Signatures of the Contravariant Form on Representations of the Hecke Algebra and Rational Cherednik Algebra associated to $G(r,1,n)$

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Abstract

The Hecke algebra and rational Cherednik algebra of the group $G(r,1,n)$ are non-commutative algebras that are deformations of certain classical algebras associated to the group. These algebras have numerous applications in representation theory, number theory, algebraic geometry and integrable systems in quantum physics. Consequently, understanding their irreducible representations is important. If the deformation parameters are generic, then these irreducible representations, called Specht modules in the case of the Hecke algebra and Verma modules in the case of the Cherednik algebra, are in bijection with the irreducible representations of $G(r,1,n)$. However, while every irreducible representation of $G(r,1,n)$ is unitary, the Hermitian contravariant form on the Specht modules and Verma modules may only be non-degenerate. Thus, the signature of this form provides a great deal of information about the representations of the algebras that cannot be seen by looking at the group representations.

In this paper, we compute the signature of arbitrary Specht modules of the Hecke algebra and use them to give explicit formulas of the parameter values for which these modules are unitary. We also compute asymptotic limits of existing formulas for the signature character of the polynomial representations of the Cherednik algebra which are vastly simpler than the full signature characters and show that these limits are rational functions in t. In addition, we show that for half of the parameter values, for each k, the degree k portion of the polynomial representation is unitary for large enough n.

Contents

1 Introduction

In mathematics and physics, a fundamental notion is that of a group. Groups are the mathematical objects that represent symmetry and as such they are defined via the main properties of symmetries of spaces. The identity is always a symmetry of any space; given two symmetries, one can compose them to define a third and if we compose more than two symmetries then the order of composition is irrelevant; given a symmetry operation, an inverse for the operation exists and is also a symmetry.

Groups are often divided into two categories. On the one hand, we have the finite groups such as S_n, the group of permutations of the set $\{1,\ldots,n\}$. The group S_n in particular has a natural action via symmetry operations on the coordinate space of n identical particles and plays an important role in the geometry of this space. On the other end of the spectrum, we have Lie groups, which are continuous groups of symmetries such as the group of invertible real (or complex) $n\times n$ matrices, GL_n, which controls the geometry of n-dimensional real (complex) Euclidean space.

Since a standard idea in both mathematics and physics is to use the symmetries of a space to understand it better, the study of groups and their actions on physical systems or geometric spaces is extremely important. Unfortunately, this can be fairly difficult. So, as is usually the case, we replace the problem with a "linear approximation". This is where representation theory comes in.

A representation of a group G is a linear action of the group on a vector space. More precisely, an n-dimensional (complex) representation of G is a homomorphism $\rho : G \to GL(V)$ (i.e., a map ρ such that $\rho(ab) = \rho(a)\rho(b)$), where V is an n-dimensional complex vector space. If the group G is continuous, then the map ρ is required to be continuous as well. Thus, a representation can be viewed as a linearization of the group. As a small miracle, this linearization process does not lose much information. For example, using representation theory, one can classify all the simple Lie groups, which are the building blocks of all Lie groups (see [FH] for example). Additionally, the representations of symmetry groups on quantum mechanical systems yield a lot of information about the system, and the general structure of the periodic table can be explained using representation theory ([CS], [Sin]). The use of spherical harmonics in the quantum theory of angular momentum comes from the (infinite dimensional) representation theory of $SO(3)$, the group of rotations on the 2-sphere, while the theory of spin can be seen to be the representation theory of $SL(2,\mathbb{C})$, the group of 2×2-matrices with determinant 1 (see [Wey] for these applications and many more.)

Finite groups are not short of applications either. One extremely important application to number theory comes from Galois theory and Galois cohomology ([Ser]). Another application arises in the representation theory of Lie groups: every reductive Lie group has an associated finite group called its Weyl group which, to a large extent, controls the representation theory of the Lie group ([FH]). In the case of GL_m, the Weyl group is S_n and there is an extremely strong correspondence between the representation theory of S_n and GL_m known as Schur-Weyl duality (see [FH]).

The objects of study in this paper arise from generalizations of these Weyl groups known as complex reflection groups. Weyl groups are real reflection groups, i.e., there exists a representation V of the Weyl group such that every element in the group is a product of reflections across some hyperplane. Complex reflection groups, on the other hand, are built out of complex reflections in the same manner, where a complex reflection is a linear operator on a vector space whose set of fixed points is a hyperplane (but whose other eigenvalue does not have to be -1). The motivation for the study of complex reflection groups is twofold. First, they are extremely natural generalizations of Weyl groups, which are central to the study of simple Lie groups. Second, a theorem of Chevalley-Sheppard-Todd ([NS]) states that if a group G acts linearly on a vector space V, then the space of orbits of the action has no singularities if and only if G is a complex reflection group. Thus, complex reflection groups have very nice geometric properties.

In this paper, we restrict our attention to a particular infinite family of complex reflection groups $G(r,1,n)$, which can be defined as the group of complex $n \times n$-matrices with all elements either zero or r^{th} roots of unity and with exactly one nonzero element in each row and column. For $r = 1$, this group is just S_n, and, in general, the representation theory of $G(r,1,n)$ is extremely similar to that of S_n (see [GJ] and references therein). Since the representation theory is so well understood, we study not the group itself but two algebras (spaces with addition and multiplication), the Hecke algebra and Cherednik algebra, that are defined using $G(r,1,n)$ as a starting point.

The Hecke algebra associated to $G(r,1,n)$ is defined by taking the group algebra of $G(r,1,n)$ and deforming the relations using some complex parameters to change the multiplication formulas. Hecke algebras play a central role in the geometric construction of representations of simple Lie groups ([CG]), in Kazhdan-Lusztig theory ([KL]), in the representation theory of p-adic groups ([CMHL]), and in the characteristic p representation theory of algebraic groups. The rational Cherednik algebra of $G(r,1,n)$ is defined by deforming an algebra associated to the orbit space of $G(r,1,n)$ on $V \oplus V^*$, with V a n-dimensional complex representation. Its representation theory plays a critical role in the proof of the Macdonald conjectures ([Che]) and in the study of the Calogero-Moser integrable systems in physics ([EM]).

In the representation theory of any algebra or group, the most important objects are the irreducible representations, which are the representations that do not have any smaller representation sitting inside them. If the parameters involved in the definition of the Hecke and Cherednik algebras avoid certain countable sets of hypertori and hyperplanes, respectively, then their irreducible representations are well-understood. In both cases, the irreducibles in the category of finite dimensional representations of the Hecke algebra and those in the category $\mathscr{O}$ for the Cherednik algebra are in one to one correspondence with those of $G(r,1,n)$ (see [GJ] for the Hecke algebra and [EM] for the Cherednik algebra). In the case of the Hecke algebra, the irreducible is called a Specht module and in the case of the Cherednik algebra, it is the Verma module associated to the irreducible representation of $G(r,1,n)$.

However, there is one key difference between the representation theory of $G(r,1,n)$ and that of the two algebras. Every irreducible representation of $G(r,1,n)$ is unitary, i.e., there exists a unique (up to scaling) positive definite Hermitian form on the representation such that $G(r,1,n)$ acts by unitary operators. There is a similar statement that holds for the Specht modules and Verma modules: there exists a unique (up to scaling) Hermitian form such that certain elements of the algebras have specified adjoints with respect to the form (see [Sto]). This form, called the contravariant form, is non-degenerate but not positive definite.

For any non-degenerate Hermitian form on a finite dimensional vector space, there is an invariant called the signature that characterizes the form. This signature is defined by taking an orthogonal basis for the form and then summing the signs of the norm squares $\langle v, v \rangle$ of the basis elements. This is independent of the choice of orthogonal basis. Since the Specht modules are finite dimensional, the signature of the contravariant forms on these representations is well-defined. On the other hand, the Verma modules are infinite dimensional but they are graded over the non-negative integers by finite dimensional subspaces. The contravariant form induces non-degenerate forms on each graded piece and so we can define the signature character of the Verma module as $\sum_{i=0}^{\infty} b_i t^i$, where b_i is the signature of the form on the i^{th} graded piece.

Computing the signature and signature character of the contravariant form is important for two reasons. The first source of motivation is that they are important invariants of the Specht modules and Verma modules that help us better understand the behaviour of these representations. The second source of motivation comes from the application of the representation theory of groups to quantum mechanics. In any such applications, the groups act via unitary representations and hence the classification of unitary representations of groups is an important problem in representation theory. For infinite groups, this problem is very difficult. In fact, the solution uses the computation of signatures of non-degenerate Hermitian forms that are not positive definite (see [AvLTVJ]). A similar situation arises here, as the computation of the signature of the Specht modules will allow us to see when these representations are unitary.

In this project, our first goal is to compute the signatures of the contravariant form on the Specht modules and to use them to compute explicit formulas for the unitary range, i.e., the range of deformation parameter values in which these representations are unitary. For $r = 1$, the signatures and signature characters of Specht modules and Verma modules have been computed in [Ven]. Additionally, the unitary ranges are also known for the Hecke algebra in the case $r = 1$. However, for $r > 1$, the signature formulas and the unitary range for the Specht modules are not known. The first main result of the paper is the computation of these formulas for all values of r and all Specht modules.

