Circular Planar Graphs and Electrical Networks

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PRIMES

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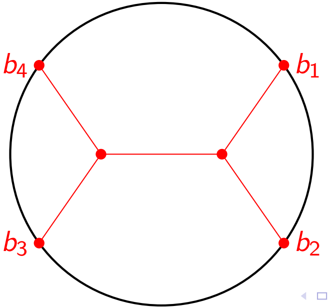
Circular Planar Graphs

Definition

A **circular planar graph** *is a collection of vertices V and edges E between vertices that can be embedded in a disc with designated boundary vertices on the circle of the disc. Edges intersect at vertices. The order n is the number of boundary vertices.*



Electrical Networks

- Modelling electrical networks.

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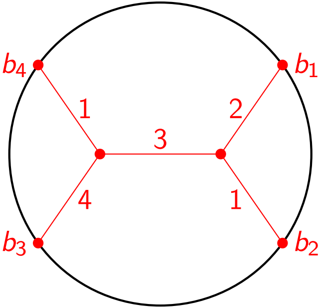
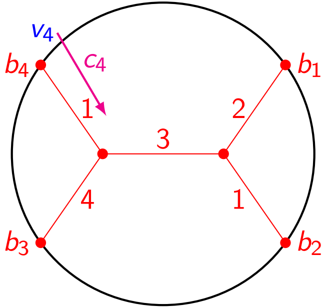
- Replace edges with resistors: to each edge e assign a positive real number $\gamma(e)$.



Experiment

- Place batteries at boundary vertices. What are the currents at the boundary nodes?



Network Response Matrix

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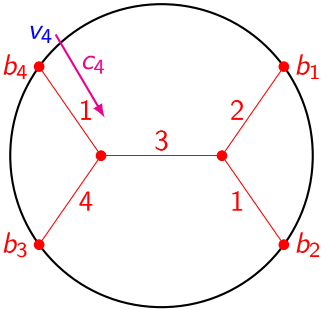
- Current through each internal vertex is zero
- Linear map, Dirichlet-to-Neumann map, sending voltages to current at boundary vertices. (Curtis, Ingerman, Morrow, '98)
- We can define a **network response matrix** for our electrical network. Example:

$$\frac{1}{39}\begin{bmatrix} -46 & 16 & 24 & 6 \\ 16 & -31 & 12 & 3 \\ 24 & 12 & -60 & 24 \\ 6 & 3 & 24 & -33 \end{bmatrix}$$

Network Response Matrix



$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \frac{1}{39} \begin{bmatrix} -46 & 16 & 24 & 6 \\ 16 & -31 & 12 & 3 \\ 24 & 12 & -60 & 24 \\ 6 & 3 & 24 & -33 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

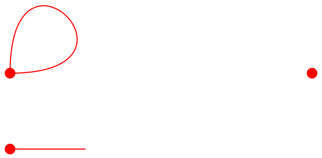
Equivalent Networks

Definition

Two electrical networks are **equivalent** *if they have the same response matrices.*

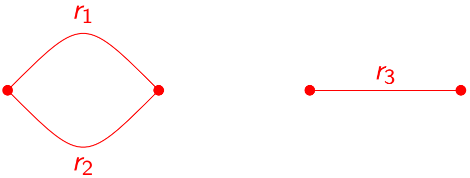
Local Equivalences

Self-loop and Spike Removal



Local Equivalences

Parallel Edges



$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3}$$

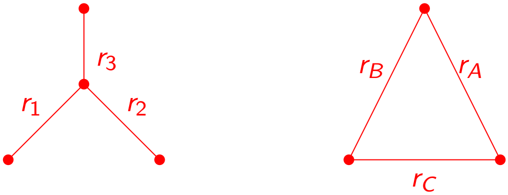
Local Equivalences

Series Edges

$$r_1 + r_2 = r_3$$

Local Equivalences

Y-Δ Transformations



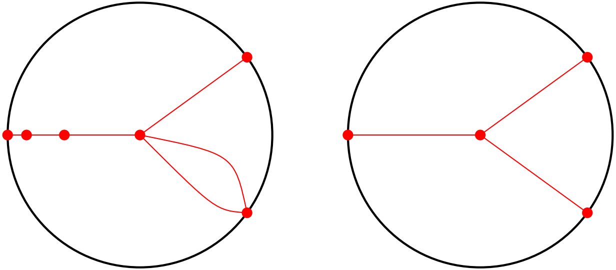
$$r_1 = \frac{r_B r_C}{r_A + r_B + r_C},\ r_2 = \frac{r_A r_C}{r_A + r_B + r_C},\ r_3 = \frac{r_A r_B}{r_A + r_B + r_C}$$

Theorem (de Verdiére, Gitler, Vertigan, '96)

Equivalent networks are related by these equivalence moves.

