The Defect Angle and the Relation to the Laplacian Matrix

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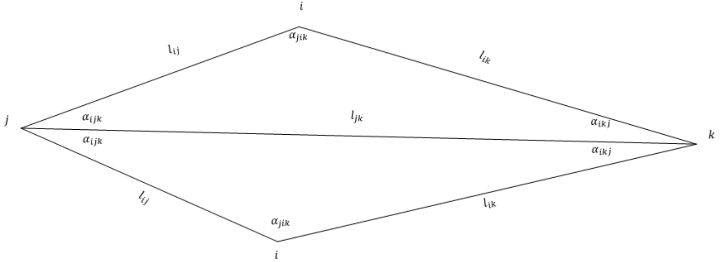
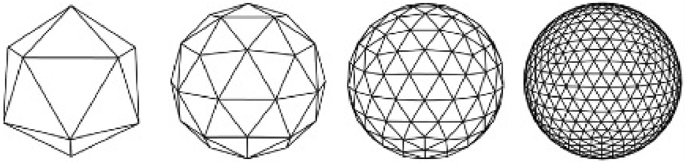
Finite Triangulation

- Tiling of any two-dimensional surface with triangles
- Pillow triangulation of a sphere



Finite Triangulation

- Approximation of smooth surface improves as number of triangles increases



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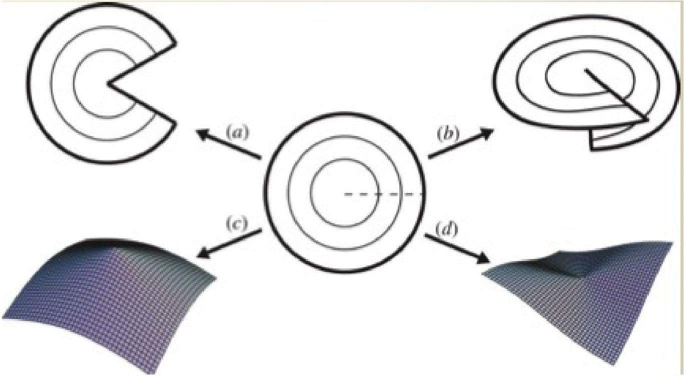
Defect Angle (ϵ_i)

- A measure of the angle "missing" from a vertex of the triangulation

$$\epsilon_i = 2\pi - \sum_{(j,k)|(i,j,k)\in F} (\alpha_{jik})$$

Defect Angle



Demonstration of angular deficit and surplus

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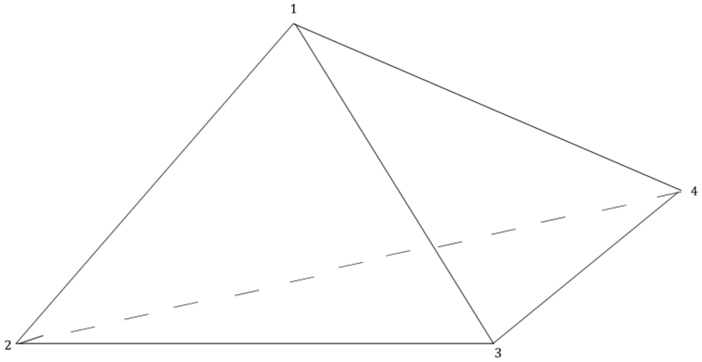
Defect Angle



Defect angles in a regular tetrahedron. The defect angle at each vertex is π because there are 3 angles measuring $\frac{\pi}{3}$.

What is Γ?

- A function on a finite triangulation, determined by A. Ko and M. Roček, equal to:

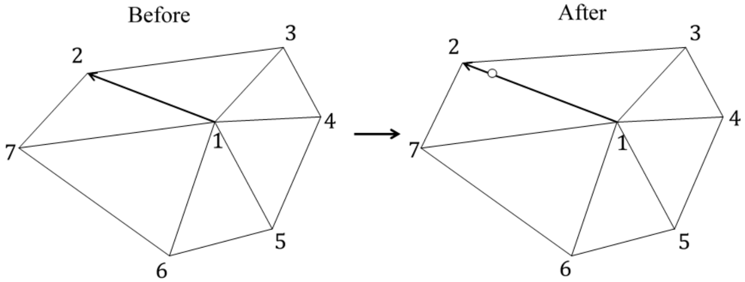
$$\Gamma = \frac{1}{12\pi}\left[\sum_{\angle ijk}\left(\int_{\frac{\pi}{2}}^{\alpha_{ijk}}\left(y-\frac{\pi}{3}\right)\cot y\, dy\right)+\sum_{\langle ij\rangle} 2k_{ij}\pi \ln\left(\frac{\ell_{ij}}{\ell_0}\right)\right]$$