Our second goal is to compute the asymptotics of the signature character of the polynomial representation of the Cherednik algebra, which is the Verma module associated to the trivial representation of $G(r,1,n)$. The full signature character for all Verma modules has already been computed (see [Gri1] [Gri2].) The signature character depends on the Euclidean space of deformation parameters used in the definition of the Cherednik algebra. If we focus our attention to the signature of the degree k part of a Verma module, then this signature is locally constant with respect to the parameters and, in fact, only changes when we cross one of finitely many affine hyperplanes. Thus, if we choose an infinite ray in our parameter space and move towards infinity, then eventually, the t^k coefficient of the signature character stabilizes. Hence, we get a well-defined power series in the limit as the parameters go to infinity, which we call the asymptotic signature character of the Verma module. This asymptotic character measures the stable limit of the signature character as the parameter values grow large and it depends on the direction in which the parameter values tend to infinity.

There are two main reasons the asymptotic signature character is an interesting object to study. The first is that the limiting formula is vastly simpler than the full signature character. This was found to be the case for $r = 1$ in [Ven] and is expected to hold for $r > 1$ as well. It is certainly true for the cases that we compute in this paper. The second reason the asymptotic character is interesting is because of its connection with the Deligne category of the Cherednik algebra (see [Eti], [Aiz]). A suitable sequence, depending on n, of Verma modules of the Cherednik algebra of $G(r,1,n)$ (such as the sequence of polynomial representations) gives an object in the Deligne category which is also equipped with a suitable notion of non-degenerate Hermitian form with associated signature character. This Deligne category object measures the stable properties of the sequence of representations, i.e. the properties that the representations have sufficiently far down the sequence. The relationship between the signature characters of the sequence of Verma modules and the signature character of the object in the Deligne category is not fully understood. However, formulas in the Deligne category setting are usually obtained from the corresponding formulas in the normal Cherednik algebra setting by interpolating the latter, i.e., by replacing the parameter n in the latter formulas by a complex variable t corresponding to the complex variable involved in the definition of the Deligne category. Additionally, it is known that, while the full signature characters of the Verma modules do not show any polynomial behaviour in n, the coefficient of t^k in the asymptotic signature characters of any suitable sequence of Cherednik Algebra representations is polynomial in n for large enough n. Thus, if we can compute what this polynomial is, it should shed light on the signature character for the corresponding object in the Deligne category.

The second main result of this paper is the computation of the asymptotic signature character of the polynomial representation of the Cherednik algebra associated to $G(r,1,n)$ for half of all possible parameter values. In addition, we also show that, for these parameter values, the coefficient of t^k in the asymptotic signature character is $\binom{n+k}{k}$ for n sufficiently large, i.e., the degree k portion of the polyomial representation is unitary for n sufficiently large. Hence, for half of all possible parameter values, the polynomial representation is stably unitary. In the coming few months, we hope to extend this result to all parameter values and to arbitrary Verma modules. We also hope to use this computation to study the signature character of the corresponding object in the Deligne category of the Cherednik algebra.

The paper is organized as follows. In section 2, we introduce the Hecke algebra, construct the Specht modules and define the contravariant form. In 2.4, we compute the signature of the contravariant form and in 2.5 we exhibit some of its consequences. In section 3, we define the Cherednik algebra and its Verma modules and rederive the signature character of the polynomial representation. In 3.5, we compute a formula the asymptotic norms of an orthogonal basis for the polynomial representations and prove, in particular, that the polynomial representation is stably unitary. In 3.6, we derive a recursion relation that the asymptotic signature character satisfies and use it to compute the asymptotic character for small values of n. In section 4, we outline some of the work we plan to do in the coming few months.

2 The Hecke Algebra

2.1 Definition of the Hecke Algebra

The Hecke Algebra is a deformation of a complex reflection group $G(r,1,n)$. A *complex reflection* is an operator on $\mathbb{C}^n$ that fixes every point of some $n-1$ dimensional complex hyperplane. A *complex reflection group* is a finite group generated by complex reflections. The specific complex reflection group that will be discussed in this paper is $G(r,1,n)$. The group $G(r,1,n)$ is a group of n by n matrices where for every matrix, each row and each column have exactly one nonzero entry that is an r^{th} root of unity ([AK]). The representation theory of $G(r,1,n)$ is given by combinatorial objects called multi-Young diagrams and standard multi-tableau which are defined below ([AK]).

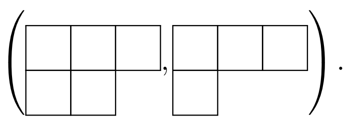
Definition. For a non-negative integer n, a *partition* $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$ of n is a tuple of integers so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 1$ and $\mu_1 + \mu_2 + \cdots + \mu_k = n$. A *Young diagram* corresponding to μ is given by k rows of boxes so that row i has μ_i boxes starting from the left. The size of μ denoted by $|\mu|$ is defined to be n.

Definition ([AK])**.** A *multi-Young diagram* with r parts and total size n is a r-tuple of Young Diagrams $Y = (Y_1, Y_2, \cdots, Y_r)$ so that $|Y_1| + |Y_2| + \cdots + |Y_r| = n$. When the context is clear, r and n will be omitted. The size of Y, denoted by $|Y|$, is defined as $|Y_1| + |Y_2| + \cdots + |Y_r|$.

Definition ([AK])**.** A *standard multi-tableau* corresponding to a multi Young diagram Y is a way of filling in the boxes of Y with 1 to n so that in every tableau, each row and each column is increasing.

Remark. To reduce verbosity, a standard multi-tableau will just be called a multi-tableau.

Example. The following is an example of a multi Young diagram of size 9 and 2 parts:



The following is an example of a multi-tableau of the Young diagram:

$$\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 8 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 4 & 5 & 9 \\ \hline 6 \\ \cline{1-1} \end{array}\right).$$

Each irreducible representation of $G(r,1,n)$ corresponds to a multi-Young diagram with r parts and size n ([AK]). Moreover, a basis for each such representation is given by all the multi-tableaux of the shape of the representation ([AK]). This classifies all representations of $G(r,1,n)$ since for any finite group, all representations are semisimple and can be decomposed into irreducibles. In order to study the Hecke Algebra, $G(r,1,n)$ needs to be given a different presentation. The group $G(r,1,n)$ will be defined by generators and relations, and the Hecke Algebra will have the same generators with some of the relations being deformed.

Definition ([AK])**.** The complex reflection group $G(r,1,n)$ is given by the generators $s_0, s_1, \cdots, s_{n-1}$ with relations $s_0^r = 1$, $s_1^2 = s_2^2 = \cdots = s_{n-1}^2 = 1$, $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$, $s_i s_j = s_j s_i$ if $|i-j| \geq 2$, and if $1 \leq i \leq n-2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The eigenvalue relations (first and second relations) will be deformed to get the Hecke Algebra.

Definition ([AK])**.** The *Hecke Algebra* of $G(r,1,n)$ over the complex numbers denoted by $H_{n,r}(q)$ is given by the generators $T_0, T_1, \cdots, T_{n-1}$ with relations $(T_0 - u_1)(T_0 - u_2)\cdots(T_0 - u_r) = 0$, $(T_i + 1)(T_i - q) = 0$ for all $i \geq 1$, $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$, $T_i T_j = T_j T_i$ if $|i-j| \geq 2$, and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ if $1 \leq i \leq n-2$, where q, $u_1, u_2, \cdots, u_r$ are complex numbers.

In order to have the representation theory of the Hecke Algebra be semisimple, the parameters q, $u_1, u_2, \cdots, u_r$ will be restricted to where $q^m \neq 1$ for $2 \leq m \leq n$ and $q^m \cdot u_i/u_j \neq 1$ for $i \neq j$ and $0 \leq m \leq n$ ([Ari]). These will be called the generic values for the parameters.

2.2 Irreducible Representations of the Hecke Algebra

Since the Hecke Algebra under these parameters is semisimple, the irreducible representations can classify all representations of the Hecke Algebra. Like $G(r,1,n)$, each irreducible representation of the Hecke Algebra corresponds to a multi-Young diagram with r parts and size n ([AK]). Define the Specht module S_λ as the representation of the Hecke Algebra corresponding to the multi-Young diagram λ. Like the case of $G(r,1,n)$, a basis for S_λ is given by all the multi-tableaux of shape λ ([AK]).

Definition. Given a multi-tableau t_p of size n, let the *find value* of i, denoted by f_i, be defined as the number of the tableau that i is placed in. The *find vector* is defined to be the vector $(f_1, f_2, \cdots, f_n)$.

Definition ([AK])**.** Given a multi-tableau t_p of size n, let the *content value* of i ,denoted by d_i, be defined as the column number minus the row number of i. The *content vector* is $(d_1, d_2, \cdots, d_n)$.