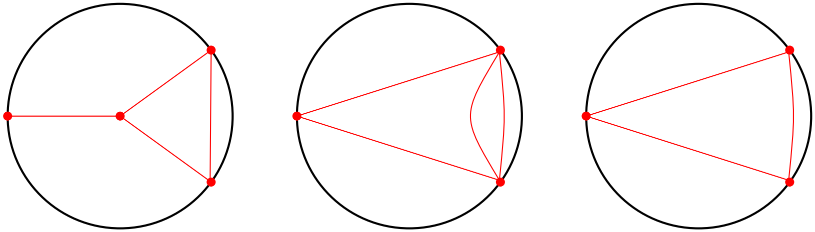
Equivalence classes for circular planar graphs

Critical Graphs



A critical graph is a graph in an equivalence class with the smallest number of edges.

Critical Graphs



Inverse Boundary Problem

Given a network response matrix and the underlying circular planar graph, can we recover the original resistances?

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Given a network response matrix and the underlying circular planar graph, can we recover the original resistances?

Exactly when the graph is critical! (Curtis, Ingerman, Morrow, '98)

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Equivalent critical graphs are related by Y-Δ transformations. How efficiently can this be done?

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Theorem

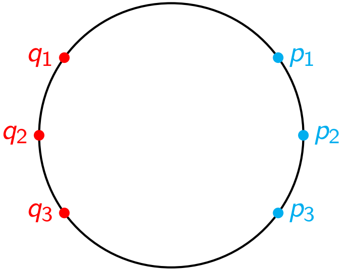
The diameter of an equivalence class is at most quartic in n.

Proof involves medial graphs and reduced decompositions.

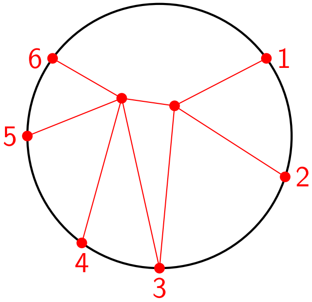
Circular Pairs

Definition

A **circular pair** *is an ordered pair of sequences of vertices* $(P, Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ *such that* $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ *are in circular order. Roughly, a circular pair is* **connected** *if there are disjoint paths from* p_i *to* q_i*.*



Circular Pairs



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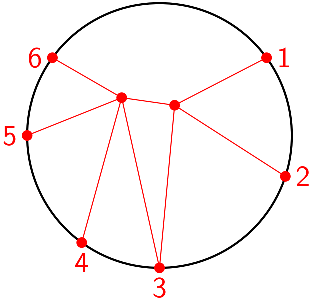

- $(5, 6; 4, 3)$ is an un-connected circular pair.

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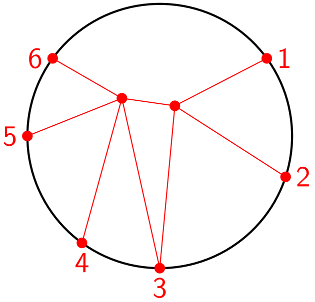

- $(5, 6; 4, 3)$ is an un-connected circular pair.
- $(5, 1; 3, 2)$ is a connected circular pair.

Circular Minors

Definition

For a circular pair (P, Q), the associated **circular minor** *is the determinant of the submatrix of the network response matrix with row set P and column set Q.*

Theorem (Curtis, Ingerman, Morrow, '98)

Minors of circular pairs that are connected are positive, and minors of those that are not connected are 0.

Positivity Tests

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Theorem (Kenyon, Wilson)

There exists a set of $\binom{n}{2}$ circular minors such that if all the minors in the set are positive, all circular minors are positive.

Positivity Tests

Conjecture (Kenyon, Wilson)

Fix a critical graph G with k edges. There exists a set S_1 of k circular minors and a set S_2 of $\binom{n}{2} - k$ minors such that if the elements of S_1 are known to be positive and the minors of S_2 are known to be 0, then the matrix is a response matrix for some electrical network with underlying graph G.

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Theorem

The conjecture holds for odd n for $k = \binom{n}{2} - 2$, $\binom{n}{2} - 1$. The minors are explicity constructed.

Future Directions

- Properties of EP_n

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- More general descriptions of positivity tests

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- More general descriptions of positivity tests
- Analogues of the totally non-negative Grassmannian: Weak separation, cluster algebras

Acknowledgements

Thanks to my mentor Carl, PRIMES, and my family.

Thanks for listening! Questions?