, we are done.

First, we notice that $\pi - \alpha - \beta$ is γ, the third angle of the triangle. Notice that, $0 < \beta < \frac{\pi}{3} < \gamma < \pi$, in the region, so for all $x = \beta$ and $x = \gamma$, $f''(x) < 0$, so $f(x)$ is concave down.

Now, consider all the possibilities for γ for a fixed β, say β_0. Well, $\frac{\pi-\beta_0}{2} < \gamma < \pi - 2\beta_0$ in the region. But, from above, we know that if $\frac{\pi}{3} < \gamma \leq \frac{\pi}{2}$, then $f(\gamma) > f(\beta)$, so now all we need to consider are values $\frac{\pi}{2} < \gamma < \pi - 2\beta_0$. But because $f(x)$ is decreasing past $\frac{\pi}{2}$, we really only need consider $\pi - 2\beta_0$. So, in order to show that $f(\gamma) > f(\beta)$ in this area, we need to show that $f(\pi - 2x) > f(x)$ for $0 < x < \frac{\pi}{3}$, or

$$2x\cos(x) > (\pi - x)\sin\frac{x}{2}$$

We see that these two are equal at $x = 0$ and $x = \frac{\pi}{3}$. Using the bound that $\frac{x}{2} > \sin\frac{x}{2}$ for $x > 0$, we know:

$$2x\cos(x) > (\pi - x)\frac{x}{2}$$

$$\Rightarrow 4\cos(x) > \pi - x$$

$4\cos(x)$ is decreasing on the interval $0 < x < \frac{\pi}{2}$ and $\pi - x$ is always increasing. Thus, as this is true at 0 (as $4 > \pi$) and $\frac{\pi}{4}$ (as $\frac{4}{\sqrt{2}} > \frac{3}{4}\pi$), it is true on the interval $[0, \frac{\pi}{4}]$. Now, it is also true a little bit past $\frac{\pi}{4}$, but it's more useful to only consider up to $\frac{\pi}{4}$, as this is easier to deal with.

Now that $2x\cos(x) > (\pi - x)\sin\frac{x}{2}$ is proven for $0 < x < \frac{\pi}{4}$, we need to prove it also for $\frac{\pi}{4} \leq x < \frac{\pi}{3}$. One way we can do this is show that the difference, lets call it $g(x) = 2x\cos(x) - (\pi - x)\sin\frac{x}{2}$ has endpoints ≥ 0 on the interval, and that over the whole interval it is concave.

$$g''(x) = \frac{d^2}{dx^2}\left(2x\cos(x) - (\pi - x)\sin\frac{x}{2}\right) = \frac{1}{4}(\pi - x)\sin\frac{x}{2} - 4\sin(x) + \cos\frac{x}{2} - 2x\cos(x)$$

On the interval $[\frac{\pi}{4}, \frac{\pi}{3}]$, we can find maximum of each term:

$$\frac{1}{4}(\pi - x)\sin\frac{x}{2} \leq \frac{1}{4}\left(\frac{3}{4}\pi\right)\left(\frac{1}{2}\right) = \frac{3}{32}\pi$$

$$4\sin(x) \geq 2\sqrt{2}$$

$$\cos\frac{x}{2} \leq \frac{\sqrt{3}}{2}$$

$$2x\cos(x) \geq 2\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

Thus, on $[\frac{\pi}{4}, \frac{\pi}{3}]$:

$$\frac{1}{4}(\pi - x)\sin\frac{x}{2} - 4\sin(x) + \cos\frac{x}{2} - 2x\cos(x) \leq \frac{3}{32}\pi - 2\sqrt{2} + \frac{\sqrt{3}}{2} - \frac{\pi}{4} = \frac{-5}{32}\pi + \frac{\sqrt{3} - 4\sqrt{2}}{2} < 0$$

Thus, as $g''(x)$ is always negative for $\frac{\pi}{4} \leq x < \frac{\pi}{3}$, $g(x)$, the difference function, is concave down on that interval.