  - Constants k_{ij} are defined such that

$$\sum_{j|\langle ij\rangle\in E} k_{ij} = 1-\frac{n_i}{6}$$

    where n_i is the number of vertices adjacent to i.
  - Defined so that if we change each edge ℓ_{ij} at i by $\alpha_i\ell_{ij}$, then Γ will change proportionally to $\alpha_i\epsilon_i$.

Rescaling a vertex



The effect of rescaling $\langle 1, 2 \rangle$ on the triangulation. Only two of the six triangles are affected, showing the locality of the rescaling and Γ.

Phi Quantities (Φ_i)

- Consider a triangulation where the edge lengths are given by

$$\ell_{ij} = \ell_{ij}^0 e^{\Phi_i + \Phi_j}$$

  - We can rescale the triangulation's edge lengths by adding constants to Φ_i and Φ_j.
  - $\frac{\partial \Gamma}{\partial \Phi_i}$ is proportional to ϵ_i.

Project objective

- The principal problem we are investigating:
  - What relations can we find between the properties of discrete triangulations and those of smooth surfaces?
- An interesting question we explored in passing:
  - What is the Taylor series of Γ, and what information about a triangulation is conveyed in its coefficients?

Methodology and Procedure

- **Methodology**
  - Multivariate calculus
- **Procedure**
  - Second-order Taylor series with respect to the Φ values
  - Laplace operator and Laplace matrix

Laplace operator (∇^2)

- The energy functional is given by an integral involving the Laplace operator $\nabla^2\Psi$ on an arbitrary function Ψ:

$$-\iint_D \Psi\nabla^2\Psi \, dx\, dy$$

- Integrating by parts we can rewrite this as:

$$\iint_D (\nabla\Psi \cdot \nabla\Psi) \, dx\, dy$$

Laplacian matrix (L)

- Discrete analogue of the Laplace operator
- Acts on a matrix Φ by matrix multiplication ($L \cdot \Phi$)
- Xianfeng Gu et al. define a modified Laplace matrix:

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum_k w_{ik} & \text{if } i = j \end{cases}, w_{ij} = \begin{cases} \frac{\cot \alpha_{ikj}}{2} & \text{if } [v_i, v_j] \in \partial M \\ \sum_{k|(i,j,k)\in F} \frac{\cot \alpha_{ikj}}{2} & \text{if } [v_i, v_j] \notin \partial M \end{cases}$$

- The w_{ij} terms are named *cotangent edge weights*.

Cotangent edge weights

Interior edge $\langle ij \rangle$ has

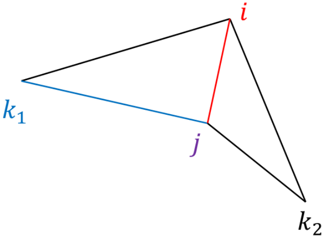
$$w_{ij} = \frac{1}{2}\left(\cot \alpha_{ik_1j} + \cot \alpha_{ik_2j}\right)$$

so $L_{ij} = -\frac{1}{2}\left(\cot \alpha_{ik_1j} + \cot \alpha_{ik_2j}\right)$

Boundary edge $\langle jk_1 \rangle$ has

$$w_{jk_1} = \frac{1}{2}\cot \alpha_{jik_1}$$

so $L_{jk_1} = -\frac{1}{2}\cot \alpha_{jik_1}$



$L_{k_1k_1} = w_{ik_1} + w_{jk_1}$
$= \frac{1}{2}\left(\cot \alpha_{jik_1} + \cot \alpha_{ijk_1}\right)$

As k_1 and k_2 are not adjacent, $L_{k_1k_2} = 0$.