Proposition ([AK])**.** *Let t_p be a multi-tableau. The following describe the actions of the generators of the Hecke Algebra on t_p:*

$$T_{i-1}t_p = \begin{cases} u_{f_1}t_p & \text{if } i=1 \\ qt_p & \text{if } i \text{ and } i-1 \text{ are in the same row of } t_p \\ -t_p & \text{if } i \text{ and } i-1 \text{ are in the same column of } t_p \\ \frac{u_{f_i}}{u_{f_i}-q^{d_{i-1}-d_i}\cdot u_{f_{i-1}}}\left((q-1)t_p + \frac{qu_{f_i}}{u_{f_i}-q^{d_{i-1}-d_i}\cdot u_{f_{i-1}}}t_q\right) & \text{otherwise} \end{cases}$$

where t_q is the multi-tableau formed by switching i and $i-1$ in t_p.

Definition. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ be a multi-Young diagram with size n. The *special multi-tableau* of shape λ is given by placing the numbers $|\lambda_1|+|\lambda_2|+\cdots+|\lambda_{i-1}|+1$ to $|\lambda_1|+|\lambda_2|+\cdots+|\lambda_i|$ in λ_i where λ is filled in left to right, then top to bottom.

Example. The following is an example of a special multi-tableau of size 9:

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 6 & 7 & 8 \\ \hline 9 \\ \cline{1-1}\end{array}\right).$$

Note that $d_3 = 3-1 = 2$ and $f_3 = 1$. The content vector is $(0,1,2,-1,0,0,1,2,-1)$ and the find vector is $(1,1,1,1,1,2,2,2,2)$.

2.3 The Contravariant Form

Under any Hermitian form $\langle , \rangle$, let the squared norm of an element t be the value of $\langle t,t\rangle$. For the purposes of this paper, the squared norm will just be called the norm because $\langle t,t\rangle$ can be negative and thus the square root is not well defined. Though there are many similarities between the representation theory of $G(r,1,n)$ and that of $H_q(n,r)$, there is one fundamental difference. There is a Hermitian form on each representation of $G(r,1,n)$ that is a positive definite inner product such that each group element acts unitarily.

It is shown in [Sto] that if the parameters are on the unit circle, there also exists a unique (up to scaling by a real number) non-degenerate Hermitian form $(\ ,\ )$ on the Specht modules of $H_q(n,r)$ such that each T_i acts by a unitary operator and the norm of the special multi-tableau is positive. The main difference between the two Hermitian forms is that the form on the group algebra is positive definite while this property does not hold for $H_q(n,r)$.

For an orthogonal basis B of a vector space V, the *signature* of V is defined to be the number of elements with positive norm in B minus the number of elements with negative norm in B. By basic linear algebra, the signature is independent of choice of orthogonal basis. The signature of each Specht module of $H_q(n,r)$ is nontrivial and therefore interesting to compute since not all elements have positive norm.

2.4 Computation of the Signature

Fix λ as a multi Young diagram and S_λ as the Specht module corresponding to λ. Define the Jucys-Murphy elements, given in [Sto], as $L_i = q^{1-i}T_{i-1}\cdots T_0T_1\cdots T_{i-1}$ for $1 \leq i \leq n$. Note that the Jucys-Murphy elements are unitary operators under the form $(\ ,\ )$ since each T_j is unitary, and q^{1-i} is unitary since q lies on the unit circle. It is shown in [OPd] that the multi-tableaux of shape λ are simultaneous eigenvectors of the Jucys-Murphy elements. Moreover, it is also shown in [OPd] that for any two distinct multi-tableaux, there exists a Jucys-Murphy element L_i so that they have distinct eigenvalues for L_i. Thus, any two multi-tableaux are orthogonal under $(\ ,\ )$ since they are eigenvectors with distinct eigenvalues of a unitary operator.

Let $\mathrm{sgn}(r)$ be the sign of the real number r. Define $N(t_p) = \mathrm{sgn}((t_p,t_p))$, the sign of the norm of t_p. Note that since the multi-tableaux of shape λ give an orthogonal basis for S_λ, the signature can be expressed as $\sum_{t_p} N(t_p)$. Thus, it suffices to compute $N(t_p)$ for every multi-tableau t_p of shape λ. Moreover, to compute the value of $N(t_p)$, only the change of sign from the special multi-tableau of shape λ is needed, since the special multi-tableau has positive norm. Let $D(t_p)$ be the set of ordered pairs (i,l) with $i > l$ so that either $f_i < f_l$ or $f_i = f_l$ and $d_i < d_l$, and let $C(t_p)$ be the set of $i \geq 2$ so that $(i,i-1) \in D(t_p)$.

Lemma 1. *Let $i \in C(t_p)$, and let t_q be the multi-tableau formed by switching i and $i-1$ in t_p. Then,*

$$\mathrm{sgn}(N(t_p)) = \mathrm{sgn}(N(t_q)) \cdot \mathrm{sgn}(|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}| - |q-1|).$$

Proof. Note that since the form is a contravariant form, and q, u_{i-1}, and u_i lie on the unit circle,

$$\begin{aligned}
N(t_p) = N(T_{i-1}t_p) &= N\left(\frac{u_{f_i}}{u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}} \cdot \left((q-1)t_p + \frac{qu_{f_i}}{u_{f_i} - q^{d_{i-1}-d_i-1}u_{f_{i-1}}}t_q\right)\right) \\
&= \mathrm{sgn}\left(\frac{|(q-1)u_{f_i}|^2}{|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}|^2}N(t_p) + \frac{|qu_{f_i}u_{f_i}|^2}{|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}|^2 \cdot |u_{f_i} - q^{d_{i-1}-d_i-1}u_{f_{i-1}}|^2}N(t_q)\right) \\
&= \mathrm{sgn}\left(\frac{|(q-1)|^2}{|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}|^2}N(t_p) + \frac{1}{|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}|^2 \cdot |u_{f_i} - q^{d_{i-1}-d_i-1}u_{f_{i-1}}|^2}N(t_q)\right).
\end{aligned}$$

Rearranging gives that $\mathrm{sgn}(|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}|^2 - |q-1|^2)N(t_p) = N(t_q)$. Dividing, taking the sign of both sides and noting that $\operatorname{sgn}(r) = \operatorname{sgn}(1/r)$ for all real numbers r gives that

$$\operatorname{sgn}(N(t_p)) = \operatorname{sgn}\left((|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}|^2 - |q-1|^2)(|u_{f_i} - q^{d_{i-1}-d_i-1}u_{f_{i-1}}|^2)N(t_q)\right)$$
$$= \operatorname{sgn}(N(t_q)) \cdot \operatorname{sgn}\left(|u_{f_i} - q^{d_{i-1}-d_i}u_{f_{i-1}}| - |q-1|\right),$$

as desired. □

Theorem 1. *The value of $N(t_p)$ for t_p as a multi-tableau in S_λ is given by*

$$\prod_{(i,l)\in D(t_p)} \operatorname{sgn}(|u_{f_i} - q^{d_l-d_i}u_{f_l}| - |q-1|).$$

Proof. Let t_1 be the special multi-tableau of λ. Let t_p be a multi-tableau and let i be in the spot that n is supposed to be in. Note that i must be in the last tableau. Suppose that $i < n$. If $i+1$ is in the last tableau also, then $d_{i+1} > d_i$ since $i+1$ must be in a lower row number and greater column number than i and thus the overall content of $i+1$ is greater than that of i. Therefore, either way $(i+1, i) \in D(t_p)$. Moreover, i and $i+1$ can be switched. Switching i and $i+1$ makes the new content vector to be $(d_1, ..., d_{i+1}, d_i, ..., d_n)$ and the new find vector to be $(f_1, ..., f_{i+1}, f_i, ..., f_n)$.