We calculate that $g(\frac{\pi}{4}) = \frac{\sqrt{2}}{4}\pi - \frac{3}{4}\pi\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = \frac{\pi}{16}(7\sqrt{2} - 3\sqrt{6}) > 0$ and $g(\frac{\pi}{3}) = 0$. Thus, as g is concave down on that interval, it must be only positive. This finally means that for $\frac{\pi}{4} \leq x < \frac{\pi}{3}$, $2x\cos(x) > (\pi - x)\sin\frac{x}{2}$, so this is true for all of $0 < x < \frac{\pi}{3}$.

This, therefore, means that $f(\pi - 2x) > f(x)$ for $0 < x < \frac{\pi}{3}$, so $f(\gamma) > f(\beta)$ over the region of interest, meaning that, finally, $\frac{d\beta}{dt} < 0$. □

Proof of Theorem 2. Through lemmas 2.1-2.3, we've shown that the behavior of all scalene triangles under curve shortening flow can be described by the behavior of those in a certain region of the phase plane, and that in that region, $\frac{d\alpha}{dt} > \frac{d\beta}{dt}$ and $\frac{d\beta}{dt} < 0$. We also know that $\frac{dc}{dt} < 0$.

As $\frac{d\beta}{dt}$ is strictly negative, we know that there are no fixed points in the region. Also, since $\frac{d\alpha}{dt} > \frac{d\beta}{dt}$ for every point in the region, it is impossible for any trajectory to approach the point $(\alpha, \beta) = (0, 0)$. This is because every vector points in the negative β direction, but cannot have $\frac{d\alpha}{dt}$ more negative than $\frac{d\beta}{dt}$, which is necessary for any vector to point toward $(0, 0)$.

Thus, every trajectory must end either on $\beta = 0$ or must go to the line $\beta = \pi - 2\alpha$ on the right boundary on the region. This boundary represents isosceles triangles with top angle less than $\frac{\pi}{3}$, which we already know result in line segments. The line $\beta = 0$ for $\alpha \neq 0$ also represents line segments, as long as $c > 0$. Thus, if c does not go to 0 before α and β reach their final values, all scalene triangles must result in line segments under DCSF.

Now we will compare $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ to $\frac{dc}{dt}$ as c gets closer to 0. For any α and β, we know that $\frac{\frac{d\alpha}{dt}}{\frac{dc}{dt}} =$

$$\frac{1}{c}\left(\frac{\frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\sin(\alpha+\beta)}{\sin\beta} + (\pi - \alpha)\sin\frac{\alpha}{2} - (\pi - \beta)\sin\frac{\beta}{2}}{-(\pi - \alpha)\cos\frac{\alpha}{2} - (\pi - \beta)\cos\frac{\beta}{2}}\right)$$

When we fix α and β, the entire expression multiplied by $\frac{1}{c}$ is constant as c is changing. Thus, $\frac{1}{c}$ determines the size of the expression, as long as $\frac{d\alpha}{dt} \neq 0$. Since we are only considering scalene triangles, it is never zero. Thus, for any fixed α and β in the region

$$\lim_{c \to 0} \frac{\left(\frac{d\alpha}{dt}\right)}{\left(\frac{dc}{dt}\right)} = \pm\infty$$

where the sign depends on the sign of $\frac{d\alpha}{dt}$. With a similar argument, we can see that

$$\lim_{c \to 0} \frac{\left(\frac{d\beta}{dt}\right)}{\left(\frac{dc}{dt}\right)} = \infty$$

if $\alpha_1 \neq \beta_1$.

This then means that the β does ultimately go to 0 and α to some angle α_1 much faster than c can go to 0. This end state, with a still nonzero value for c once β has gone to 0, is a line segment, and therefore triangles in this region go to lines. But, we have shown that all triangles can be represented by a point in this region, so all scalene triangles go to lines under DCSF. $\square$

References

[1] Gage M, Hamilton R S. The heat equation shrinking convex plane curves[J]. Journal of Differential Geometry, 1986, 23(1): 69-96.

[2] Grayson M A. The heat equation shrinks embedded plane curves to round points[J]. Journal of Differential geometry, 1987, 26(2): 285-314.

[3] Ramanujam A. Properties Of Triangles When They Undergo The Curve-Shortening Flow. Research Science Institute Project 2016.