Calculation of the Taylor series

- We need to compute the first and second derivatives of Γ with respect to the Φ quantities.
  - By construction, the first derivative of Γ with respect to Φ_i is:

$$\frac{\partial \Gamma}{\partial \Phi_i} = \epsilon_i$$

Calculation of the Taylor series

- There are two cases of second derivatives of Γ with respect to Φ :
  - "On-diagonal": the second derivatives of the form

$$\frac{\partial^2 \Gamma}{\partial \Phi_i^2}$$

  - "Off-diagonal": the second derivatives of the form

$$\frac{\partial^2 \Gamma}{\partial \Phi_i \cdot \partial \Phi_j}$$

  - We use the defect angle to compute these derivatives.

Calculation of the Taylor series

- We start by differentiating an angle of a triangle, α_{jik}.
- We use the Law of Cosines:

$$\cos\alpha_{jik} = \frac{\ell_{ij}^2 e^{2\Phi_i+2\Phi_j} + \ell_{ik}^2 e^{2\Phi_i+2\Phi_j} - \ell_{jk}^2 e^{2\Phi_j+2\Phi_k}}{2\ell_{ij}e^{\Phi_i+\Phi_j}\ell_{ik}e^{\Phi_i+\Phi_k}}$$

- By differentiating both sides with respect to a Φ value, we can isolate the derivative of the angle.

Calculation of the Taylor series

- To recapitulate, we have the following derivatives:

$$\frac{\partial \Gamma}{\partial \Phi_i} = \epsilon_i$$

$$\frac{\partial^2 \Gamma}{\partial \Phi_i^2} = -\sum_{i|(i,j,k)\in F} \cot \alpha_{ikj} + \cot \alpha_{ijk}$$

$$\frac{\partial^2 \Gamma}{\partial \Phi_i \cdot \partial \Phi_j} = -\sum_{i,j|(i,j,k)\in F} \cot \alpha_{ikj}$$

- Substituting these into the general formula of a multivariable Taylor series, we finish the derivation.

Taylor series

- The Taylor series of Γ to the second order is calculated to be:

$$\Gamma = \Gamma_0 + \frac{1}{12\pi}\left(\sum_{i\in V}(\epsilon_i \Phi_i)\right.$$
$$+\sum_{i\in V}\left(\sum_{j,k|(i,j,k)\in F}\left(\frac{\cot\alpha_{ijk}+\cot\alpha_{ikj}}{2}\right)\Phi_i^2\right)$$
$$\left.+\sum_{\langle ij\rangle\in E}\left(\sum_{k|(i,j,k)\in F}(-\cot\alpha_{ikj})\,\Phi_i\Phi_j\right)\right)$$

- This series had not been calculated previously.

Laplace operator

- M. Roček and R. M. Williams calculated the previous integral

$$\int\int_D (\nabla\Phi \cdot \nabla\Phi)\, dx\, dy$$

- They determined that this is equal to:

$$\frac{1}{2}\Bigg(\left(\frac{\cot\alpha_2 + \cot\alpha_3}{2}\right)\Phi_1^2 + \left(\frac{\cot\alpha_1 + \cot\alpha_3}{2}\right)\Phi_2^2 + \left(\frac{\cot\alpha_1 + \cot\alpha_2}{2}\right)\Phi_3^2 + (-\cot\alpha_3)\,\Phi_1\Phi_2 + (-\cot\alpha_2)\,\Phi_1\Phi_3 + (-\cot\alpha_1)\,\Phi_2\Phi_3 \Bigg)$$

which is proportional to the second-order terms of the Taylor series of Γ for a pillow triangulation by a factor of 6π.

Results

- The three quantities we have discussed are all equal!
  - The coefficients of second-order terms of the Taylor series of Γ
  - The entries of the Laplace matrix
  - The coefficients of the expansion of the energy functional

Further Research

- In general, further research topics are those which discretize other continuous concepts.
  - Cauchy-Riemann Equation - useful tool in the continuous case; holomorphic criteria for complex functions. Holomorphic functions give solutions to the Liouville equation.
  - Liouville theory-concerns solutions to the Liouville equation in the continuous case.

Conclusion

- We determine the Taylor series expansion of Γ with respect to Φ quantities.
- We verify that the Taylor series expansion, the gradient integral, and the Laplace matrix are (up to proportionality factors) equivalent.

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