Switching i and $i+1$ in t_p gives a new multi-tableau t_q with $i+1$ in the spot that n is supposed to be in. Moreover the sign that was picked up is $\operatorname{sgn}(|u_{f_{i+1}} - q^{d_i-d_{i+1}}u_{f_i}| - |q-1|)$ according to Lemma 1. Now switch $i+1$ and $i+2$ and so on until n is supposed to be in the spot that it is in t_1. Now n is in the spot that it is supposed to be and the total sign that was picked up $\prod_{k=i+1}^{n} \operatorname{sgn}(|u_{f_k} - q^{d_i-d_k}u_{f_i}| - |q-1|)$. Apply this and get $n-1$, $n-2$,..., 1 in the correct spot gives the desired formula for the norm. □

Lemma 2. *Let $q = \mathrm{q}^2$. The value $|\frac{a_1 - q^d \cdot a_2}{q-1}| - 1$ has the same sign as*

$$\left(\mathrm{q}^{d-1} \cdot \frac{a_2}{a_1} - \mathrm{q}^{-(d-1)}\right)\left(\mathrm{q}^{d+1} - \frac{a_1}{a_2} \cdot \mathrm{q}^{-(d+1)}\right).$$

Proof. Note that since $(\mathrm{q} - \mathrm{q}^{-1})^2$ is a positive real number,

$$\operatorname{sgn}(|\frac{a_1 - \mathrm{q}^{2d} \cdot a_2}{\mathrm{q}^2 - 1}| - 1) = \operatorname{sgn}(|\frac{\mathrm{q}^{-d} \cdot a_1 - \mathrm{q}^d \cdot a_2}{\mathrm{q} - \mathrm{q}^{-1}}| - 1)$$
$$= \operatorname{sgn}(\frac{\mathrm{q}^{-d} \cdot a_1 - \mathrm{q}^d \cdot a_2}{\mathrm{q} - \mathrm{q}^{-1}} \cdot \frac{\frac{\mathrm{q}^d}{a_1} - \frac{\mathrm{q}^{-d}}{a_2}}{\mathrm{q} - \mathrm{q}^{-1}} - 1)$$
$$= \operatorname{sgn}(\mathrm{q}^{-2d} \cdot \frac{a_1}{a_2} + 2 - \mathrm{q}^{2d} \cdot \frac{a_2}{a_1} - (\mathrm{q} - \mathrm{q}^{-1})^2)$$
$$= \operatorname{sgn}((\mathrm{q}^{d-1} \cdot \frac{a_2}{a_1} - \mathrm{q}^{-(d-1)})(\mathrm{q}^{d+1} - \frac{a_1}{a_2} \cdot \mathrm{q}^{-(d+1)})),$$

as desired. □

Therefore, the value of $N(t_p)$ for a multi-tableau t_p is given by

$$\prod_{(i,l)\in D(t_p)} \operatorname{sgn}((\mathrm{q}^{d_l-d_i-1}\cdot\frac{u_{f_l}}{u_{f_i}}-\mathrm{q}^{-(d_l-d_i-1)})(\mathrm{q}^{d_l-d_i+1}-\frac{u_{f_i}}{u_{f_l}}\cdot\mathrm{q}^{-(d_l-d_i+1)})).$$

Theorem 2. *The value of the signature of S_λ is given by*

$$\sum_{t_p}\prod_{(i,l)\in D(t_p)} \operatorname{sgn}((\mathrm{q}^{d_l-d_i-1}\cdot\frac{u_{f_l}}{u_{f_i}}-\mathrm{q}^{-(d_l-d_i-1)})(\mathrm{q}^{d_l-d_i+1}-\frac{u_{f_i}}{u_{f_l}}\cdot\mathrm{q}^{-(d_l-d_i+1)}))$$

where the sum is taken over all the multi-tableaux t_p of shape λ.

2.5 Consequences of the Signature

2.5.1 Comparison With Previous Results

In [Ven], the signatures of representations of the Hecke Algebra are computed when $r = 1$. When $r = 1$, $G(r,1,n)$ becomes S_n and therefore $H_q(n,r)$ becomes the Hecke Algebra of S_n. The results obtained in this paper are consistent with [Ven].

Corollary 1 ([Ven])**.** *When $r = 1$, the norm of each multi-tableau t_p is given by*

$$\prod_{\substack{i>l\\ d_i<d_l}} \operatorname{sgn}((\mathrm{q}^{d_i-d_l-1}-\mathrm{q}^{-(d_i-d_l-1)})(\mathrm{q}^{d_i-d_l+1}-\mathrm{q}^{-(d_i-d_l+1)})).$$

Proof. Note that the find value for all $1\le i\le n$ is the same since $r = 1$. Thus, $D(t_p)$ is the set of ordered pairs (i,l) such that $i > l$ and $d_i < d_l$. Thus, the norm of a given element is

$$\prod_{\substack{i>l\\ d_i<d_l}} \operatorname{sgn}((\mathrm{q}^{d_i-d_l-1}-\mathrm{q}^{-(d_i-d_l-1)})(\mathrm{q}^{d_i-d_l+1}-\mathrm{q}^{-(d_i-d_l+1)}))$$

which is the expression in [Ven]. □

2.5.2 The Restriction Problem

Given $H_q(n,r)$, the subalgebra generated by $T_0,T_1,\cdots,T_{n-2}$ is the Hecke Algebra of $G(r,1,n-1)$. Let S_λ be a representation of $H_q(n,r)$. Note that S_λ can be restricted to a representation of $H_q(n-1,r)$ by taking only the actions of $T_0,T_1,\cdots,T_{n-2}$. Let $R(\lambda)$ be the set of Young multi-diagrams of size $n-1$ that can be obtained by removing a square of λ. The restriction of S_λ to $H_q(n-1,r)$ can be seen to be $S_\lambda = \bigoplus_{\mu\in R(\lambda)} S_\mu$. This is the branching rule where λ gets restricted to a representation of $H_q(n-1,r)$ ([AK]). This can be generalized for $H_q(n,r)$ restricting to $H_q(n-m,r)$ for any $0\le m\le n$ by restricting one square at a time. Now, just as the representation can be restricted, so can the contravariant form. The restriction of the contravariant form to S_μ can be the same form, or the opposite form where the special multi-tableau has negative norm because the generators of the algebra are still unitary and thus the form can only restrict to one of the two. The goal of the restriction problem is to find out when the restricted form is the original form and when it is the opposite form by looking at the special multi-tableau of S_μ as an element of S_λ.

Corollary 2. *Let $\mu \in R(\lambda)$. Let s be a special multi-tableau of μ, and let t be the multi-tableau of λ where it is s with n being put in the remaining square. The restriction of the contravariant form to the form on S_μ is the opposite form if and only if*

$$\prod_{(i,l)\in D(t)} sgn((\mathrm{q}^{d_l-d_i-1}\cdot\frac{u_{f_l}}{u_{f_i}}-\mathrm{q}^{-(d_l-d_i-1)})(\mathrm{q}^{d_l-d_i+1}-\frac{u_{f_i}}{u_{f_l}}\cdot\mathrm{q}^{-(d_l-d_i+1)}))$$

is negative.

Proof. Note that the form is the opposite if and only if $N(t)<0$ and the conclusion holds. □

2.5.3 The Unitary Range

The *unitary range* is the range of values of the parameters of $H_q(n,r)$ for which the contravariant form is actually positive definite. For this to happen, all that is needed is for all the norms of nonzero elements to be positive. Moreover, this is an important consequence of the computation of the norms because it gives more information about certain representations of the Hecke Algebra.

Corollary 3. *Suppose that $\mathrm{q}=e^{i\theta_1}$ where $2\theta_1\in[0,2\pi)$. Suppose that two squares in the i^{th} and l^{th} tableau have a content difference d, and $u_{f_l}/u_{f_i}=e^{i\theta_2}$. The representation is unitary if and only if for all θ_2, there exists some integer k so that*

$$2\theta_1(d-1)\le 2k\pi-\theta_2\le 2\theta_1(d+1).$$

Proof. Note that the norm change factor must be positive. Thus,

$$\mathrm{q}^{2d}\frac{u_{f_l}}{u_{f_i}}-\mathrm{q}^{-2d}\frac{u_{f_i}}{u_{f_l}}>\mathrm{q}^2+\mathrm{q}^{-2}$$

which is equivalent to saying that $\cos(2d\theta_1+\theta_2)>cos(2\theta_1)$. This implies the conclusion. □

3 The Cherednik Algebra

3.1 Definition of the Cherednik Algebra

The group $G(r,1,n)$ has a fundamental representation $V=\mathbb{C}^n$ on which it acts by complex reflections. There is a classical construction of the semidirect tensor product $S(V\oplus V^*)\otimes G(r,1,n)$ where the multiplication is given by $gvg^{-1}=g(v)$, and $gv^*g^{-1}=g(v^*)$. This algebra is closely related to the algebra of regular functions on the space $(V\oplus V^*)/\mathbb{C}[G(r,1,n)]$ and thus is interesting in algebraic geometry ([EM]). The Cherednik Algebra is, essentially, a quantum deformation of this space.

Before the Cherednik Algebra can be defined, some notation is needed. Let $s_{ij}\in S_n$ be the transposition (i,j) and let s_i be the transposition $(i,i+1)$. Also, let $\zeta=e^{2\pi i/r}$ and let ζ_i^l be the element of $G(r,1,n)$ with all 1's on its diagonal except for the i^{th} row where it has ζ^l.

Definition ([Gri3])**.** The *Cherednik Algebra* of $W = G(r,1,n)$ with parameters $\kappa, c_0, c_1, \cdots, c_{r-1}$ is the algebra generated by $\mathbb{C}[x_1, \cdots, x_n]$, $\mathbb{C}[y_1, \cdots, y_n]$, and t_w for $w \in W$ with $t_w t_v = t_{wv}$, $t_w x = (w \cdot x) t_w$, and $t_w y = (w \cdot y) t_w$ for $w,\ v \in W$, $x \in V^*$, and $y \in V$,

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}$$

for $1 \le i \ne j \le n$, and

$$y_i x_i = x_i y_i + \kappa - \sum_{j=0}^{r-1} (d_j - d_{j-1}) \varepsilon_{ij} - c_0 \sum_{j \ne i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}$$

where $d_j = \sum_{l=1}^{r-1} \zeta^{lj} c_l$ and $\varepsilon_{ij} = \frac{1}{r} \sum_{l=0}^{r-1} \zeta^{-lj} t_{\zeta_i^l}$. The Cherednik Algebra will be denoted by $\mathbb{H}$.

For generic parameters, $\mathbb{H}$ is semisimple. By the PBW Theorem (see [EM]), the Cherednik Algebra is isomorphic to $S(V) \otimes \mathbb{C}[G(r,1,n)] \otimes S(V^*)$ as a vector space. In fact, the semidirect tensor product $S(V) \otimes \mathbb{C}[G(r,1,n)]$ is a subalgebra of the Cherednik Algebra and the PBW theorem shows that the Cherednik Algebra is free as a module over $S(V) \otimes \mathbb{C}[G(r,1,n)]$ and freely generated by $S(V^*)$.

3.2 Construction of the Verma Module

For every multi Young diagram λ, define V_λ to be the representation of $G(r,1,n)$ associated to λ. We define the Verma module associated to an irreducible representation V_λ of $G(r,1,n)$. We first extend this representation to a representation of the subalgebra $S(V) \otimes \mathbb{C}[G(r,1,n)]$ by letting polynomials in V act by their constant terms. Then, we induce it up to the Cherednik algebra by setting $M_\lambda = \mathbb{H} \otimes_{S(V) \otimes \mathbb{C}[G(r,1,n)]} V_\lambda$ ([EM]). By the PBW theorem, this is isomorphic as a vector space to $S(V^*) \otimes V_\lambda$. In particular, this representation is infinite dimensional but is non-negatively graded with the degree i piece corresponding to degree i polynomials of $S(V^*)$ tensored with V_λ where each piece has finite dimension.

At generic values of parameters, the Verma modules are irreducible ([EM]). Additionally, it is shown in [Sto] that if the values d_i are real, there exists a unique (up to scaling by a real number) non-degenerate Hermitian form that extends a $G(r,1,n)$-invariant positive definite form on V_λ such that y_i and x_i are adjoint. This is called the *contravariant form*.

The decomposition into degree i pieces of the Verma module is an orthogonal decomposition for this contravariant form and hence there is a non-degenerate Hermitian form on each graded piece. So, the signature character can be defined as $\sum_{i=0}^{\infty} b_i \cdot t^i$ where b_i is the signature of the i^{th} graded piece. The norms of basis elements for any representation of the Cherednik Algebra is given in [Gri2], and the signature character easily follows as a consequence. The signature character of the polynomial representation will be computed in a different manner in order to come up with asymptotic formulas.

3.3 Construction of Orthogonal Basis and Intertwining Operators

Define $z_i = y_i x_i + c_0 \sum_{1\le j<i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}$. Define the operator π_i as $\sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}}$, and intertwiners

$$\sigma_i = t_{s_i} + \frac{c_0}{z_i - z_{i+1}} \pi_i,$$

$\Phi = x_n t_{s_{n-1}s_{n-2}\cdots s_1}$, and $\Psi = y_1 t_{s_1 s_2 \cdots s_{n-1}}$ as defined in [Gri3].

A basis for the polynomial representation is given by f_μ where $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in \mathbb{Z}_{\ge 0}$, where f_μ is equal to $x^\mu +$ lower terms as given in [Gri3]. Note that the operators z_i are self-adjoint. It is shown in [Gri3] that f_μ are simultaneous eigenvectors for z_i, and for any two different f_μ, there exists an element z_i such that the eigenvalues of these elements are different. Thus, the basis given by f_μ is orthogonal.

In order to get a full description of the polynomial representation, the action of the intertwiners on the Verma modules is needed. Let $v_\mu(i) = |\{j < i | \mu_j < \mu_i\}| + |\{j \ge i | \mu_j \le \mu_i\}|$.

Proposition ([Gri3])**.** *The action of the intertwiners on f_μ is given by $\sigma_i f_\mu = f_{s_i\mu}$ if $\mu_i \not\equiv \mu_{i+1} \pmod r$ or $\mu_i < \mu_{i+1}$, $\sigma_i f_\mu = 0$ if $\mu_i = \mu_{i+1}$, and*

$$\sigma_i f_\mu = \frac{\delta^2 - r^2 c_0^2}{\delta^2} f_{s_i\mu}$$

otherwise, where $\delta = \kappa(\mu_i - \mu_{i+1}) - c_0 r (v_\mu(i) - v_\mu(i+1))$, $\Phi f_\mu = f_{\phi\mu}$, $\Psi f_\mu = 0$ if $\mu_n = 0$, and if $\mu_n \ne 0$,

$$\Psi f_\mu = (\kappa\mu_n - (d_0 - d_{-\mu_n}) - c_0 r (v_\mu(n) - 1)) f_{\psi\mu}.$$

3.4 Computing the Signature Character

The sign of the norm of each of these elements will be computed and then the sum will give the signature character. For convenience, let $L(\delta) = \operatorname{sgn}(\delta^2 - r^2 c_0^2)$. Also, for $\tau \in S_n$, define the ordered set

$$R_i = \{1, 2, \cdots, \tau(i) - 1\} \setminus \{\tau(1), \tau(2), \cdots, \tau(i-1)\}.$$

Let $N(\mu)$ denote the sign of the norm of μ. Also, given two tuples μ and μ', let $\operatorname{Chg}(\mu, \mu') = N(\mu)/N(\mu')$. Moreover, let $P_\mu = \{\tau \in S_n | \mu_i = \mu_j, i < j \implies \tau(i) < \tau(j)\}$. For $\tau \in P_\mu$, let

$$\tau(\mu) = (\mu_{\tau(1)}, \mu_{\tau(2)}, \cdots, \mu_{\tau(n)}).$$

Lemma 3. *Let $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ and suppose that $\mu_i \ne \mu_{i+1}$. Let $\mu' = (\mu_1, \cdots, \mu_{i+1}, \mu_i, \cdots, \mu_n)$. Then, $v_{\mu'}(j) = v_\mu(j)$ if $j \ne i$ or $i+1$, $v_{\mu'}(i) = v_\mu(i+1)$, and $v_{\mu'}(i+1) = v_\mu(i)$.*

Proof. Note that the value of $v_{\mu'}(j)$ for $j \ne i, i+1$ is $v_\mu(j)$ since nothing is affected by the swap of μ_i and μ_{i+1}. Note that μ_i either contributes to $v_\mu(i+1)$ or doesn't, and the same holds after the swap. Thus, $v_{\mu'}(i) = v_\mu(i+1)$. The other condition is similar. □

Fix μ with $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ and let $F_\mu(i) = |\{k \ge i | \mu_k \le \mu_i\}|$.

Theorem 3. *If $\tau \in P_\mu$, then,*

$$\mathrm{Chg}(\mu,\tau(\mu)) = \prod_{i=1}^{n} \prod_{\substack{j\in R_i \\ \mu_{\tau(i)}\equiv\mu_j \pmod r}} L\left(\kappa(\mu_{\tau(i)}-\mu_j) - c_0 r(v_\mu(\tau(i)) - v_\mu(j))\right).$$

Proof. Let $\alpha_i = (\mu_{\tau(1)},\mu_{\tau(2)},\cdots,\mu_{\tau(i-1)},\mu_{h_1},\cdots,\mu_{h_l})$, where $h_1 < h_2 < \cdots < h_l$ is the ordered set

$$\{1,2,\cdots,n\}\setminus\{\tau(1),\tau(2),\cdots,\tau(i-1)\}.$$

Note that $\alpha_1 = \mu$ and $\alpha_n = \tau(\mu)$. Suppose that h_k and h_{k-1} are switched in α_i. If $\mu_{h_k} = \mu_{h_{k-1}}$, then it was as if the switch did not occur, and thus there is no change in the norm. If $\mu_{h_k} \not\equiv \mu_{h_{k-1}} \pmod r$, then there is no change in the norm. Otherwise, the change is $L(\kappa(\mu_{h_k}-\mu_{h_{k-1}}) - c_0 r(v_\mu(h_k) - v_\mu(h_{k-1})))$ because $\mu_{h_k} > \mu_{h_{k-1}}$ and $\mu_{h_k} \equiv \mu_{h_{k-1}} \pmod r$. Thus, in α_i, the place where $\mu_{\tau(i)}$ is needs to be moved to the place where μ_{h_1} is by adjacent swaps to get to α_{i+1}. This creates a change in the norm of

$$\prod_{\substack{j\in R_i \\ \mu_{\tau(i)}\equiv\mu_j \pmod r}} L(\kappa(\mu_{\tau(i)}-\mu_j) - c_0 r(v_\mu(\tau(i)) - v_\mu(j)))$$

because $\mu_j < \mu_{\tau(i)}$ for $j \in R_i$ since $\tau \in P_\mu$. Thus, this value is equal to $\mathrm{Chg}(\alpha_i,\alpha_{i+1})$. Therefore, multiplying over all i gives the desired value for $\mathrm{Chg}(\mu,\tau(\mu))$. □

Theorem 4. *The value of $N(\mu)$ is the sign of*

$$\prod_{i=1}^{n}\prod_{m=0}^{\mu_i-1}\big((\kappa(m+1) - (d_0 - d_{-m-1}) - c_0 r(i-1))\cdot \prod_{\substack{j<i \\ r|m \\ m>0}} L(\kappa\cdot m - c_0 r j)\cdot$$
$$\prod_{\substack{i+1\le j\le n \\ \mu_j\equiv m+1 \pmod r \\ \mu_j > m+1}} L(\kappa(\mu_j - m - 1) - c_0 r F_\mu(j))\big)$$

Proof. Note that $\mathrm{Chg}(\mu,(0,0,\cdots,0)) = N(\mu)$ since $(0,0,\cdots,0)$ has positive norm by convention. Let $\alpha = (0,0,\cdots,m,\mu_{i+1},\cdots,\mu_n)$ and let $\beta = (0,0,\cdots,m+1,\mu_{i+1},\cdots,\mu_n)$ where $m < \mu_i$. First, m will be moved to the front of α by adjacent transpositions. Note that $m < m+1 \le \mu_{i+1}$. Thus, $v_\mu(i) = i$. Similarly, for $j < i$, $v_\mu(j) = i - j$ since $\mu_j = 0$. Moreover, when m is swapped with a previous value, the value of v_μ does not change for values before that. Thus, by applying the transformation that brings m to the first component of α, the norm changes by

$$\prod_{\substack{j<i \\ m\equiv 0 \pmod r \\ m>0}} L(\kappa m - c_0 r j).$$

Then, Φ is applied to go up the grading. The norm change is $\kappa(m+1) - (d_0 - d_{-m-1}) - c_0 r(i-1)$. The tuple now looks like $\alpha' = (0,0,\cdots,\mu_{i+1},\mu_{i+2},\cdots,m+1)$. It is desired to move $m+1$ to its correct place by adjacent swaps. To do this, $m+1$ only has to be swapped with values that are higher than it. Note that $v_{\alpha'}(n) = i$, and for $i \le j \le n-1$, $v_{\alpha'}(j) = i + |\{j \ge i | \mu_j \ge \mu_i\}| = i + F_\mu(j)$. Thus, the norm change when $m+1$ is moved to the correct position is

$$\prod_{\substack{i+1\le j\le n \\ \mu_j \equiv m+1 \pmod r \\ \mu_j > m+1}} L(\kappa(\mu_j - m - 1) - c_0 r F_\mu(j))$$

as desired. Multiplying over all i and m gives the desired value of the change of norm. □

3.5 Asymptotic Signature

In this section, we wish to compute the asymptotic behaviour of the signature character formula we derived in the previous section. To do so, we need to pick a ray in our parameter space and then move towards infinity. However, note that if we scale $(\kappa, c_0, \ldots, c_{r-1})$ all by a positive real number, then the value of the signature is unchanged. Thus, the actual parameter space of the Cherednik algebra is the ratio of the parameters $(\frac{c_0}{\kappa}, \ldots, \frac{c_{r-1}}{\kappa})$. Thus, taking a ray and moving towards infinity is the same as fixing $c_0, \ldots, c_{r-1}$ and taking the limit of $\kappa \to 0^+$. Thus, we make the following definition.

Definition. The *asymptotic signature character* of the polynomial representation of $\mathbb{H}$ in the direction $(c_0, \ldots, c_{r-1})$ is the limit as $\kappa \to 0^+$ of the signature character of the polynomial representation with parameters $(\kappa, c_0, c_1, \ldots, c_{r-1})$.

This definition was motivated in the introduction. For the purposes of the current paper, the reason we study this asymptotic character is because it is much simpler than the full signature character and because if we fix our attention on the t^k coefficient in the signature character, then for κ small enough, the signature character stops changing and the asymptotic signature character and the signature character agree up the coefficient of t^k for any smaller κ.

Lemma 4. *If a and b are real numbers, then*

$$\lim_{\kappa\to 0^+} L(\kappa \cdot a + c_0 r \cdot b) = \begin{cases} -1 & \text{if } |b| < 1 \\ 1 & \text{if } |b| > 1 \\ \operatorname{sgn}(abc_0) & \text{if } |b| = 1 \end{cases}$$

Proof. Note that

$$\begin{aligned}\lim_{\kappa\to 0^+} L(\kappa \cdot a + c_0 r \cdot b) &= \lim_{\kappa\to 0^+} \operatorname{sgn}\left((\kappa \cdot a + c_0 r \cdot b)^2 - r^2 c_0^2\right) \\ &= \lim_{\kappa\to 0^+} \operatorname{sgn}\left((\frac{\kappa \cdot a}{c_0 r} + b)^2 - 1\right) \\ &= \lim_{\kappa\to 0^+} \operatorname{sgn}\left(\frac{\kappa^2 a^2}{c_0^2 r^2} + \frac{2\kappa ab}{c_0 r} + b^2 - 1\right).\end{aligned}$$

If $|b| \neq 1$, then $b^2 - 1 \neq 0$. Thus, if $|b| \neq 1$, then

$$\begin{aligned}\lim_{\kappa \to 0^+} L(\kappa \cdot a + c_0 r \cdot b) &= \lim_{\kappa \to 0^+} \operatorname{sgn}\left(\frac{\kappa^2 a^2}{c_0^2 r^2} + \frac{2\kappa ab}{c_0 r} + b^2 - 1\right) \\ &= \operatorname{sgn}(b^2 - 1)\end{aligned}$$

which is 1 if $|b| > 1$ and -1 if $|b| < 1$. If $|b| = 1$, then $b^2 - 1 = 0$. If $|b| = 1$, since $\kappa > 0$ and $r > 0$,

$$\begin{aligned}\lim_{\kappa \to 0^+} L(\kappa \cdot a + c_0 r \cdot b) &= \lim_{\kappa \to 0^+} \operatorname{sgn}\left(\frac{\kappa^2 a^2}{c_0^2 r^2} + \frac{2\kappa ab}{c_0 r} + b^2 - 1\right) \\ &= \lim_{\kappa \to 0^+} \operatorname{sgn}(\kappa) \cdot \operatorname{sgn}\left(\frac{\kappa a^2}{c_0^2 r^2} + \frac{2ab}{c_0 r}\right) \\ &= \operatorname{sgn}(abc_0),\end{aligned}$$

as desired. □

Let $M(a,b) = \lim_{\kappa \to 0^+} L(\kappa \cdot a + c_0 r \cdot b)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ with $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$.

Corollary 4. *The value of $N(\tau(\mu))$ is the sign of*

$$\left(\prod_{i=1}^{n} \prod_{\substack{j \in R_i \\ \mu_{\tau(i)} \equiv \mu_j \pmod r}} M\left(\mu_{\tau(i)} - \mu_j, v_\mu(j) - v_\mu(\tau(i))\right)\right) \times$$

$$\prod_{i=1}^{n} \prod_{m=0}^{\mu_i - 1} \left((\kappa(m+1) - (d_0 - d_{-m-1}) - c_0 r(i-1)) \times \prod_{\substack{j<i \\ r|m \\ m>0}} M(m, -j) \times \prod_{\substack{i+1 \leq j \leq n \\ \mu_j \equiv m+1 \pmod r \\ \mu_j > m+1}} M(\mu_j - m - 1, -F_\mu(j))\right)$$

as $\kappa \to 0^+$.

Proof. This follows straight from applying Lemma 4 to Theorems 3 and 4. The terms with κ are still included because when $i = 1$ and $m + 1$ is divisible by r, then

$$\kappa(m+1) - (d_0 - d_{-m-1}) - c_0 r(i-1) = \kappa(m+1)$$

which is positive. □

For the rest of this discussion, let $c_0 < 0$ and $\kappa \to 0^+$. Let g_j be the greatest positive integer that is at most n so that $d_j < d_0 + rc_0(g_j - 1)$ for $1 \leq j \leq r-1$. If no such positive integer exists, then let $g_j = 0$.

Lemma 5. *The value*

$$\lim_{\kappa \to 0^+} N(\tau(\mu)) = \prod_{j=1}^{r-1} \prod_{s=1}^{g_j} (-1)^{\left\lfloor \frac{\mu_s + j}{r} \right\rfloor}.$$

Proof. Note that if $a > 0$ and $b < -1$, then $M(a,b) = 1$ by Lemma 4. Also, if $a > 0$ and $b = -1$, then since $c_0 < 0$, $M(a,b) = \operatorname{sgn}(abc_0) = 1$ by Lemma 4. If $j \in R_i$, then $\mu_{\tau(i)} > \mu_j$ since $j < \tau(i)$ and $\tau(i)$ is the least index k so that $\mu_k = \mu_{\tau(i)}$ because $\tau \in P_\mu$. Since $\mu_{\tau(i)} > \mu_j$, it is easy to see from the definition of v_μ that $v_\mu(\tau(i)) > v_\mu(j)$. Thus, for all $j \in R_i$, it must be that

$$M(\mu_{\tau(i)} - \mu_j, v_\mu(j) - v_\mu(\tau(i))) = 1.$$

Moreover, if $m > 0$ and $j \geq 1$, then $M(m, -j) = 1$. Note that $F_\mu(j) \geq 1$ for all j. So, if $\mu_j > m+1$, then $M(\mu_j - m - 1, -F_\mu(j)) = 1$. Therefore, by Corollary 4,

$$\begin{aligned}\lim_{\kappa\to 0^+} N(\tau(\mu)) &= \lim_{\kappa\to 0^+} \prod_{i=1}^{n} \prod_{m=0}^{\mu_i - 1} \left((\kappa(m+1) - (d_0 - d_{-m-1}) - c_0 r(i-1)\right) \\ &= \lim_{\kappa\to 0^+} \prod_{i=1}^{n} \prod_{m=1}^{\mu_i} \left((\kappa(m) - (d_0 - d_{-m}) - c_0 r(i-1)\right)\end{aligned}$$

Fix some $i \leq n$, and let $1 \leq m \leq \mu_i$. Suppose that $-m \equiv j \pmod r$ where $0 \leq j \leq r-1$. If $j = 0$, then $d_{-m} = d_0$ and

$$\kappa(m) - (d_0 - d_{-m}) - c_0 r(i-1) = \kappa(m) - c_0 r(i-1) > 0.$$

Otherwise, if $i > g_j$, then

$$\kappa(m) - (d_0 - d_{-m}) - c_0 r(i-1) > \kappa(m) - (d_0 - d_j) - c_0 r \cdot g_j > 0,$$

and if $i \leq g_j$, then

$$\kappa(m) - (d_0 - d_{-m}) - c_0 r(i-1) \leq \kappa(m) - (d_0 - d_j) - c_0 r(g_j - 1) < 0.$$

Thus, $\lim_{\kappa\to 0^+} N(\tau(\mu))$ is equal to

$$\prod_{j=1}^{r-1} \prod_{s=1}^{g_j} \prod_{\substack{1 \leq m \leq \mu_i \\ -m \equiv j \pmod r}} (-1) = \prod_{j=1}^{r-1} \prod_{s=1}^{g_j} (-1)^{\left\lfloor \frac{\mu_s + j}{r} \right\rfloor},$$

since there are $\left\lfloor \frac{\mu_s + j}{r} \right\rfloor$ values of m that are at most μ_i and equivalent to $-j$ modulo r. ☐

Corollary 5. *Suppose that $k \geq 0$ is an integer. If $n > k + \max_{1 \leq i \leq r-1} g_i$, the coefficient of t^k in the signature character as $\kappa \to 0^+$ is*

$$\binom{n+k}{k}.$$

Proof. Let $\mu \in \mathbb{Z}_{\geq 0}^n$ and suppose that $\mu_1 + \mu_2 + \cdots + \mu_n = k$. Since $n > k + \max_{1 \leq i \leq r-1} g_i$, there must be at least $\max_{1 \leq i \leq r-1} g_i$ of the values $\mu_1, \mu_2, \cdots, \mu_n$ that are 0. Thus, by Lemma 5, the norm of μ must be positive. Therefore, the coefficient of t^k in the signature character is the number of μ with $\mu_1 + \mu_2 + \cdots + \mu_n = k$ which is easily seen to be $\binom{n+k}{k}$. ☐

Note that the greatest integer g_j so that $d_j < d_0 + rc_0(g_j - 1)$ is bounded above by $\frac{d_j - d_0 + rc_0}{rc_0}$. Therefore, for all

$$n > \max_{1 \le j \le r-1} \max(0, \frac{d_j - d_0 + rc_0}{rc_0}) + k,$$

the coefficient of t^k in the asymptotic signature is $\binom{n+k}{k}$.

Thus, we see that if we look at the signature character of the polynomial representations of the Cherednik algebra for $G(r,1,n)$, then for each k the t^k coefficient ofthe asymptotic signature character for $c_0 < 0$, grows as $\binom{n+k}{k}$, a polynomial in n for large enough n. In particular, this computation shows that for large enough n, the degree k portion of the polynomial representation is unitary. Thus, we have proved the claim that we made in the introduction regarding the stable unitarity of the polynomial representation, if $c_0 < 0$.

3.6 Computation of the Asymptotic Signature for Certain Values of g and n

It is interesting to see a concrete formula for the asymptotic signature character. In this section, we derive a recurrence relation for the asymptotic signature when $r = 2$ and $c_0 < 0$, which allows us to compute the value of the asymptotic signature for small values of n. Additionally, we use this recurrence to give an alternative proof in the case of $r = 2, c_0 < 0$ for the known fact that the asymptotic signature character is a rational function in t.

Let $p_{i,n} = 1 - t^n - \binom{n}{1}t^{n-1}(1-t) - \cdots - \binom{n}{i}t^{n-i}(1-t)^i$ for $0 \le i \le n$. Let $a_{i,n}(t)$ and $b_{i,n}(t)$ be rational functions of t so that $a_{0,n} = b_{0,n} = 1/(1-t)^n$ and

$$a_{i,n} = \sum_{j=1}^{i} (-1)^i \binom{n}{j} \cdot \frac{t^{n-j}}{1-(-1)^i t^{2n}} \cdot (t^n a_{i-j,n-j} + (-1)^j b_{i-j,n-j}) + \frac{p_{i,n}((-1)^i t^n + 1)}{(1-t)^n(1-(-1)^i t^{2n})}$$

$$b_{i,n} = \sum_{j=1}^{i} \binom{n}{j} \frac{t^{n-j}}{1-(-1)^i t^{2n}} ((-1)^{i-j} t^n \cdot b_{i-j,n-j} + a_{i-j,n-j}) + \frac{p_{i,n}(t^n+1)}{(1-t)^n(1-(-1)^i t^{2n})},$$

for $1 \le i \le n$.

Theorem 5. *Suppose that $\kappa \to 0^+$, $c_0 < 0$ and $r = 2$. Moreover, let g be the greatest integer less than or equal to n so that $d_1 < d_0 + 2c_0(g-1)$. Then, the asymptotic signature is given by $a_{g,n}$.*

Proof. Define generating functions $h_{i,n}$ and $g_{i,n}$ as

$$h_{i,n}(t) = \sum_{\mu=(\mu_1 \le \mu_2 \le \cdots \mu_n)} \sum_{\tau \in P_\mu} t^{\mu_1+\mu_2+\cdots+\mu_n} (-1)^{\lfloor \frac{\mu_1+1}{2} \rfloor + \lfloor \frac{\mu_2+1}{2} \rfloor + \cdots + \lfloor \frac{\mu_i+1}{2} \rfloor}$$

and

$$g_{i,n}(t) = \sum_{\mu=(\mu_1 \le \mu_2 \le \cdots \mu_n)} \sum_{\tau \in P_\mu} t^{\mu_1+\mu_2+\cdots+\mu_n} (-1)^{\lfloor \frac{\mu_1}{2} \rfloor + \lfloor \frac{\mu_2}{2} \rfloor + \cdots + \lfloor \frac{\mu_i}{2} \rfloor}.$$

It will be shown that $h_{i,n} = a_{i,n}$ and $g_{i,n} = b_{i,n}$. Note that $h_{0,n} = g_{0,n} = 1/(1-t)^n$ since the representations are unitary when $i = 0$, and in that case the generating functions just reduce down to the generating function of $1/(1-t)^n$. Now, suppose that $i \ge 1$.

Let $\mu \in \mathbb{Z}_{\ge 0}^n$ and let $\mu_{g(1)}, \mu_{g(2)}, \cdots, \mu_{g(i)}$ be the i smallest elements of μ. Suppose that l is the smallest element of μ and that there are j copies of l in μ. Let $\mu' \in \mathbb{Z}_{\ge 0}^{n-j}$ be the tuple formed by removing the

j values of l from μ and reducing all the other elements by $l+1$. Let $\mu_{h(1)}, \mu_{h(2)}, \cdots, \mu_{h(i-j)}$ be the $i-j$ smallest elements of μ'. Note that if $j \leq i$ and $l = 2a$ for some integer a, then

$$(-1)^{\left\lfloor \frac{\mu_{g(1)}+1}{2} \right\rfloor + \left\lfloor \frac{\mu_{g(2)}+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\mu_{g(i)}+1}{2} \right\rfloor} = (-1)^{ja} \cdot (-1)^{(i-j)(a+1)} \cdot (-1)^{\left\lfloor \frac{\mu'_{h(1)}}{2} \right\rfloor + \left\lfloor \frac{\mu'_{h(2)}}{2} \right\rfloor + \cdots + \left\lfloor \frac{\mu'_{h(i-j)}}{2} \right\rfloor}$$

while if $i > j$, then

$$(-1)^{\left\lfloor \frac{\mu_{g(1)}+1}{2} \right\rfloor + \left\lfloor \frac{\mu_{g(2)}+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\mu_{g(i)}+1}{2} \right\rfloor} = (-1)^{ia}.$$

Now, if $l = 2a+1$ for some integer a and $i \leq j$, then

$$(-1)^{\left\lfloor \frac{\mu_{g(1)}+1}{2} \right\rfloor + \left\lfloor \frac{\mu_{g(2)}+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\mu_{g(i)}+1}{2} \right\rfloor} = (-1)^{j(a+1)} \cdot (-1)^{(i-j)(a+1)} \cdot (-1)^{\left\lfloor \frac{\mu'_{h(1)}+1}{2} \right\rfloor + \left\lfloor \frac{\mu'_{h(2)}+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\mu'_{h(i-j)}+1}{2} \right\rfloor}$$

while if $i > j$, then

$$(-1)^{\left\lfloor \frac{\mu_{g(1)}+1}{2} \right\rfloor + \left\lfloor \frac{\mu_{g(2)}+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\mu_{g(i)}+1}{2} \right\rfloor} = (-1)^{i(a+1)}.$$

For a given l and j, there are $\binom{n}{j}$ ways to choose the positions in μ to be l. Also,

$$\mu_1 + \mu_2 + \cdots + \mu_n = \mu'_1 + \mu'_2 + \cdots + \mu'_{n-j} + (l+1)(n-j) + lj = \mu'_1 + \mu'_2 + \cdots + \mu'_{n-j} + ln + n - j.$$

Therefore,

$$\begin{aligned} h_{i,n}(t) = \sum_{a=0}^{\infty} \Big(& \sum_{j=1}^{i} (-1)^{j(a+1)} \binom{n}{j} (-1)^{(i-j)(a+1)} h_{i-j,n-j}(t) \cdot t^{2an+2n-j} \\ & + \sum_{j=i+1}^{n} (-1)^{i(a+1)} \binom{n}{j} \frac{1}{(1-t)^{n-j}} \cdot t^{2an+2n-j} \Big) \\ + \sum_{a=0}^{\infty} \Big(& \sum_{j=1}^{i} (-1)^{ja} \binom{n}{j} (-1)^{(i-j)(a+1)} g_{i-j,n-j} \cdot t^{2an+n-j} \\ & + \sum_{j=i+1}^{n} (-1)^{ia} \binom{n}{j} \frac{1}{(1-t)^{n-j}} \cdot t^{2an+n-j} \Big). \end{aligned}$$

The first sum over a and j is the case where the lowest value in μ is $l = 2a+1$ and there are j copies of l in μ. The second sum over a and j is where $l = 2a$ and there are j copies of l in μ. Simplifying the expression gives that

$$h_{i,n} = \sum_{j=1}^{i} (-1)^i \binom{n}{j} \cdot \frac{t^{n-j}}{1-(-1)^i t^{2n}} \cdot (t^n h_{i-j,n-j} + (-1)^j g_{i-j,n-j}) + \frac{p_{i,n}((-1)^i t^n + 1)}{(1-t)^n (1-(-1)^i t^{2n})}.$$

By a similar analysis, the recurrence for $g_{i,n}$ is

$$g_{i,n} = \sum_{j=1}^{i} \binom{n}{j} \frac{t^{n-j}}{1-(-1)^i t^{2n}} ((-1)^{i-j} t^n \cdot g_{i-j,n-j} + h_{i-j,n-j}) + \frac{p_{i,n}(t^n+1)}{(1-t)^n (1-(-1)^i t^{2n})}.$$

Note that since $h_{0,n} = a_{0,n}$, $g_{0,n} = b_{0,n}$, and $h_{i,n}$ and $g_{i,n}$ satisfy the same recurrence relations as $a_{i,n}$ and $b_{i,n}$, respectively, $h_{i,n} = a_{i,n}$ and $g_{i,n} = b_{i,n}$. Since $h_{g,n}$ is exactly the value of the asymptotic signature, $a_{g,n}$ is the asymptotic signature in this case. □

Corollary 6. *The functions $a_{i,n}$ and $b_{i,n}$ are rational functions of t.*

Proof. The result will be proven by induction on i. Note that for $i = 0$, $a_{0,n} = b_{0,n} = 1/(1-t)^n$ for all $n \geq 0$. Now, suppose that for all $i \leq m$, $a_{i,n}$ and $b_{i,n}$ are rational functions of t for all $n \geq i$. Now, suppose that $i = m+1$ and $n \geq i$. Note that by the recurrence relation, $a_{i,n}$ and $b_{i,n}$ are sums of rational functions of t by the inductive assumption. Therefore, $a_{i,n}$ and $b_{i,n}$ must be rational functions of t. □

Example. Table 1 shows the asymptotic signature characters for small values of n (computed in SAGE).

Table 1: Asymptotic Signature for Small Values

n	g	$a_{g,n}$ (Asymptotic Signature)
1	1	$\frac{1-t}{t^2+1}$
2	1	$\frac{(t+1)^2}{t^4+1}$
2	2	$\frac{(t-1)^2}{(t^2+1)^2}$
3	1	$\frac{-(t^2+t+1)^2}{(t^4-t^2+1)(t^2+1)(t-1)}$
3	2	$\frac{t^6+2t^5-2t^4-2t^2+2t+1}{(t^4+1)(t^2+t+1)(t^2-t+1)(1-t)}$
3	3	$\frac{(1-t)^3}{(t^2+1)^3}$

4 Conclusion and Future Studies

In this paper, we have computed the signatures of all the Specht modules for the Hecke algebra. We also computed a formula for the asymptotic signature character of the polynomial representation of the Cherednik Algebra for half of all parameter values. The immediate next step would be to extend the results to the other half of the parameter values, and also extend the recurrence relation obtained for $r = 2$ to higher values of r and refine the recurrence to get a better algorithm for computing the asymptotic signature character for small values of n.

The full signature character of arbitrary Verma modules has already been computed by Steven Griffeth but this formula is very messy. Simplifying those formulas would be useful in computing formulas for the asymptotic signature character of arbitrary Verma modules similar to those we obtained here. This could then be used to study the stability properties of the asymptotic signature. More precisely, given a Verma module of the Cherednik algebra of $G(r,1,n)$ associated to some multiparition λ of n, we can define a multipartition λ_m of $n+m$ by adding m boxes to the first row of the first tableau in the multitableau. For example, if $r = 2$, $n = 5$ and the multiparition λ is $((2,1),(1,1))$ then, λ_m is $((2+m,1),(1,1))$. Each λ_m defines a Verma module of the Cherednik algebra of $G(r,1,n+m)$ and we call such a sequence of

Verma modules a stable sequence. The dependence of the asymptotic signature character of such stable sequences in n is interesting to study. In this paper, we showed that for half of the paramater values, each coefficient of the asymptotic signature character of the polynomial representation was eventually polynomial in n, and moreover, for large enough n the signature in degree k was unitary (with the cutoff value of n depending linearly with k). It is expected that similar results for other stable sequences of Verma modules and for all parameter values hold, namely that each coefficient of the asymptotic signature character is a polynomial in m for large enough m. This should be true for general reasons, but a constructive proof would give a better understanding of the actual value of the polynomial. The range of parameters for which the asymptotic signature character is stably unitary would be an interesting consequence of knowing the value of the polynomial.

This stable behavior of the asymptotic signature character has connections with the Deligne category of the Cherednik Algebra. Any stable sequence of Verma modules as defined above gives an object in the Deligne category, which has an associated signature character. There should exist some relationship between the asymptotic signature character of the stable sequence and the signature character of the object in the Deligne category but this is not well understood. The computation of the polynomial in m that represents the stable limit of the coefficient of t^k would help to better understand the relationship and to compute the signature character of the object in the Deligne category.

In the polynomial representation studied here, and in general for all Verma modules of the Cherednik Algebra, we can decompose each graded piece into irreducible representations of $G(r,1,n)$. This decomposition is orthogonal for the contravariant form and hence the form induces a non-degenerate form on the multiplicity space of each irreducible of $G(r,1,n)$. Refining the signature character by computing the signatures of these forms on the multiplicity spaces would make the signature formula nicer.

Another direction is to see what happens at other values of parameters than the generic values. In this case, the contravariant form on the Verma module is possibly degenerate, but the kernel of the form coincides with the largest proper submodule of the Verma module. Thus, on the quotient module, there exists a non-degenerate Hermitian form, and computing the signature of this form would be interesting.

5 Acknowledgements

I sincerely thank Siddharth Venkatesh for his guidance throughout this project. I also thank Prof. Pavel Etingof for suggesting the project and many helpful conversations. Additionally, I am grateful to Prof. Steve Griffeth for his advice on the project. Furthermore, I thank Dr. Tanya Khovanova for suggestions regarding this paper. I express my gratitude to Vidya Venkateswaran for her time spent explaining her results. I thank MIT mathematics student Saarik Kalia for working with me on the part of the project computing the asymptotic signature. Finally, I am grateful to MIT PRIMES for providing me with an opportunity to perform research.

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